Supermartingales as Radon-Nikodym densities, Novikov’s and Kazamaki’s criteria, and the distribution of explosion times

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This presentation is based on joint papers with Nicolas Perkowski, Martin Larsson, and Ioannis Karatzas

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This presentation has three parts

1. **Supermartingales as Radon-Nikodym densities.**
2. Canonical proof that certain conditions of the form
   \[ \sup_{\sigma \in \mathcal{T}} \mathbb{E}[H_{\sigma}] < \infty \]
   are sufficient for the uniform integrability / martingale property of a nonnegative local martingale \( Z \).
3. Computation of the distribution of the explosion time \( S \) of the diffusion
   \[ dX(t) = s(X(t)) (dW(t) + b(X(t))dt), \quad X(0) = \xi. \]

**Unifying theme:** A one-to-one correspondence of
- the lack of martingale property of a nonnegative local martingale;
- a positive probability of explosions of a related process.
(due to McKean and Föllmer)
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Supermartingales as Radon-Nikodym densities

1. Given:
   - Filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$;
   - nonnegative, right-continuous $P$–supermartingale $Z = (Z_t)_{t \geq 0}$ with $E_P[Z_0] = 1$.

2. “Can we somehow assign a measure $Q$ to $Z$, in the sense of a change of measure?”

3. Possible applications:
   - Deriving certain properties of $Z$ (martingale property, decompositions, ...);
   - Duality;
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   - Financial mathematics: study of arbitrage opportunities.
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Constructions of measures associated to $Z$

- Construction of finitely additive measure on $(\Omega \times [0, \infty], \mathcal{A})$, where $\mathcal{A} \subset \mathcal{P}$ is a suitable algebra, and where $\mathcal{P}$ denotes the predictable sigma algebra. (Doléans; Metivier & Pellaumail).

- Under certain topological assumptions on the filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$, construction of countably additive measure on $(\Omega \times [0, \infty], \mathcal{P})$ (3 different constructions: Fölmer; Meyer; Stricker). If $Z$ is local martingale then construction on $(\Omega, \mathcal{F})$.

- If $Z$ is the pointwise limit of a family of uniformly integrable martingales, then existence of a finitely additive measure on $(\Omega, \mathcal{F})$. (Cvitanić & Schachermayer & Wang; Karatzas & Žitković).
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Definitions: countably additive case

If $Q$ and $\tau$ are a probability measure and a stopping time on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$, then $(Q, \tau)$ is called a Föllmer pair for $Z$ if

$$P[\tau = \infty] = 1 \quad \text{and} \quad Q[A \cap \{\rho < \tau\}] = \mathbb{E}_P[Z_\rho 1_A] \quad \text{for all } A \in \mathcal{F}_\rho \text{ and finite s.t. } \rho.$$

(1)

We also call $Q$ a Föllmer (countably additive) measure for $(Z, \tau)$, or, slightly abusing notation, a Föllmer (countably additive) measure for $Z$.

- Inspired by Kunita-Yoeurp decomposition.
- Föllmer pair for $Z$ is unique if given $Q$, $\tilde{Q}$ and $\tau$, $\tilde{\tau}$ such that $(Q, \tau)$ and $(\tilde{Q}, \tilde{\tau})$ both satisfy (1), we have $Q = \tilde{Q}$ and $Q[\tau = \tilde{\tau}] = 1$.
- If $\tau$ is a stopping time, then Föllmer (c.a.) measure for $(Z, \tau)$ is unique if, given $Q$, $\tilde{Q}$ such that $(Q, \tau)$ and $(\tilde{Q}, \tau)$ both satisfy (1), we have $Q = \tilde{Q}$.
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If \( Q \) and \( \tau \) are a probability measure and a stopping time on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})\), then \((Q, \tau)\) is called a *Föllmer pair* for \( Z \) if

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- $\text{ba}_1(\Omega, \mathcal{F}, P)$: space of finitely additive set functions $Q$ on $\mathcal{F}$, weakly absolutely continuous with respect to $P$, with $Q[\Omega] = 1$.
- $Q$ can be uniquely decomposed as $Q = Q^r + Q^s$. Here, $Q^r$ is sigma-additive and $Q^s$ is purely finitely additive.
- $\text{ba}(\Omega, \mathcal{F}, P)$ can be identified with the dual space $L^\infty(\Omega, \mathcal{F}, P)^*$ of $L^\infty(\Omega, \mathcal{F}, P)$.

