Option overlay strategies for an existing portfolio

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Introduction

- Wealth management does not have a clearly defined objective.
- One recognizes that ultimately the final outcome has a random component and the theory of rational behavior under uncertainty suggests that risk attitudes be addressed by attempting to maximize the expected utility of final wealth.
- But what is this final date and is it reasonable to focus such a single minded attention on a fixed date.
- After all there may be strategies that trade wealth at other dates for an accumulation at the fixed date that are possibly undesirable in their consequences for the other dates.
- Such considerations have led to the use of the expected utility of a consumption stream through time as an objective.
- This could be fine from the perspective of a single individual, but when wealth is managed on behalf of generations to come there is no consumption in sight.
- The potential consumers are as yet unborn.
The Overlay Problem

- The problem we consider here is that of using options within limits to enhance the performance of an existing position without altering the basic position of being long with respect to the underlier.
- We add to this position some exposure via options.
- However, this exposure is limited by bounding the absolute difference between the final delta and the delta of the existing position.
- The problem is then one of an optimal constrained option overlay as opposed to an optimal positioning in derivatives.
With regard to an objective function, we are considering wealth management from the perspective of a large fund operating in the interests of generations to come.

There are then no consumption streams and certainly no terminal wealth dates matching option maturities.

Instead our focus turns on maximizing the market value of the position taken.
There is a current market value that may be accessed via a risk neutral density.

The objective focuses attention on a future market value that has to be modeled.

For this we turn to recent developments in the theory of two price economies as set out in Cherny and Madan (2010), and Madan (2012, 2013).

Such economies have also been studied in Jouini and Kallal (1995), Bion Nadal (2009), and Guasoni, Lepinette and Rasonyi (2012).
Bid and Ask in Two Price Economies

- In such economies all assets must be sold to market at a lower bid price while liabilities may be unwound at a higher ask price.
- Furthermore the bid and ask price functionals are related by the ask price being the negative of the bid price for the negative cash flow.
- This relationship is a consequence of the recognition that selling a cash flow must be equivalent to buying its negative, (Madan (2013)).
- The bid price functional is itself a nonlinear concave functional arriving at a conservative valuation that is the infimum of a set of expectations taken with respect to a convex set of test valuations or scenario probabilities.
- All the scenario probabilities are equivalent to the base probability in the sense of agreeing on the events with positive probability.
If this bid price or conservative valuation is further taken to depend on just the cash flow’s probability distribution function under the base probability and one also demands additivity for comonotone risks then Kusuoka (2001) has shown that such a bid price is just an expectation taken under a concave distortion of the distribution function under the base probability.

Such distorted valuations are well known and have been consistently applied in the insurance literature for some time (Wang (1995), (1996), (2000)), Hamada and Sherris (2003)). The approach also has connections to the work of Yaari (1987).

The concave distortion is constructed by just composing the base distribution function with another concave distribution function on the unit interval.
Summary of Result

- We are then led maximizing such a distorted expectation subject to a delta constraint on the final position.
- We obtain a closed form expression for this final position.
- It turns out that the final position is based on a comparison between the risk neutral and distorted distribution functions, as opposed to densities.
- The latter is involved in expected utility maximization.
- The final position for conservative value maximization is therefore more robust.
The first step in constructing the designed option overlay is to describe the initial position being overlaid.

We take by way of example a rebalanced portfolio that is preset to rebalance to an equity cash ratio of 1.5 whenever this ratio reaches predefined upper and lower bounds.

Such a rebalance strategy can be observed to be one that accesses a concave function of the underlying asset price at the option maturity.

The strategy sells a rising market, reducing delta and buys a falling market or increasing delta on the way down.
The second step is the construction of the base probability used in evaluating the option overlaid cash flow to be accessed.

One may employ time series analysis on past daily returns to estimate the daily return distribution.

However, the probability of interest is that for returns at the option maturity and this will typically be a few months away.

We combine the effects of independent shocks and the effects of self similarity.
The third step is the construction of the risk neutral density at the option maturity.

This is done parsimoniously by fitting the four parameter Sato process based on the variance gamma model to option prices at a range of selected maturities.

The adequacy of this model for synthesizing option prices was demonstrated in Carr, Geman, Madan and Yor (2007).
The fourth step puts together the base probability, its distortion, and the risk neutral probability to construct the constrained bid price maximizing position.

