Oligopolies & Mean Field Games

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Joint with Patrick Chan (Princeton)
Motivating Problem: Energy Production from different sources

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  - players have asymmetric exhaustible resources;
  - market transitions from $N$-opoly $\rightarrow (N-1)$-opoly $\rightarrow \cdots \rightarrow$ duopoly $\rightarrow$ monopoly.

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Here, we focus on Bertrand competition (in the continuum limit, Bertrand and Cournot are the same).

In the simplest one-period case, each player $i$ maximizes profit:

$$\Pi_i = q_i(p_i - s_i)$$

where $s_i$ is per-unit cost of production.

- **Cournot**: choose $q_i$ and $p_i = P_i(q)$;
- **Bertrand**: choose $p_i$ and $q_i = D_i(p)$.

In nonzero sum (stochastic) differential games, value functions solve nonlinear Hamilton-Jacobi-Bellman system of $N$ PDEs.

In the continuum mean field game approximation: two PDEs. Lasry-Lions ’07, Huang-Malhamé-Caines ’06, Carmona-Delarue ’12, Bensoussan-Frehse-Yam ’13.
Market Structure

- The market model is specified by linear inverse demand functions, and are the basis of Cournot competition: For $q \in \mathbb{R}_+^N$, the price received by player $i$ is

\[
P_i(q) = 1 - (q_i + \epsilon \bar{q}_i), \quad \bar{q}_i = \frac{1}{N - 1} \sum_{j \neq i} q_j, \quad i = 1, \ldots, N.
\]

The parameter $\epsilon > 0$ measures the degree of interaction.

- The case $\epsilon = 0$ corresponds to independent goods. We also have $\epsilon < N - 1$ (substitutable goods).

- The dependence of the price player $i$ receives depends on the quantities produced by his competitors through their mean $\bar{q}_i$. That is, the interaction is of mean field type.
Bertrand System

Demand function obtained by inversion a rank-one update to the identity matrix (Sherman-Morrison formula):

\[
a_n = \frac{1}{1 + \epsilon \frac{n-1}{N-1}}, \quad b_n = \frac{1 + \epsilon \frac{n-2}{N-1}}{(1 + \epsilon \frac{n-1}{N-1}) \left(1 - \frac{\epsilon}{N-1}\right)},
\]

\[
c_n = \frac{\epsilon}{(1 + \epsilon \frac{n-1}{N-1}) \left(1 - \frac{\epsilon}{N-1}\right)}.
\]

For \(p_1 \leq \cdots \leq p_N\), the resulting demands are given by

\[
D_i(p) = \begin{cases} 
    a_n - b_n p_i + c_n \bar{p}_i^n, & i = 1, 2, \cdots, n \\
    0, & i = n + 1, \cdots, N
\end{cases}
\]

\[
\bar{p}_i^n = \frac{1}{n-1} \sum_{j \neq i} p_j,
\]

where we use the largest \(n\) such that \(D_n^p(p) \geq 0\). These demand functions are the basis of Bertrand competition.
In the continuum limit $N \to \infty$: $\eta \in [0, 1]$ is the proportion of players who receive positive demand (the continuous variable corresponding to $n/N$).

The demand function of such a player who sets price $p$ is:

$$D(p, \bar{p}) = a(\eta) - b(\eta)p + c(\eta)\bar{p}, \quad \eta > 0$$

where $\bar{p}$ is the mean price of all the players, and the continuum limits are

$$a(\eta) = \frac{1}{(1 + \epsilon \eta)}, \quad b(\eta) = 1, \quad c(\eta) = \frac{\epsilon}{(1 + \epsilon \eta)}.$$

The actual demand is given by $D(p, \bar{p})$ provided that it is positive, and zero otherwise.
**N-player Game (Static)**

- There are \( N \) players who have constant marginal costs of production \( 0 \leq s_1 \leq \cdots \leq s_N \).
- The optimization problem faced by each firm \( i \) is

\[
\max_{p_i} \left( a_N - b_N p_i + c_N \bar{p}_i \right) (p_i - s_i), \quad \bar{p}_i = \frac{1}{N-1} \sum_{j \neq i} p_j.
\]

- We obtain

\[
p_i^* = \frac{1}{2b_N + \frac{c_N}{N-1}} \left( b_N s_i + a_N + \frac{c_N}{2b_N - c_N} \frac{N}{N-1} (b_N \bar{s} + a_N) \right).
\]

Using the limits of \( a_N, b_N, c_N \) as \( N \to \infty \)

\[
p_i^* \to \frac{1}{2} \left( s_i + \frac{2}{2 + \epsilon} + \frac{\epsilon}{2 + \epsilon} \bar{s} \right), \quad \bar{s} = \frac{1}{N} \sum_{j=1}^{N} s_j.
\]

