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Stochastic Perron for Stochastic Target Games

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September 3, 2014
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1. Introduction
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Consider a stochastic target game:

- Two players: controller and nature.
- A state process $X$ can be manipulated by the nature through the selection of $\alpha$. Another process $Y$ is driven by both players, where the controller reacts to the nature.
- The controller tries to find a strategy such that the controlled state process almost-surely reaches a given target.
- The nature may choose a parametrization of the model to be totally adverse to the controller.
The formulation in our paper has been considered by Bouchard and Nutz (2013).

Related literature on the use of Perron's method:
Our goal

- Characterize the inf (sup) of the stochastic super-solutions (sub-solutions) $v^+$ ($v^-$) as the viscosity sub-solution (super-solution) of the HJB equation without going through the geometric dynamic programming principle first (hence avoiding the abstract measurable selection theorem).

- Give a more elementary proof to the result. As a result, we are able to avoid using Krylov’s method of shaken coefficients and hence avoid assuming the concavity of the Hamiltonian, which was required in Bouchard and Nutz’s paper.
Methodology

- Define the set of stochastic super- and sub-solutions appropriately, denoted by $U^+$ and $U^-$ respectively.
- Show: $v^+ := \inf_{w \in U^+}$ is a viscosity sub-solution.
- Show: $v^- = \sup_{w \in U^-}$ is a viscosity super-solution.
- Since $v^+$, $v$ and $v^-$ compare by definition, then

$$v^+ = v = v^-$$

holds under a comparison principle.
Notations

- Time horizon: $T > 0$;
- $\mathcal{D} := [0, T] \times \mathbb{R}^d$, $\mathcal{D}_{< T} := [0, T) \times \mathbb{R}^d$, $\mathcal{D}_T := \{ T \} \times \mathbb{R}^d$;
- $\Omega := C([0, T]; \mathbb{R}^d)$;
- $W_t(\omega) = \omega_t$: the canonical process;
- $\mathbb{F} = (\mathcal{F}_s)_{0 \leq s \leq T}$: the augmented filtration generated by $W$;
- $\mathbb{P}$: the Wiener measure on $\Omega$;
- $\mathbb{F}^t = (\mathcal{F}_s^t)_{t \leq s \leq T}$: the augmented filtration generated by $(W_s - W_t)_{s \geq t}$, for $t \leq s \leq T$;
**Notations**

- $U$: Borel subset of $\mathbb{R}^d$;
- $A$: compact subset of $\mathbb{R}^d$;
- $\mathcal{S}^t$: the set of $\mathcal{F}^t$-stopping times valued in $[t, T]$;
- $\mathcal{U}^t (\mathcal{A}^t)$: the collection of all $\mathbb{F}^t$-predictable processes in $L^p(\mathbb{P} \otimes dt)$ with values in $U(A)$;
Assumptions on $\mu_X, \sigma_X, \mu_Y$ and $\sigma_Y$

For $u \in \mathcal{U}^t$, $\alpha \in \mathcal{A}^t$, let $(X_{t,x}^\alpha, Y_{t,x,y}^{u,\alpha})$ denote processes satisfying

$$\begin{align*}
    dX_s &= \mu_X(s, X_s, \alpha_s)ds + \sigma_X(s, X_s, \alpha_s)dW_s, \\
    dY_s &= \mu_Y(s, X_s, Y_s, u_s, \alpha_s)ds + \sigma_Y(s, X_s, Y_s, u_s, \alpha_s)dW_s.
\end{align*}$$

with initial data $(X_t, Y_t) = (x, y)$.

**Assume:** $\mu_X, \mu_Y, \sigma_Y$ and $\sigma_X$ are continuous in all variables. Moreover there exists $K > 0$ such that, for all $(x, y), (x', y') \in \mathbb{R}^d \times \mathbb{R}$ and $u \in U$, 