The option position is then the swap of the initial position for this bid price maximizing position.

We set up a back test that takes up option positions in maturities near two months and after a rebalance monitoring date.

The option positioning date is a fixed number of days prior to a rebalance date.

The option positions are held to their maturity and unwound by exercise.
The Rebalanced Cash Flow

- Denote by \( T \) the time to maturity of the options and let \( S \) be the level of the equity index at this maturity.
- Let the current date on which we take the option positions be time 0 with the initial index level being \( S_0 \).
- The cash flow to be overlaid is some function of the index level at the option maturity that we denote by \( c(S) \).
- Here we consider a relatively simple trading strategy that seeks to rebalance to an equity ratio of \( \eta \) if at an intermediate monitoring date \( t \) the equity ratio exceeds \( \eta + h \) or falls below \( \eta - h \).
- Let the index value at the monitoring date \( t \) be \( s \).
- The initial cash account is \( B \) dollars.
- Accounting for the rebalance one may show that the portfolio value at time \( T \) is

\[
V = (1 - \eta)B + \eta nS + \eta B \frac{S}{S} + (1 - \eta)ns.
\]
Reverse Conditional Expectations

- We may simplify the expression for $V$ further by evaluating the reverse conditional expectation or

$$E[s|S] = E[S(t)|S(T)].$$

- For a log normal process we show in the appendix that

$$E[s|S] = S^\theta$$

for $\theta < 1$.

- For a variety of other processes we also observe in the appendix empirically that the dependence has this shape to first order.

- Evaluating the final value at this conditional expectation we approximate

$$c(S) = (1 - \eta)B + \eta nS + \eta BS^{1-\theta} + (1 - \eta)nS^\theta.$$

- The rebalanced portfolio value is then seen as a concave function of the level of the index at the option maturity.
Identifying the base probability at the option maturity

- For a prospective option positioning day with a particular maturity selected we have to estimate the probability density or distribution function for the level of the underlying asset at this maturity.
- Typically such a maturity will be between one to three months.
- There are very few observations of returns over such a long time interval to allow for a direct estimation of the distribution.
- We therefore estimate a distribution over a much shorter interval like a day for which we have ample data and then address how to build the longer horizon return distribution from the estimated shorter one.
- We note up front however that mean returns are very difficult to estimate from data.
- Furthermore, we do not wish to position in options with any conjectured or estimated directional focus.
- Additionally, for positioning over horizons like a month or two, one does not expect to be able to realize any mean.
Given data on daily returns we wish to employ a parsimonious distributional model capable of capturing movements in skewness and kurtosis in addition to volatility.

We may model the demeaned daily log price relative for the index level as a centered variance gamma variate. Given a gamma variable $G$, with unit mean and variance $\nu$ the centered variance gamma variable $X$ has the distribution of

$$X = \theta (G - 1) + \sigma \sqrt{G} Z$$

where $Z$ is a standard normal variable independent of $G$.

The parameter $\sigma$ reflects the underlying volatility, skewness is captured by $\theta$ and $\nu$ calibrates the excess kurtosis or the volatility of volatility.
The centered variance gamma law has a simple characteristic function given by

\[ \phi_X(u) = E \left[ \exp (iuX) \right] = e^{-iu\theta} \left( \frac{1}{1 - iu\theta + (1/2)\sigma^2\nu u^2} \right)^{1/\nu} \]
Given that our interest is in getting the probabilities of being in different intervals relatively correct and not as such in the parameters. Also knowing that the data in probability do not come from this model the superiority of maximum likelihood estimation is called into question.

Madan (2013a) shows that from the perspective matching the probabilities of intervals it may be better to estimate parameters by matching the observed digital moments. These are bounded moments while the score function of maximum likelihood often involves unbounded moments.