A weakly absolutely continuous, finitely additive probability measure $Q \in \text{ba}_1(\Omega, \mathcal{F}, P)$, such that

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Comparison of Föllmer f.a. and c.a. measures

- If $Z$ is a uniformly integrable $P$–martingale and if $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t$, then each Föllmer countably additive measure for $Z$ is a Föllmer finitely additive measure for $Z$. Moreover, the class of Föllmer finitely additive measures is strictly larger than the class of Föllmer countably additive measures (in most cases).

- If $Z$ is not a uniformly integrable $P$–martingale, then the sets of Föllmer countably additive measures for $Z$ and of Föllmer finitely additive measures for $Z$ are disjoint.

- In general, the existence of a Föllmer countably additive measure does not imply the existence of a Föllmer finitely additive measure (e.g., finite probability space), nor does the opposite implication hold (e.g., under “usual assumptions”).
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Existence and uniqueness: c.a. case (preparation)

Assumption $\mathcal{P}$: Let $E$ be a state space, and let $\Delta \notin E$ be a cemetery state. For all $\omega \in (E \cup \{\Delta\})^{[0,\infty)}$ define

$$\zeta(\omega) = \inf\{t \geq 0 : \omega(t) = \Delta\}.$$  

Let $\Omega \subset (E \cup \{\Delta\})^{[0,\infty)}$ be the space of paths $\omega : [0, \infty) \to E \cup \{\Delta\}$, for which $\omega$ is càdlàg on $[0, \zeta(\omega))$, and for which $\omega(t) = \Delta$ for all $t \geq \zeta(\omega)$.

For all $t \geq 0$ define $X_t(\omega) = \omega(t)$ and the sigma algebras $\mathcal{F}_t^0 = \sigma(X_s : s \in [0, t])$ and $\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s^0$. Moreover, set $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t^0 = \bigvee_{t \geq 0} \mathcal{F}_t$.

$$\hat{\tau}^Z_n = \inf\{t \geq 0 : Z_t \geq n\} \wedge n; \quad \hat{\tau}^Z = \lim_{n \uparrow \infty} \hat{\tau}^Z_n.$$
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Assumption \( \mathcal{P} \): Let \( E \) be a state space, and let \( \Delta \notin E \) be a cemetery state. For all \( \omega \in (E \cup \{\Delta\})^{[0,\infty)} \) define

\[
\zeta(\omega) = \inf\{t \geq 0 : \omega(t) = \Delta\}.
\]

Let \( \Omega \subset (E \cup \{\Delta\})^{[0,\infty)} \) be the space of paths \( \omega : [0, \infty) \to E \cup \{\Delta\} \), for which \( \omega \) is càdlàg on \([0, \zeta(\omega))\), and for which \( \omega(t) = \Delta \) for all \( t \geq \zeta(\omega) \).

For all \( t \geq 0 \) define \( X_t(\omega) = \omega(t) \) and the sigma algebras \( \mathcal{F}_0^t = \sigma(X_s : s \in [0, t]) \) and \( \mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s^0 \). Moreover, set \( \mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t^0 = \bigvee_{t \geq 0} \mathcal{F}_t \).

\[
\hat{\tau}_n^Z = \inf\{t \geq 0 : Z_t \geq n\} \wedge n; \quad \hat{\tau}^Z = \lim_{n \uparrow \infty} \hat{\tau}_n^Z.
\]
Existence and uniqueness: c.a. case (preparation)

Assumption \( \mathcal{P} \): Let \( E \) be a state space, and let \( \Delta \notin E \) be a cemetery state. For all \( \omega \in (E \cup \{\Delta\})^{[0,\infty)} \) define

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For all \( t \geq 0 \) define \( X_t(\omega) = \omega(t) \) and the sigma algebras \( \mathcal{F}_t^0 = \sigma(X_s : s \in [0, t]) \) and \( \mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s^0 \). Moreover, set \( \mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t^0 = \bigvee_{t \geq 0} \mathcal{F}_t \).

\[
\hat{\tau}^Z_n = \inf\{ t \geq 0 : Z_t \geq n \} \wedge n; \quad \hat{\tau}^Z = \lim_{n \uparrow \infty} \hat{\tau}^Z_n.
\]
Existence and uniqueness: c.a. case (theorem)

Under Assumption \( (\mathcal{P}) \), suppose that one of the following conditions hold:

- \( Z \) is a \( P \)–local martingale;
- \( P \) satisfies \( \mathbb{E}_P[Z \zeta \mathbf{1}_{\{\zeta < \infty\}}] = 0 \).