Assumptions on $\mu_X, \sigma_X, \mu_Y$ and $\sigma_Y$

\[
|\mu_X(\cdot, x, \cdot) - \mu_X(\cdot, x', \cdot)| + |\sigma_X(\cdot, x, \cdot) - \sigma_X(\cdot, x', \cdot)| \leq K|x - x'|,
\]
\[
|\mu_X(\cdot, x, \cdot)| + |\sigma_X(\cdot, x, \cdot)| \leq K,
\]
\[
|\mu_Y(\cdot, y, \cdot) - \mu_Y(\cdot, y', \cdot)| + |\sigma_Y(\cdot, y, \cdot) - \sigma_Y(\cdot, y', \cdot)| \leq K|y - y'|,
\]
\[
|\mu_Y(\cdot, y, u, \cdot)| + |\sigma_Y(\cdot, y, u, \cdot)| \leq K(1 + |u| + |y|).
\]

This implies that for $\forall(t, x, y) \in \mathcal{D}_{\leq T} \times \mathbb{R}$ and any $u \in \mathcal{U}^t, \alpha \in \mathcal{A}^t$, (2.1) admits a unique strong solution.

Also assume $g: \mathbb{R}^d \to \mathbb{R}$ is a bounded and measurable function.
Definition 1 (Admissible strategies)

A map $u : \mathcal{A}^t \to \mathcal{U}^t$, $\alpha \mapsto u[\alpha]$ is a $t$-admissible strategy if

$$\{ \omega \in \Omega : \alpha(\omega)|_{[t,s]} = \alpha'(\omega)|_{[t,s]} \} \subset \{ \omega \in \Omega : u[\alpha](\omega)|_{[t,s]} = u[\alpha'](\omega)|_{[t,s]} \} \text{ -a.s.}$$

for all $s \in [t, T]$ and $\alpha, \alpha' \in \mathcal{A}^t$. Denote $u \in \mathcal{U}(t)$.

Definition 2 (Value function)

$$\nu(t, x) := \inf \{ y \in \mathbb{R} : \exists u \in \mathcal{U}(t) \text{ s.t. } Y_{t,x,y}^u(T) \geq g(X_{t,x}^\alpha(T)) \text{ -a.s.} \}.$$
Non-anticipating family of stopping times

**Definition 3**

Let $\{\tau^\alpha\}_{\alpha \in A^t} \subset \mathcal{S}^t$ be a family of stopping times. This family is t-non-anticipating if

\[
\{\omega \in \Omega : \alpha(\omega)|_{[t,s]} = \alpha'(\omega)|_{[t,s]}\} \subset \\
\{\omega \in \Omega : t \leq \tau^\alpha(\omega) = \tau^{\alpha'}(\omega) \leq s\} \cup \{\omega \in \Omega : s < \tau^\alpha, s < \tau^{\alpha'}\}-a.s.
\]

Denote the set of t-non-anticipating families of stopping times by $\mathcal{G}^t$. 
Strategies starting at a non-anticipating family of stopping times

Definition 4

Fix $t$ and let $\{\tau_\alpha\} \in \mathcal{G}^t$. We say that a map $u : \mathcal{A}^t \to \mathcal{U}^t$, $\alpha \mapsto u[\alpha]$ is a $(t, \{\tau_\alpha\})$-admissible strategy if it is non-anticipating in the sense that

$$\{\omega \in \Omega : \alpha(\omega)_[t,s] = \alpha'(\omega)_[t,s]\} \subset \{\omega \in \Omega : s < \tau_\alpha, \ s < \tau_\alpha'\} \cup \{\omega \in \Omega : t \leq \tau_\alpha = \tau_\alpha' \leq s, \ u[\alpha](\omega)_[\tau_\alpha(\omega),s] = u[\alpha'](\omega)_[\tau_\alpha'(\omega),s]\},$$

for all $s \in [t, T]$ and $\alpha, \alpha' \in \mathcal{A}^t$, denoted by $u \in \mathcal{U}(t, \{\tau_\alpha\})$. 
Definition 5 (Concatenation)

Let \( \alpha_1, \alpha_2 \in \mathcal{A}^t \), \( \tau \in \mathcal{S}^t \) is a stopping time. The concatenation of \( \alpha_1, \alpha_2 \) is defined as follows:

\[
\alpha_1 \otimes_{\tau} \alpha_2 := \alpha_1 \mathbb{1}_{[t, \tau)} + \alpha_2 \mathbb{1}_{[\tau, T]}. 
\]

The concatenation of elements in \( \mathcal{U}^t \) is defined in the similar fashion.