We therefore use in this study, digital moment estimation by choosing parameters to minimize the sum of squared deviations between observed tail probabilities and the tail probabilities computed from the model.
Digital Call Price Function

- For the model probabilities we used a digital option price computation marginally generalizing the procedures of Carr and Madan (1999).
- For completeness one may define the price, \( c(k) \), of a digital call in log strike \( k \) in terms of the density \( f(x) \) for the log price relative \( \ln(S/S_0) \) as \( c(k) = \int_k^{\infty} f(x) \, dx \).
- The modified digital call price is defined by \( e^{\alpha k} c(k) \) for \( \alpha > 0 \) and is a square integrable function for small positive \( \alpha \).
- The Fourier transform of the modified call price \( \gamma(u) \) is given by

\[
\gamma(u) = \int_{-\infty}^{\infty} e^{iku} \gamma(u) \, du
\]

\[
= \frac{\phi(u - i\alpha)}{(\alpha + iu)}
\]

where \( \phi(u) \) is the characteristic function for the log price relative. Fourier inversion of \( \gamma(u) \) using the fast Fourier transform yields the the modified digital call price and hence the call price.
Employing daily return data each day from December 13 1984 to April 30 2013, using five years of daily return data immediately preceding the estimation date we estimated the $VG$ parameters for the S&P index by such a digital moment estimation procedure.

We present the results for the last day of April 30 2013. The reported parameter values are annualized.
We need to construct the two month physical return distribution using parameters for the $VG$ estimated from daily returns by digital moment estimation.

Let $X$ be the $VG$ law for the logarithm of the stock’s martingale component.

$$X = \log\left(\frac{S}{S_0}\right) + \omega$$

where $\omega$ is set to ensure $E[S] = S_0$.

Let $X_h$ be the law for the log return at a longer time step $h$ with

$$E[\exp(X_h)] = 1.$$
By self decomposability of the variance gamma law we have that for every \( c, 0 < c < 1 \)

\[
X \overset{law}{=} cX + X^{(c)}
\]

where \( X^{(c)} \) is independent of \( X \).

We define the long horizon return by running \( cX \) as a Lévy process for \( h \) units of time and by scaling the independent component \( X^{(c)} \).

Specifically we write

\[
X_h \overset{law}{=} (cX)(h) + h^\gamma X^{(c)}.
\]
We apply this procedure to construct long horizon returns that permit skewness and excess kurtosis to fall but not as fast as they do for a Lévy process.

We define

\[ S_h = S_0 \exp (X_h) . \]

For the constructions employed in this paper we use the value for \( c, \gamma \) as recommended in Eberlein and Madan (2010), \( c = \gamma = 0.5. \)
Some examples

- This procedure gives us access to the physical probability at arbitrary horizons from time series data on the daily returns of an underlier.
- We present a sample of physical and risk neutral two month densities for the S&P 500 index as estimated on December 17, 2007, October 16, 2008 and August 1, 2011.
- The risk neutral densities are extracted from option surfaces as described later.
- One may observe the considerable width of the risk neutral density on October 16, 2008 a month after the Lehman default.
The risk neutral density or pricing probability may be recovered from any probability model consistent with option prices across strikes and maturities.

It is also known from Carr and Madan (2005) and Davis and Hobson (2007) that option price quotations are free of static arbitrage opportunities just if they are consistent with a one dimensional Markov martingale process for the discounted price process for the underlying asset.

A number of exponential Lévy processes, including the variance gamma model have been observed to be consistent option prices at a single maturity.

However, it was observed in Konikov and Madan (2002) that all Lévy processes had a structural rate of decay in skewness and excess kurtosis that was market inconsistent.

Carr, Geman, Madan and Yor (2007) then developed the Sato process as a market consistent alternative.
For the Sato process one begins with the law of a special Lévy process at unit time.

One requires that the law at unit time be not only infinitely divisible as it is for a Lévy process, but that it also be self decomposable.

This property requires that the Lévy density when scaled by the absolute value of the jump size, be monotone decreasing for positive jumps and monotone increasing for negative jumps.

The variance gamma Lévy density when scaled by the absolute jump size is a negative exponential for positive jumps and a positive exponential for negative jumps, and hence is self decomposable.
Sato (1991) showed that for any self decomposable law at unit time one may define an additive process with independent but inhomogeneous increments with marginal laws that are a time scaling of the law at unit time.

For example, given a variance gamma law at unit for the random variable $X$, one may define marginals laws for $X(t)$ at all times $t$ with

$$X(t) \overset{(d)}{=} t^\gamma X.$$ 

The Sato process then has these marginal laws.
The Sato process based on the variance gamma at unit time then has four parameters, $\sigma$, $\nu$, $\theta$ and $\gamma$.