Then there exist \( \tau \) and \( Q \) such that (1) holds. If \( Z \) is a \( P \)–local martingale, then we can use \( \tau = \hat{\tau}^Z \). Moreover:

(I) The following conditions are equivalent:
   
   \( a \) the set \( \{ \tau < \zeta \} \) is \( Q\|F_\tau \)–negligible;
   
   \( b \) there is a unique Föllmer countably additive measure for \( (Z, \tau) \).

(II) If \( \bar{\tau} \) is a stopping time such that the pair \( (Q, \bar{\tau}) \) also satisfies (1), then \( Q[\tau = \bar{\tau}] = 1 \).

(III) The following statement in (c) always implies the one in (d). The reverse implication holds if \( E \) is uncountable.

- \( c \) The \( P \)–supermartingale \( Z \) is a \( P \)–local martingale and the set \( \{ \hat{\tau}^Z < \zeta \} \) is \( \hat{Q}^Z\|F_{\hat{\tau}^Z} \)–negligible;
- \( d \) there is a unique Föllmer pair for \( Z \).
Existence and uniqueness: c.a. case (theorem)

Under Assumption (\(\mathcal{P}\)), suppose that one of the following conditions hold:

- \(Z\) is a \(P\)-local martingale;
- \(P\) satisfies \(\mathbb{E}_P[Z\zeta 1_{\{\zeta<\infty\}}] = 0\).

Then there exist \(\tau\) and \(Q\) such that (1) holds. If \(Z\) is a \(P\)-local martingale, then we can use \(\tau = \hat{\tau}^Z\). Moreover:

(I) The following conditions are equivalent:
   (a) the set \(\{\tau < \zeta\}\) is \(Q|_{\mathcal{F}_\tau}\)-negligible;
   (b) there is a unique Föllmer countably additive measure for \((Z, \tau)\).

(II) If \(\tilde{\tau}\) is a stopping time such that the pair \((Q, \tilde{\tau})\) also satisfies (1), then \(Q[\tau = \tilde{\tau}] = 1\).

(III) The following statement in (c) always implies the one in (d).
   The reverse implication holds if \(E\) is uncountable.
   (c) The \(P\)-supermartingale \(Z\) is a \(P\)-local martingale and the set \(\{\hat{\tau}^Z < \zeta\}\) is \(Q^F|_{\mathcal{F}_{\hat{\tau}^Z}}\)-negligible;
   (d) there is a unique Föllmer pair for \(Z\).
Existence and uniqueness: c.a. case (theorem)

Under Assumption $(\mathcal{P})$, suppose that one of the following conditions hold:

- $Z$ is a $P$–local martingale;
- $P$ satisfies $\mathbb{E}_P[Z \zeta 1_{\{\zeta < \infty\}}] = 0$.

Then there exist $\tau$ and $Q$ such that (1) holds. If $Z$ is a $P$–local martingale, then we can use $\tau = \hat{\tau}^Z$. Moreover:

(I) The following conditions are equivalent:

(a) the set $\{\tau < \zeta\}$ is $Q|_{\mathcal{F}_{\tau^-}}$–negligible;
(b) there is a unique Föllmer countably additive measure for $(Z, \tau)$.

(II) If $\tilde{\tau}$ is a stopping time such that the pair $(Q, \tilde{\tau})$ also satisfies (1), then $Q[\tau = \tilde{\tau}] = 1$.

(III) The following statement in (c) always implies the one in (d). The reverse implication holds if $E$ is uncountable.