Lemma 2.1

Fix \( t \) and let \( \{\tau^\alpha\} \in \mathcal{G}^t \). For \( u \in \mathcal{U}(t) \) and \( \tilde{u} \in \mathcal{U}(t, \{\tau^\alpha\}) \), define
\[
u_*[\alpha] := u[\alpha] \otimes_{\tau^\alpha} \tilde{u}[\alpha]. \]
Then \( \nu_* \in \mathcal{U}(t) \). Use \( u \otimes_{\tau^\alpha} \tilde{u}[\alpha] \) to represent \( u[\alpha] \otimes_{\tau^\alpha} \tilde{u}[\alpha] \).
Definition 6 (Stochastic super-solutions)

A function $w : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ is called a stochastic super-solution of (2.2) if

1. it is bounded, continuous and $w(T, \cdot) \geq g(\cdot)$,

2. for fixed $(t, x, y) \in D \times \mathbb{R}$ and $\{\tau^\alpha\} \in \mathcal{G}^t$, there exists a strategy $\tilde{u} \in \mathcal{U}(t, \{\tau^\alpha\})$ such that, for any $u \in \mathcal{U}(t)$, $\alpha \in A^t$ and each stopping time $\rho \in \mathcal{S}^t$, $\tau^\alpha \leq \rho \leq T$ with the simplifying notation $X := X^\alpha_{t,x}$, $Y := Y^u_{t,x,y} \tilde{u}[\alpha]$, we have

$$Y(\rho) \geq w(\rho, X(\rho)) \mathbb{P} - \text{a.s. on } \{Y(\tau^\alpha) > w(\tau^\alpha, X(\tau^\alpha))\}.$$
The set of stochastic super-solutions is denoted by $\mathcal{U}^+$. Assume it is nonempty and $\nu^+ := \inf_{w \in \mathcal{U}^+} w$. For any stochastic super-solution $w$, choose $\tau^\alpha = t$ for all $\alpha$ and $\rho = T$, then there exists $\tilde{u} \in \mathcal{U}(t)$ such that, for any $\alpha \in \mathcal{A}^t$ and $Y_{t,x,y}(T) \geq w(T, X^\alpha_{t,x}(T)) \geq g(X^\alpha_{t,x}(T))$ -a.s. on $\{y > w(t, x)\}$.

Hence, $y > w(t, x)$ implies $y \geq \nu(t, x)$. This gives $w \geq \nu$ and $\nu^+ \geq \nu$. 
Definition 7 (Stochastic sub-solutions)

A function \( w : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) is called a stochastic sub-solution of (2.2) if

1. it is bounded, continuous and \( w(T, \cdot) \leq g(\cdot) \),

2. for fixed \((t, x, y) \in \mathcal{D} \times \mathbb{R} \) and \( \{\tau^{\alpha}\} \in \mathcal{S}^t \), for any \( u \in \mathcal{U}(t) \), \( \alpha \in \mathcal{A}^t \), there exists \( \tilde{\alpha} \in \mathcal{A}^t \) (may depend on \( u \), \( \alpha \) and \( \tau^{\alpha} \)) such that for each stopping time \( \rho \in \mathcal{S}^t \), \( \tau^{\alpha} \leq \rho \leq T \) with the simplifying notation \( X := X_{t,x}^{\alpha}, Y := Y_{t,x,y}^{u,\alpha \otimes \tau^{\alpha} \tilde{\alpha}} \), we have

\[
P \left( Y(\rho) < w(\rho, X(\rho)) \mid B \right) > 0,
\]

for any \( B \subset \{ Y(\tau^{\alpha}) < w(\tau^{\alpha}, X(\tau^{\alpha})) \} \), \( B \in \mathcal{F}_t^{\tau^{\alpha}} \) and \( P(B) > 0 \).
The set of stochastic sub-solutions is denoted by $\mathcal{U}^-$. Assume it is nonempty and let $v^- := \sup_{w \in \mathcal{U}^-} w$. For any stochastic sub-solution $w$, choose $\tau^\alpha = t$ for all $\alpha$ and $\rho = T$. Hence for any $u \in \mathcal{U}(t)$, there exists $\tilde{\alpha} \in \mathcal{A}^t$, such that

$$
\mathbb{P} \left( Y_{t,x,y}^{u,\tilde{\alpha}}(T) < w(T, X_{t,x}^{\tilde{\alpha}}(T)) \leq g(X_{t,x}^\alpha(T)) \mid y < w(t, x) \right) > 0.
$$