Carr, Geman, Madan and Yor (2007) showed that this four parameter model for the logarithm of the stock was market consistent.

We extract the risk neutral density at each date from market option prices by fitting this four parameter model to the quoted prices of options for all traded and liquid strikes and maturities.
Sato Fit Results

- The Figure presents a graph of the fit to data on out of the money options as at market close on April 30, 2013.
- There were 424 options across 13 maturities fit with the four parameters of the VGSSD Sato process.
- The parameter estimated were $0.1476$, $1.3221$, $-0.0963$, and $0.6312$ for $\sigma$, $\nu$, $\theta$ and $\gamma$ respectively.
- The fit statistics were $1.3868$, $1.0417$ and $0.0548$ for the root mean square error, the average absolute error and the average percentage error respectively.
- The estimations have been conducted for each day between October 22, 2007 and April 30, 2013.
Designing the optimal positions

- We have extracted from daily return data a physical density for the stock at an option maturity, typically around two months, that we denote by \( p(S) \).
- The corresponding distribution function is \( P(S) \).
- Additionally we have estimated from option price quotations the risk neutral density \( q(S) \) with distribution function \( Q(S) \).
- Finally we have modeled an existing initial position of rebalanced portfolio as a concave function of the underlying stock price that we denote by \( c(S) \).
- The task of now is to combine these three inputs to form a desired optimal position denoted here by \( c^*(S) \). The actual cash flow to be accessed by taking positions in options is then

\[
a(S) = c^*(S) - c(S).
\]
Without any loss of generality we may scale the problem to a discussion of gross returns by setting the initial level of the underlying asset at unity.

Before developing the procedures for a proposed alternative of maximizing a conservative market valuation of the position we review the classical solution of maximizing expected utility for a selected utility function.

The classic expected utility maximization problem seeks to choose \( c^*(S) \) with a view to maximizing the expected utility

\[
\int_0^\infty U(c^*(S)) p(S) dS
\]

for a utility function \( U(W) \) where \( W \) represents the terminal wealth.

This constraint is then given by

\[
\int_0^\infty c^*(S) q(S) dS = \int_0^\infty c(S) q(S) dS.
\]
The solution is given by equating the expected marginal utility per initial dollar expended across all states to the implied marginal utility of wealth $\lambda$ or by the condition

$$\frac{U'(c^*(S))p(S)}{q(S)} = \lambda.$$ 

Equivalently the optimal cash flow to be accessed may be written as

$$c^*(S) = (U')^{-1} \left( \frac{\lambda q(S)}{p(S)} \right).$$

The constant $\lambda$ is solved for by satisfying the budget constraint.
Some EU results

- Figures illustrate such expected utility maximizing optimal designs for March 11, 2010 and August 3, 2010.
Comments on the EU solution

- From the structure of the solution it may be noticed that the solution is driven by the ratio of two densities, as it is the decreasing monotone function of inverse marginal utility applied to the ratio of the risk neutral to the physical density.
- As such the solution seeks cash flow when probability density exceeds the pricing density and sells cash flow otherwise.
- As typically price exceeds probability in the tails the solution sells tail cash flows.
- It also completely replaces the initial position by the utility maximizing position and will often take positions that short the underlying asset to the point of attaining a negative final delta in many regions of exposure.
Conservative Value Maximization

- Expected utility theory is based on an axiomatic analysis of rational behavior under uncertainty in abstract.
- In particular the theory abstracts from the existence of a financial market that every day values many human and economic activities.
- An alternative criterion may be sought by attempting to directly maximize the market value of the position taken.
- Of course the market value at initiation is given by the risk neutral or pricing measure and there is no point in a valuation under this measure.
- One has to model a prospective probability that may possibly be prevailing at or near maturity, in case the position had to be unwound at this later date.
- The exercise then shifts to modeling a future market valuation.
We consider here the valuation operators of two price economies.

Such economies have been studied in Cherny and Madan (2010), Madan and Schoutens (2012), Madan (2012) and Eberlein, Madan, Pistorius, Schoutens and Yor (2013).

In two price economies, the law of one price fails as the market requires a positive alpha under a whole set of test probabilities to do a trade and is not generous enough to accept the other side of a trade for a positive alpha under a single pricing rule.