(c) The $P$–supermartingale $Z$ is a $P$–local martingale and the set $\{\hat{\tau}^Z < \zeta\}$ is $\hat{Q}^Z|_{\mathcal{F}_{\hat{\tau}^Z^-}}$–negligible;
(d) there is a unique Föllmer pair for $Z$. 
Existence and uniqueness: c.a. case (theorem)

Under Assumption (\(\mathcal{P}\)), suppose that one of the following conditions hold:

- \(Z\) is a \(P\)–local martingale;
- \(P\) satisfies \(\mathbb{E}_P[Z\xi 1_{\{\xi<\infty\}}] = 0\).

Then there exist \(\tau\) and \(Q\) such that (1) holds. If \(Z\) is a \(P\)–local martingale, then we can use \(\tau = \hat{\tau}^Z\). Moreover:

(I) The following conditions are equivalent:

(a) the set \(\{\tau < \xi\}\) is \(Q|\mathcal{F}_{\tau}\)–negligible;
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  - (c) The $P$–supermartingale $Z$ is a $P$–local martingale and the set $\{\hat{\tau}^Z < \zeta\}$ is $\hat{Q}^Z|\mathcal{F}_{\hat{\tau}^Z^-}$–negligible;
  - (d) there is a unique Föllmer pair for $Z$. 
Existence and uniqueness: c.a. case (theorem)

Under Assumption (P), suppose that one of the following conditions hold:

- \( Z \) is a \( P \)-local martingale;
- \( P \) satisfies \( \mathbb{E}_P[Z\zeta \mathbf{1}_{\{\zeta<\infty\}}] = 0 \).

Then there exist \( \tau \) and \( Q \) such that (1) holds. If \( Z \) is a \( P \)-local martingale, then we can use \( \tau = \hat{\tau}^Z \). Moreover:

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- (c) The \( P \)-supermartingale \( Z \) is a \( P \)-local martingale and the set \( \{ \hat{\tau}^Z < \zeta \} \) is \( \hat{Q}^Z|_{\mathcal{F}_{\hat{\tau}^Z^-}} \)-negligible;
- (d) there is a unique Föllmer pair for \( Z \).
Existence and uniqueness: c.a. case (theorem)

Under Assumption ($\mathcal{P}$), suppose that one of the following conditions hold:

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Existence and uniqueness: c.a. case (proof)

Rough outline:

- **Existence:**
  
  - Motivation: Interpret $Z$ as the reciprocal of a local martingale that can jump to zero under $Q$. (e.g. $Z_t = e^{-t}$).
  
  - Use multiplicative decomposition: $Z = MD$.
  
  - Construct measure for local martingale $M$ (use an extension theorem).
  
  - Interpret $D$ as survival function (proceed as when constructing killed diffusions).

- **Uniqueness:**
  
  - Read lots of papers and don't give up.
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  - Read lots of papers and don’t give up.
Existence and non-uniqueness: f.a. case

Assumption $\mathcal{B}$: $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ supports a Brownian motion $W = (W_t)_{t \geq 0}$.

Under Assumption $(\mathcal{B})$, there exists a Föllmer finitely additive measure for $Z$. The Föllmer finitely additive measure is never unique.
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Existence and uniqueness: f.a. case (proof)

Rough outline:

• **Existence:**
  - Prove: There exists a family \( (L^{(i,j,k,m,n)})_{i,j,k,m,n \in \mathbb{N}} \) of u.i. nonnegative \( P \)-martingales with \( \mathbb{E}_P[L_0^{(i,j,k,m,n)}] = 1 \), s.t.
    \[
    \lim_{i \to \infty} \lim_{j \to \infty} \lim_{k \to \infty} \lim_{m \to \infty} \lim_{n \to \infty} L^{(i,j,k,m,n)}_\rho = Z_\rho
    \]
    for all finite s.t. \( \rho \). (Use suitable “suicide strategies” and localize.)
  - \( Q \) is then a cluster point in the family of the (c.a.) probability measures generated by \( L^{(i,j,k,m,n)}_\infty \).

• **Uniqueness:** Consider two cases:
  - (A) \( P[Z_\infty > 0] > 0 \) (destroy some mass of \( Q^r \) at time \( \infty \))
  - (B) \( P[\rho < \infty] = 1 \), where \( \rho = \inf\{t \geq 0 : Z_t \leq 1/2\} \). (Play around with \( Q^s \) after \( \rho \) — “supported on the set where \( L^{(i,j,k,m,n)}_\rho \) is large”.)
Existence and uniqueness: f.a. case (proof)

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Existence and uniqueness: f.a. case (proof)

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\[
\lim_{i \to \infty} \lim_{j \to \infty} \lim_{k \to \infty} \lim_{m \to \infty} \lim_{n \to \infty} L_{\rho}^{(i,j,k,m,n)} = Z_{\rho}
\]

for all finite s.t. \(\rho\). (Use suitable “suicide strategies” and localize.)