As we will see shortly, we recover the same result from solving the limit **continuum mean field game**.
Continuum MFG (Static)

- An infinite number of players labeled by $x > 0$, with associated density $M(x)$ and marginal cost of production $s(x)$.
- A player at location $x$ optimizes his profit as though he is unable to affect the mean price $\bar{p}$:

$$\max_p \left( a(1) - p + c(1)\bar{p} \right) \left( p - s(x) \right), \quad \bar{p} = \int_0^\infty p^*(x)M(x) \, dx.$$ 

- Given $\bar{p}$, the first order condition gives

$$p^*(x) = \frac{1}{2} \left( s(x) + a(1) + c(1)\bar{p} \right)$$

from which we get:

$$p^*(x) = \frac{1}{2} \left( s(x) + \frac{2}{2 + \epsilon} + \frac{\epsilon}{2 + \epsilon} \bar{s} \right), \quad \bar{s} = \int s(x)M(x) \, dx.$$ 

which is the continuum analog of above.
Dynamic Mean Field Game with Exhaustible Resources

- An infinite number (continuum) of producers setting prices for their similar, but differentiated, goods.
- At time $t = 0$, the density of players with remaining capacity $x > 0$ is given by $M(x)$, where $\int M = 1$. Initial capacity $x$ allows us to distinguish between bigger and smaller players.
- As time evolves, some players exhaust their capacity by selling all their goods, and we denote by $m(t, x)$ the “density” of firms with positive capacity at time $t > 0$.
- Let $\eta(t)$ be the fraction of active firms remaining:

$$\eta(t) = \int_{0^+}^{\infty} m(t, x) \, dx.$$ 

In general, we expect $\eta(t) < 1$ for large enough $t > 0$. 
The remaining capacity (or reserves) of the producers \((X_t)\) follows the dynamics

\[
dX_t = -q_t \, dt + \sigma \, dW_t, \quad X_t > 0,
\]
for some Brownian motion \(W\).

A firm who starts with capacity \(x > 0\) at time \(t \geq 0\) sets prices to maximize lifetime discounted profits discounted. Value function:

\[
u(t, x) = \sup_{p} \mathbb{E} \left\{ \int_{t}^{\infty} e^{-r(s-t)} p_s q_s \mathbb{1}_{\{X_s > 0\}} ds \mid X_t = x \right\}, \quad x > 0.
\]

The associated HJB equation is

\[
\partial_t u + \frac{1}{2} \sigma^2 \partial_{xx} u - ru + \max_{p \geq 0} \left( a(\eta(t)) - p + c(\eta(t)) \bar{p}(t) \right) (p - \partial_x u) = 0.
\]

The internal optimization is the static MFG equilibrium problem with (shadow) cost \(s(x) \mapsto \partial_x u(t, x)\).
Continuum MFG Equations

- The feedback strategy is
  
  \[ p^*(t, x) = \frac{1}{2} \left( a(\eta(t)) + \partial_x u(t, x) + c(\eta(t))\bar{p}(t) \right), \]

  and the HJB equation becomes

  \[ \partial_t u + \frac{1}{2} \sigma^2 \partial_{xx} u - ru + \frac{1}{4} \left( a(\eta(t)) - \partial_x u + c(\eta(t))\bar{p}(t) \right)^2 = 0, \quad x > 0, \]

  with \( u(t, 0) = 0. \)

- The average price \( \bar{p} \) is computed from the density \( m(t, x) \) of the distribution of reserves \( X_t \):
  
  \[ \bar{p}(t) = \frac{1}{\eta(t)} \int_{\mathbb{R}_+} p^*(t, x)m(t, x) \, dx. \]

- The forward Kolmogorov equation for \( m \) is:
  
  \[ \partial_t m - \frac{1}{2} \sigma^2 \partial_{xx} m + \partial_x \left( -\frac{1}{2} \left( a(\eta(t)) - \partial_x u + c(\eta(t))\bar{p}(t) \right) m \right) = 0, \]

  with \( m(0, x) = M(x). \)
Monopoly with Deterministic Demand

- Look at the case when $\epsilon = 0$, ($\implies c \equiv 0$): the players are monopolists in their own markets, and when $\sigma = 0$.
- Let $(u_0, m_0)$ be the value function and density in this case. Then

$$\partial_t u_0 - ru_0 + \frac{1}{4}(1 - \partial_x u_0)^2 = 0, \quad x > 0,$$

with $u_0(t, 0) = 0$. The solution is time-independent and given by

$u_0(t, x) = u_0(x)$ solving

$$\frac{1}{4}(1 - u_0')^2 = ru_0, \quad u_0(0) = 0.$$

The solution is

$$u_0(x) = \frac{1}{4r} \left(1 + \mathcal{W}(\theta(x))\right)^2, \quad \theta(x) = -e^{-2rx-1},$$

where $\mathcal{W}$ is the Lambert W-function defined by $x = \mathcal{W}(x)e^{\mathcal{W}(x)}$. 
Monopoly: Capacity Trajectories