The set of acceptable risks is reduced from a half space to smaller convex set containing the nonnegative cash flows.

As a consequence the market buys at a bid or lower price and sells at a higher or ask price.

The ask price is also just the negative of the bid price for the negative cash flow as selling $X$ is equivalent to buying $-X$.

So there is just one pricing functional, say the bid pricing functional.
If the market requires a positive alpha under a set $\mathcal{M}$ of test or pricing probabilities then the bid price $b(X)$ for a random cash flow $X$ is given by

$$b(X) = \inf_{Q \in \mathcal{M}} E^Q[X].$$

The bid price of a two price economy is then a concave pricing functional and we take such a functional as our objective in designing the optimal cash flow.
It is shown in Kusuoka (2001) that if the bid price functional for a random cash flow is simply a function of the distribution function for the cash flow and additionally we require additivity for comonotone risks $X, Y$ or that

$$b(X + Y) = b(X) + b(Y),$$

then the bid price functional takes a simpler and more specific form.

Under these conditions there must exist a concave distribution function $\Psi$ on the unit interval such that for all random variables $X$ with distribution function $F_X(x)$ it is the case that

$$b(X) = \int_{-\infty}^{\infty} xd\Psi(F_X(x)).$$
Concave Distortion and Measure Change

- It is worthwhile noticing that such a distorted expectation is also an expectation under a change of measure as

\[ b(X) = \int_{-\infty}^{\infty} x \Psi'(F_X(x)) f_X(x) \, dx, \]

where \( f_X(x) \) is the probability density for the cash flow.

- The change of measure is given by \( \Psi'(F_X(x)) \) and by virtue of the concavity of \( \Psi \), the lower quantile reflecting losses are reweighted upwards while the upper quantiles are reweighted downwards.

- The nonlinearity is reflected in the dependence of the change of probability on the cash flow being valued as we evaluate \( \Psi' \) at the quantile \( F_X(x) \).

- We call the objective function a conservative value maximization and study the design of option positions under such an objective.
Consider the choice of $c^*$ to maximize a conservative financial value modeled by a distorted expectation using a fixed concave distribution function $\Psi(u)$, $0 \leq u \leq 1$.

The objective function is then

$$
\int_{-\infty}^{\infty} xd\Psi(F(x))
$$

where

$$
F(x) = \int_{c^*(S) \leq x} p(S) dS.
$$
Using the inverse distribution function

- Define the inverse distribution function $G(x)$ by
  \[ G(F(x)) = x. \]

- We may write the objective function as
  \[
  \int_0^1 G(u) d\Psi(u) = \int_0^1 \int_0^u g(v) dv d\Psi(u) \\
  = \int_0^1 g(v) dv \int_v^1 d\Psi(u) \\
  = \int_0^1 g(v) (1 - \Psi(v)) dv.
  \]
Monotone Cash Flows

- We may choose the optimal cash flow to be comonotone with a long position in the stock and define

\[ c^*(S) = G(P(S)) \]

- in which case we have that

\[
\begin{align*}
\Pr(c^*(S) \leq x) & = \Pr(G(P(S)) \leq x) \\
& = \Pr(P(S) \leq F(x)) \\
& = F(x).
\end{align*}
\]

- We then have accessed the distribution function \( F(x) \).
Transforming the budget constraint

- The budget constraint may be written as
  \[ \int_0^\infty G(P(S))q(S)\,dS = \int_0^\infty c(S)q(S)\,dS = C_0 \]

- The budget constraint may be rewritten as
  \[
  C_0 = \int_0^1 G(u)\frac{q(P^{-1}(u))}{p(P^{-1}(u))}\,du \\
  = \int_0^1 \int_0^u g(v)\,dv \frac{q(P^{-1}(u))}{p(P^{-1}(u))}\,du \\
  = \int_0^1 g(v)\,dv \int_v^1 \frac{q(P^{-1}(u))}{p(P^{-1}(u))}\,du \\
  = \int_0^1 g(v)\,dv \int_{P^{-1}(v)}^\infty q(S)\,dS \\
  = \int_0^1 g(v) \left(1 - Q\left(P^{-1}(v)\right)\right)\,dv
  \]
The Delta constraint