• \(Q\) is then a cluster point in the family of the (c.a.) probability measures generated by \(L_{\infty}^{(i,j,k,m,n)}\).

• Uniqueness: Consider two cases:
  
  (A) \(P[Z_\infty > 0] > 0\) (destroy some mass of \(Q^r\) at time \(\infty\))
  
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\[
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Definition: local martingale

A stochastic process $Z$ is a *local martingale* if there exists a sequence of stopping times $(\tau_n)$ with $\lim_{n \uparrow \infty} \tau_n = \infty$ such that $Z^{\tau_n} := (Z(t \wedge \tau_n))$ is a martingale.

Example (Stochastic exponential)

For a continuous local martingale $Y$, the process

$$\mathcal{E}(Y)(\cdot) = \exp \left( Y(\cdot) - \frac{1}{2} [Y, Y](\cdot) \right)$$

is a nonnegative local martingale.
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Stochastic exponentials

- Consider an arbitrary local martingale $M$ with jump measure $\mu$ (having compensator $\nu$).
- Thus,

$$M = M_0 + M^c + x \ast (\mu - \nu).$$

- Assume that $M_0 = 1$ and jumps of $M$ are bounded from below by $-1$.
- Define the stochastic exponential of $M$ by

$$\mathcal{E}(M)_t = \exp \left( M_t - \frac{1}{2}[M, M]_t^c - (x - \log(1 + x)) \ast \mu_t \right).$$

- Then $\mathcal{E}(M)$ is nonnegative local martingale with $\mathcal{E}(M)_0 = 1$ since it is the unique strong solution of the SDE

$$Z = 1 + Z_\cdot M.$$
Stochastic exponentials

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- Thus,

$$M = M_0 + M^c + x \ast (\mu - \nu).$$

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Novikov-Kazamaki-type conditions

- Provide sufficient conditions that $\mathcal{E}(M)$ is a uniformly integrable martingale.
- For sake of simplicity, assume that jumps of $M$ are strictly larger than $-1$.
- Conditions were first provided by Lepingle and Mémin.
- E.g.:
  $$\sup_{\sigma \in \mathcal{T}} \mathbb{E}_P \left[ \exp(A_{\sigma}) \mathbf{1}_{\{\mathcal{E}(M)_{\sigma} > 0\}} \right] < \infty,$$
  where, for some $a \in \mathbb{R} \setminus \{0\}$,
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Idea of a novel proof

• Recall Föllmer countably additive measure $Q$ along with stopping time $\hat{\tau}$ (which can be interpreted as explosion time):

\[ Q[\rho < \hat{\tau}] = \mathbb{E}_P[\mathcal{E}(M)_{\rho} \mathbf{1}_{\{\rho<\infty\}}] \]

• Thus: $\mathcal{E}(M)$ is a martingale if and only if $\mathcal{E}(M)$ does not explode under a new measure $Q$.

• The new measure $Q$ corresponds to the “Radon-Nikodym derivative” $\mathcal{E}(M)$.

• Thus, it is sufficient to check that Novikov-Kazamaki conditions guarantee that the $Q$–local martingale $1/\mathcal{E}(M)$ does not tend to zero under $Q$. 
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Convergence of local martingale

**Theorem (Limiting behaviour of a local martingale)**

Let $\tau$ be a positive predictable time, and $X$ a local martingale on $[0, \tau)$ with $\Delta X \geq -1$. Denote the jump measure of $X$ by $\mu$, and let $\nu$ be its compensator. The following equalities hold almost surely:

$$\begin{align*}
\left\{ \lim_{t \uparrow \tau} X_t \text{ exists in } \mathbb{R} \right\} &= \left\{ \liminf_{t \uparrow \tau} \left( X_t - \frac{1}{2}[X, X]_t^c - (x - \log(1 + x))1_{x \neq -1} \ast \mu_t \right) > -\infty \right\} \\
&= \left\{ [X, X]_{\tau^-} < \infty \right\} \bigcap \left\{ \limsup_{t \uparrow \tau} X_t > -\infty \right\} \\
&= \left\{ [X, X]_t^c + (|x| \wedge x^2) \ast \nu_{\tau^-} < \infty \right\} \\
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Novikov-Kazamaki conditions: outline of the proof