Dynamics

\[ x'(t) = -\frac{1}{2} \left( 1 + \mathbb{W}(\theta(x(t))) \right), \quad x(0) = x_0. \]

Explicit solution

\[ x(t; x_0) = x_0 - \frac{1}{2} t + \frac{1}{2r} (1 - e^{rt}) \mathbb{W}(\theta(x_0)). \]

Hotelling’s rule: shadow cost grows at the discount rate

\[ \partial_x u(t, x(t)) = \partial_x u(0, x_0) e^{rt}. \]

\[ \frac{d}{dt} \left( p^*(t, X(t)) - \frac{1}{2} \right) = r \left( p^*(t, X(t)) - \frac{1}{2} \right), \quad \Rightarrow \quad p^*(t, X(t)) = \frac{1}{2} \left( 1 - e^{rt} \right) + p^*(0, x_0) e^{rt}. \]

Hitting Times

The hitting time \( \tau \) is the first time a firm’s capacity hits zero, given by

\[ \tau(x) = 2x + \frac{1}{r} \left( 1 + \mathbb{W}(\theta(x)) \right) \]

\[ \tau^{-1}(t) = \frac{t}{2} - \frac{1}{2r} \left( 1 - e^{-rt} \right). \]
Monopoly: Density

Forward Kolmogorov equation

\[ \partial_t m_0 - \frac{1}{2} \partial_x \left( (1 - u'_0) m_0 \right) = 0, \quad m_0(0, x) = M(x). \]

Explicit Solution with method of characteristics

\[ m_0(t, x) = \frac{1 + \mathbb{W}(\theta(x)) e^{-rt}}{1 + \mathbb{W}(\theta(x))} M(X(t; x)) \]

\[ X(t; x) = x + \frac{t}{2} + \frac{1}{2r} (1 - e^{-rt}) \mathbb{W}(\theta(x)). \]

Closed-form integration

\[ \eta_0(t) = \int_0^\infty m_0(t, x) \, dx = 1 - F \left( \tau^{-1}(t) \right), \]

where \( F \) denotes the CDF of the initial distribution \( M \).
Figure: Various descriptive statistics of the monopoly problem, where $r = 0.2$ and initial capacity follows a beta distribution with shape parameters $\alpha = 2, \beta = 4$. 
Expansion for the (Deterministic) MF Game

- Look for approximation to the PDE system of the form
  \[ u(t, x) = u_0(t, x) + \epsilon u_1(t, x) + \epsilon^2 u_2(t, x) + \cdots, \]
  \[ m(t, x) = m_0(t, x) + \epsilon m_1(t, x) + \epsilon^2 m_2(t, x) + \cdots. \]

- Zeroth order is the monopoly problem.

- From order \( \epsilon \): a first-order transport equation for \( u_1 \) whose characteristics are the trajectories of monopoly problem:
  \[ \partial_t u_1 - r u_1 + \frac{1}{2} (1 - u_0')( -\eta_0 - \partial_x u_1 + \bar{p}_0) = 0, \quad u_1(t, 0) = 0. \]

Its solution is given by

\[ u_1(t, x) = \frac{1}{2} \int_0^{\tau(x)} \left( e^{-rs} + \mathbb{W}(\theta(x)) \right) \cdot (-\eta_0 + \bar{p}_0)(t + s) \, ds. \]

Moreover, \( u_1(t, x) \leq 0 \) for all \((t, x)\): competition reduces firms’ value.
Density: First-order Correction

- Zeroth order is the monopoly problem.
- From order $\epsilon$: a first-order transport equation for $m_1$ whose characteristic curves are again the capacity trajectories of monopoly problem:

$$
\partial_t m_1 - \frac{1}{2} \partial_x ((1 - u_0') m_1) = g(t, x),
$$

$$
g(t, x) = \frac{1}{2} \partial_x \left( (-\eta_0 + \ddot{p}_0 - \partial_x u_1) m_0 \right).
$$