- We may impose the constraint
  \[ \alpha c(S) \leq c^*(S) \leq \beta c(S) \]
- by requiring
  \[ \alpha c(S) \leq G(P(S)) \leq \beta c(S) \]
- or
  \[ \alpha c(P^{-1}(v)) \leq G(v) \leq \beta c(P^{-1}(v)) \]
- Instead we impose a stronger condition at the derivative level and require that
  \[ \frac{\alpha c'(P^{-1}(v))}{p(P^{-1}(v))} \leq g(v) \leq \frac{\beta c'(P^{-1}(v))}{p(P^{-1}(v))} \]
- The objective function, budget constraint and the delta constraints are now all expressed in terms of the derivative of the inverse distribution function \( g(v) \), \( 0 \leq v \leq 1 \).
First Order Conditions

- The first order condition then yields

\[ c^*(S) = g(P(S))p(S) = \beta c'(S)1_{1-\lambda+\lambda Q(S)-\Psi(P(S)) \geq 0} + \alpha c'(S)1_{1-\lambda+\lambda Q(S)-\Psi(P(S)) < 0} \]

- The Lagrange multiplier \( \lambda \) is again determined to satisfy the budget constraint whereby the cost of \( c^* \) equals the cost of \( c \).

- Unlike expected utility maximization, the optimal cash flow depends on a comparison of distribution functions as opposed to densities.

- The result is therefore considerably more stable.
The distortion employed is \textit{minmaxvar} for which

\[ \Psi(u) = 1 - (1 - u^{1/(1+\gamma)})^{1+\gamma} \]

The value of \( \gamma \) employed is 0.1.

As a sample we present graphs for the option positions for March 11, 2010 and August 3, 2010.
Back Test of Positioning

- For the implementation of a back test on S&P 500 options for a base portfolio that rebalances every 21 days option positioning was undertaken 10 days prior to every rebalance date for a maturity closest to two months from the option positioning date.
- The option position is determined to have the same cost as the rebalanced portfolio and the overlay is designed to be delta constrained to between a lower and upper percentage of the base positions’s delta or the delta of the rebalanced portfolio.
- We present a graph of the value of a long position in the S&P 500 index at 100 million, the rebalanced portfolio starting at the same value with an equity ratio of .6 and three option overlaid portfolios.
- The overlays have different delta ranges as indicated for the delta constraint.
- For this set of results a hedge was not implemented in specific strikes and maturities and it was supposed that the desired position theoretical zero cost option position could be perfectly captured and unwound at the designed optimal cash flow.
We next implemented a specific hedge using traded options at the selected maturities and collected or paid any up front premiums.

The next Figure presents the result for the delta constraint of 0.7, 1.3.

Shown are the base SPX, the rebalanced portfolio, the theoretical unwind with no premiums and the unwind with premiums.
The problem of overlaying an option position on that of a rebalanced portfolio in a single underlier is formulated, solved, and back tested for the S&P 500 index as the underlier over the period October 22, 2007 to April 30, 2013.

It is first observed that automatic rebalancing by reducing delta on up moves and raising it on down moves engineers a concave position in the underlying asset.

Inputs for the overlay design are the physical and risk neutral densities for the underlying asset at the option maturity.

Daily return distributions are estimated by matching digital moments with a focus on matching critical probabilities.
Distributions for the option maturity are then generated by combining the principles of independent increments and self similarity with a view to curtailing the speed at which skewness and kurtosis decay with maturity.

Risk neutral densities are extracted by estimating the Sato process based on the variance gamma model as described in Carr, Geman, Madan and Yor (2007).
The objective for the overlay design maximizes a conservative market valuation of the position taken.

This conservative market value is modeled as the bid price of a two price economy and is defined as the infimum of a multitude of stressed expectations.

As such it is a nonlinear and concave valuation operator.

When modeled as a functional of the risk distribution function along with demanding additivity for comonotone risks the objective reduces to maximizing a distorted expectation with a concave distribution function constituting the distortion.
The distortion *minmaxvar* introduced in Cherny and Madan (2009) is used for the purpose.

The design is further constrained to be monotone increasing in the underlier with a prespecified minimal departure of the final delta from the original rebalanced portfolio delta.

The program is implemented on the S&P 500 index rebalanced every 21 days with option positions taken 10 days prior to a rebalance date with a maturity near two months.