- Starting from the Novikov-Kazamaki condition

\[ \sup_{\sigma \in T} \mathbb{E}_P \left[ \exp(A_\sigma) \mathbf{1}_{\{\mathcal{E}(M)_\sigma > 0\}} \right] < \infty, \]

change the measure from \( P \) to \( Q \) via “Radon-Nikodym derivative” \( \mathcal{E}(M) \).

- Observe that \( 1/\mathcal{E}(M) \) is a \( Q \)–local martingale, in particular, a stochastic exponential \( \mathcal{E}(N) \) of some \( Q \)–local martingale \( N \).

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- By means of the last theorem, obtain a contradiction to the Novikov-Kazamaki condition.

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Distribution of explosion time: setup

- **Interval** \( I = (\ell, r) = \bigcup_{n \in \mathbb{N}} (\ell_n, r_n) \) with \(-\infty \leq \ell < r \leq \infty\).
- Consider

\[
dX(t) = s(X(t)) \left( dW(t) + b(X(t)) dt \right), \quad X(0) = \xi \in I
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- Assume that

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- Engelbert & Schmidt prove the existence of a weak solution \( X \), unique in the sense of the probability distribution, and defined up until the “explosion time”

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S := \inf \{ t \geq 0 : X(t) \notin (\ell, r) \} = \lim_{n \uparrow \infty} \inf \{ t \geq 0 : X(t) \notin (\ell_n, r_n) \}.
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- Due to uniqueness, \( X \) is Markovian.
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Feller’s test of explosions

“Feller’s test” function:

\[ v(x) := \int_c^x \left( \int_c^y \exp \left( -2 \int_z^y f(u)du \right) \frac{1}{\sigma^2(z)}dz \right) dy \]

- Feller’s test states that \( P[S = \infty] = 1 \) holds if and only if \( v(\ell+) = v(r-) = \infty \) holds.
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Feller’s test: special cases

\[ v(x) := \int_c^x \left( \int_c^y \exp \left( -2 \int_z^y f(u) \, du \right) \frac{1}{s^2(z)} \, dz \right) \, dy \]

By simplifying \( v(\cdot) \) via Tonelli, criterion of explosions is sometimes easier to check:

- If \( s(\cdot) \) is differentiable and \( b(\cdot) = s'(\cdot)/2 \):
  \[ v(x) = \int_c^x \left( \frac{1}{s(y)} \int_c^y \frac{1}{s(z)} \, dz \right) \, dy = \frac{1}{2} \left( \int_c^x \frac{1}{s(z)} \, dz \right)^2. \]

- If \( I = (0, \infty) \) and \( b(\cdot) = 0 \):
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Therefore, \( P[S = \infty] = 1 \) holds if and only if
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Feller’s test: special cases

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A diffusion in natural scale

\[ X^o(\cdot) = \xi + \int_0^\cdot s(X^o(t))dW^o(t) \]

- Again, a weak solution, unique in the sense of the probability distribution, exists up until an explosion time \( S^o \).
- \( P^o[S^o = \infty] = 1 \) holds if \( \ell = -\infty \) and \( r = \infty \).
Transformation of probability

Theorem (Generalized Girsanov theorem (a special case))

Assume that the mean/variance ratio $f(\cdot)$ is locally square-integrable on $I$. For any given $T \in (0, \infty)$ and any Borel set $\Delta \in \mathcal{B}_T(C([0, \infty)))$, we have

$$P [X(\cdot) \in \Delta, S > T] = \mathbb{E}^\circ \left[ \mathcal{E} \left( \int_0^T b(X^\circ(t)) \, dW^\circ(t) \right) \mathbf{1}_{\{X^\circ(\cdot) \in \Delta, S^\circ > T\}} \right]$$

where $\mathcal{E}(\cdot)$ denotes the stochastic exponential.