The first-order correction to density $m_1$ is given by

$$
m_1(t, x) = \int_{0}^{t} \frac{1 + \mathbb{W}(\theta(x)) e^{-r(t-s)}}{1 + \mathbb{W}(\theta(x))} g(s, X(t - s; x)) \, ds,
$$

where

$$
X(t; x) = x + \frac{t}{2} + \frac{1}{2r} \left( 1 - e^{-rt} \right) \mathbb{W}(\theta(x)).
$$
Demand: First-order Correction

Up to leading order in $\epsilon$, the equilibrium demand $q^*$ is given by

$$q^*(t, x) = \frac{1}{2}(\alpha - \partial_x u + \gamma \bar{p})$$

$$= \frac{1}{2}(1 - \partial_x u_0) + \epsilon \frac{1}{2}(-\eta_0 + \bar{p}_0 - \partial_x u_1) + O(\epsilon^2).$$

**Proposition**

*The first order correction to the equilibrium demand $q$ is negative, that is*

$$-\eta_0 + \bar{p}_0 - \partial_x u_1 \leq 0$$

*for $t \geq 0, x \geq 0$.***
Capacity Trajectories

Figure: Capacity trajectories up to order $\epsilon$, for various values of $\epsilon$. Here the initial capacity is chosen to follow the beta distribution with shape parameters $\alpha = 2, \beta = 4$. 
Iteration Scheme for Stochastic MFG

We begin an iteration with initial guesses \((\eta_0, \bar{p}_0)\) for \((\eta, \bar{p})\). Then for 
\(n = 0, 1, 2, \ldots\) we follow

**Step 1.** Given \((\eta_n, \bar{p}_n)\), solve HJB equation to calculate \(u_n\) and \(p_n^*\)
\[
\partial_t u_n + \frac{1}{2} \sigma^2 \partial_{xx} u_n - ru_n + \frac{1}{4} \left( a(\eta_n(t)) + \partial_x u_n(t, x) + c(\eta_n(t))\bar{p}_n(t) \right)^2 = 0,
\]
\[
p_n^*(t, x) = \frac{1}{2} \left( a(\eta_n(t)) + \partial_x u_n(t, x) + c(\eta_n(t))\bar{p}_n(t) \right),
\]
\[
q_n^*(t, x) = \frac{1}{2} \left( a(\eta_n(t)) - \partial_x u_n(t, x) + c(\eta_n(t))\bar{p}_n(t) \right).
\]

**Step 2.** Given \(p_n^*\) and \(q_n^*\), solve forward Kolmogorov to generate a new guess 
\((\eta_{n+1}, \bar{p}_{n+1})\)
\[
\partial_t m_{n+1} - \frac{1}{2} \sigma^2 \partial_{xx} m_{n+1} - \frac{1}{2} \partial_x \left( q_n^*(t, x)m_{n+1}(t, x) \right) = 0,
\]
\[
\eta_{n+1}(t) = \int m_{n+1}(t, x) \, dx, \quad \bar{p}_{n+1}(t) = \int p_n^*(t, x)m_{n+1}(t, x) \, dx.
\]

When \((\eta_{n+1}, \bar{p}_{n+1})\) is close enough to \((\eta_n, \bar{p}_n)\), we call \((u_n, m_n)\) a 
solution to the MFG.
Figure: Mean field Bertrand game with deterministic demand, where $r = 0.2$ and initial capacity follows a beta distribution with shape parameters $\alpha = 2, \beta = 4.$
MFG Approximation

Idea: approximate an $N$-player game by an initial density

$$M(x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_0^i),$$

where $x_0^i$ is the initial capacity of the $i$th player.

Example

Game with 4 firms, with initial capacities 1,2,3,4, would be approximated using the initial distribution

![Initial CDF for approximating a 4-player game.](image-url)
Discretized MFG: Numerical Examples

Figure: Average equilibrium price and total production in discretized MFG, where 10 players whose initial sizes are chosen to approximate a beta distribution.
Comparison with 2-player Asymptotics

MFG approximation compared to asymptotic expansion for 2-player game. Large and small players start from 0.75 and 0.25 respectively.

Figure: MFG approximation for 2-player game (solid), compared with the asymptotic expansion (dashed) and the monopoly solution (dotted).
Concluding Remarks

▶ The mean field game approximation can provide an efficient way to compute equilibrium.
▶ Asymptotic analysis about the well-understood monopoly problem can provide useful insights.
▶ Asymptotic analysis suggests that firms are more cautious and slow down production in the presence of product substitutability.
▶ This is confirmed by numerical solutions to MFG equations.
▶ Ongoing Issues
  ▶ Application to energy production: different capacity and different costs. Energy trilemma of exhaustibility, emissions and production costs.