- Appears in McKean (1969) under stronger assumptions.
- Local square-integrability of $f(\cdot)$ and occupation time formula yield that

$$\int_0^T b^2(X^\circ(t)) \, dt < \infty$$

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- With $\Delta = C([0, T])$:

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- Assume for the moment that $f(\cdot)$ is continuously differentiable.
- Denote by $F(\cdot) = \int_c f(x) \, dx$ its antiderivative.
- Then $P[S > T] =$

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\exp(-F(\xi)) \cdot \mathbb{E}^o \left[ \exp \left( F(X^o(T)) - \int_0^T V(X^o(t)) \, dt \right) \cdot 1\{S > T\} \right],
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where $V(x) := \frac{1}{2} (b^2(x) + f'(x)s^2(x))$.

- Thus, the distribution of the explosion time is determined completely by the joint distributions of $X^o(T)$ and $\int_0^T V(X^o(t)) \, dt$ on $\{S > T\}$, for all $T \in (0, \infty)$. 

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Transformation of space

- Based on ideas of Lamperti, Doss, Sussmann.
- Assume that $s(\cdot)$ is continuously differentiable and $s(\cdot) > 0$.
- Consider the function
  \[ h(x) = \int_c^x \frac{1}{s(z)} \, dz. \]
- Then $X$ explodes from $I = (\ell, r)$ exactly when $Y = h(X)$ explodes from $\tilde{I} = (h(\ell), h(r))$.
- Thus, $X$ and $Y$ have the same explosion time $S$.
- If $b(\cdot) = \nu + s'(\cdot)/2$, then $Y$ takes the simple form
  \[ Y(t) = h(\xi) + \nu t + W(t). \]
- Distribution of $S$ for BM with drift is well known.
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Analytic properties

Define the function $U : (0, \infty) \times I \rightarrow [0, 1]$ via

$$U(T, \xi) := P_{\xi}[S > T].$$

Theorem

The function $U(\cdot, \cdot)$ is jointly continuous in $[0, \infty) \times I$. 
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*The function $U(\cdot, \cdot)$ is jointly continuous in $[0, \infty) \times I$.***
Parabolic PDE

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**Theorem**

Assume that the functions \( s(\cdot) \) and \( b(\cdot) \) are locally uniformly Hölder-continuous on \( I \). Then the function \( U(\cdot, \cdot) \) is of class \( C([0, \infty) \times I) \cap C^{1,2}((0, \infty) \times I) \) and satisfies the Cauchy problem

\[
\frac{\partial U}{\partial \tau} (\tau, x) = \frac{s^2(x)}{2} \frac{\partial^2 U}{\partial x^2} (\tau, x) + b(x)s(x) \frac{\partial U}{\partial x} (\tau, x)
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\[ U(0, x) = 1. \]

Moreover, the function \( U(\cdot, \cdot) \) is dominated by every nonnegative classical supersolution of that Cauchy problem.
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Second-order ODE

Consider the Laplace transform or “resolvent” of the function $U(\cdot, \xi)$:

$$
\hat{U}_\lambda(\xi) = \int_0^\infty \exp(-\lambda T) U(T, \xi) dT = \frac{1}{\lambda} \left( 1 - \mathbb{E}_\xi [\exp(-\lambda S)] \right).
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**Theorem**

If the functions $b(\cdot)$ and $s(\cdot)$ are continuous on $I$, then the function $\hat{U}_\lambda(\xi)$ is of class $C^2(I)$ and satisfies the second-order ordinary differential equation

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\frac{s^2(x)}{2} u''(x) + b(x)s(x)u'(x) - \lambda u(x) + 1 = 0, \quad x \in I.
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Equivalent formulations for the Feller test

Theorem (Equivalence of the following conditions)

(i) \( P[S = \infty] = 1 \).

(ii) \( v(\ell+) = v(r-) = \infty \) for the “Feller test” function \( v \).

If \( f(\cdot) \) is locally square-integrable, (i)-(ii) are equivalent to:

(iii) the truncated exponential \( P^o \)-supermartingale
\[
\mathcal{E} \left( \int_0^T b(X^o(t))dW^o(t) \right) \cdot 1\{S^o > T\} \text{ is a } P^o \text{-martingale.}
\]

If \( s(\cdot) \) and \( b(\cdot) \) are continuous on \( I \), (i)–(iii) are equivalent to:

(iv) the smallest nonnegative classical solution of the second-order differential equation above is \( u(\cdot) \equiv 1/\lambda \).

If \( s(\cdot) \) and \( b(\cdot) \) are locally uniformly Hölder-cont., (i)–(iv) a.e.to:

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Circular proof??
Equivalent formulations for the Feller test

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Equivalent formulations for the Feller test

Theorem (Equivalence of the following conditions)

(i) $P[S = \infty] = 1$.

(ii) $\nu(\ell+) = \nu(r-) = \infty$ for the “Feller test” function $\nu$.

If $f(\cdot)$ is locally square-integrable, (i)-(ii) are equivalent to:

(iii) the truncated exponential $P^0$-supermartingale

$$E \left( \int_0^T b(X^0(t))dW^0(t) \right) \cdot 1\{S^0 > T\}$$

is a $P^0$-martingale.

If $s(\cdot)$ and $b(\cdot)$ are continuous on $I$, (i)–(iii) are equivalent to:

(iv) the smallest nonnegative classical solution of the second-order differential equation above is $u(\cdot) \equiv 1/\lambda$.

If $s(\cdot)$ and $b(\cdot)$ are locally uniformly Hölder-cont., (i)–(iv) a.e.to:

(v) the smallest nonnegative classical solution of the Cauchy problem above is $U(\cdot, \cdot) \equiv 1$.  

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If \( f(\cdot) \) is locally square-integrable, (i)-(ii) are equivalent to:

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\mathcal{E} \left( \int_0^T b(X^o(t))dW^o(t) \right) \cdot 1\{S^o>T\} \text{ is a } P^o\text{-martingale.}
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(v) the smallest nonnegative classical solution of the Cauchy problem above is \( U(\cdot, \cdot) \equiv 1 \).

Circular proof??
Example (slightly cheating)

With \( I = (0, \infty) \), \( \kappa \in [1/2, \infty) \), consider

\[
X(\cdot) = \xi - \int_0^\cdot (X(t))^2 dW(t) + \kappa \int_0^\cdot (X(t))^3 dt.
\]

- Corresponds to \( s(x) = -x^2 \) and \( b(x) = -\kappa x \).
- Thus, by the Feynman-Kac representation,

\[
P[S > T] = \mathbb{E}^\circ \left[ \left( \frac{X^\circ(T)}{\xi} \right)^\kappa \exp \left( -\frac{\kappa^2 - \kappa}{2} \int_0^T (X^\circ(t))^2 dt \right) \right]
= \frac{1}{T} \xi^{-\nu} \exp \left( -\frac{1}{2T \xi^2} \right) \int_0^\infty x^{1-\nu} \exp \left( -\frac{x^2}{2T} \right) I_\nu \left( \frac{x}{\xi T} \right) dx,
\]

where \( \nu := \kappa - 1/2 \).
- This is the smallest nonnegative solution of

\[
\frac{\partial U}{\partial \tau}(\tau, x) = \frac{x^4}{2} \frac{\partial^2 U}{\partial x^2}(\tau, x) + \kappa x^3 \frac{\partial U}{\partial x}(\tau, x)
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subject to \( U(0+, x) = 1 \).
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With $I = (0, \infty)$, $\kappa \in [1/2, \infty)$, consider

$$X(\cdot) = \xi - \int_0^\cdot (\mathcal{X}(t))^2 dW(t) + \kappa \int_0^\cdot (\mathcal{X}(t))^3 dt.$$ 

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Special case of example

Set $\kappa = 1$:

$$X(\cdot) = \xi - \int_0^\cdot (X(t))^2 dW(t) + \int_0^\cdot (X(t))^3 dt.$$  

- Here:

$$P[S > T] = 2 \int_0^{1/(\xi \sqrt{T})} \frac{1}{\sqrt{2\pi}} \exp \left( - \frac{r^2}{2} \right) \, dr.$$  

- Observe that

$$dX(t) = s(X(t)) \left( dW(t) + \frac{1}{2} s'(X(t)) dt \right), \quad X(0) = \xi.$$  

- Thus, we can express $X$ pathwise as

$$X(t) = \frac{1}{W(t) + (1/\xi)}, \quad 0 \leq t < S.$$
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Thank you!