Hence, $y < w(t, x)$ implies $y < v(t, x)$. This gives $w \leq v$ and $v^- \leq v$. Clearly,

$$
v^- \triangleq \sup_{w \in \mathcal{U}^-} w \leq v \leq \inf_{w \in \mathcal{U}^+} w \triangleq v^+.
$$
Assumptions

Assumption 2.1

\[
\sup_{u \in U} \frac{|\mu_Y(\cdot, u, \cdot)|}{1 + |\sigma_Y(\cdot, u, \cdot)|}
\text{ is locally bounded.}
\]

Assumption 2.2

1. There exists \( u \in U \) such that \( \mu_Y(t, x, y, u, a) = 0 \), \( \sigma_Y(t, x, y, u, a) = 0 \) for all \((t, x, y, a) \in D_{<T} \times \mathbb{R} \times A\).
2. There exists \( a \in A \), \( \mu_Y(t, x, y, u, a) = 0 \), \( \sigma_Y(t, x, y, u, a) = 0 \) for all \((t, x, y, u) \in D_{<T} \times \mathbb{R} \times U\).
Assumptions

**Assumption 2.3**

Let \( N(t, x, y, z, a) = \{ u \in U : \sigma_Y(t, x, y, u, a) \} \). There exists a measurable map \( \hat{u} : D \times \mathbb{R} \times \mathbb{R}^d \times A \to U \) such that \( N = \{ \hat{u} \} \). Moreover, the map \( \hat{u} \) is continuous.

**Assumption 2.4**

The map \((y, z) \mapsto \mu_{\hat{Y}} := \mu_Y(t, x, y, \hat{u}(t, x, y, z, a), a)\) is Lipschitz continuous and has linear growth, uniformly in \((t, x, a)\).

**Assumption 2.5 (Comparison principle)**

Let \( v \) (resp. \( \bar{v} \)) be a LSC (resp. USC) bounded viscosity super-solution (resp. sub-solution) of (2.2). Then, \( v \geq \bar{v} \) on \( D \).
Define \((t, x, y, p, M) \in \mathcal{D} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{M}^d\),

\[
H(t, x, y, p, M) := \sup_{a \in A} \left\{ -\mu \hat{u}(t, x, y, \sigma_X(t, x, a)p, a) + \mu_X(t, x, a)\top p \right. \\
+ \frac{1}{2} \text{Tr}[\sigma_X\sigma_X\top(t, x, a)M]\right\},
\]

where \(\mu \hat{u}(t, x, y, z, a) := \mu_Y(t, x, y, \hat{u}(t, x, y, z, a), a), \ z \in \mathbb{R}^d\).

The HJB equation is

\[
-\phi_t - H(t, x, \phi, D\phi, D^2\phi) = 0 \quad \text{on} \quad \mathcal{D}_{<T}
\]

\[
\phi - g = 0 \quad \text{on} \quad \mathcal{D}_T
\]

(2.2)
Preparatory lemmas

**Lemma 2.2**

Under Assumption 2.2 the sets $\mathcal{U}^+$ and $\mathcal{U}^-$ are nonempty.

**Proof.**

We will only prove $\mathcal{U}^+$ is nonempty under Assumption 2.2 (1). Choose $\tilde{u}[\alpha] = u$. For any given $\{\tau^\alpha\} \in \mathcal{S}^t$, we have $\tilde{u} \in \mathcal{U}(t, \{\tau^\alpha\})$ and $\forall \alpha \in \mathcal{A}^t$ and $\rho \in \mathcal{S}^t$ such that $\tau^\alpha \leq \rho \leq T$,

$$
Y_{t,x,y}^{u \otimes \tau^\alpha, \tilde{u}[\alpha], \alpha}(\rho) = Y_{t,x,y}^{u \otimes \tau^\alpha, \tilde{u}[\alpha], \alpha}(\tau^\alpha).
$$

From the boundedness of $g$, there exists an $C$, such that $g(x) < C$. Now take $w(t, x) \equiv C$. On the set $\{Y(\tau^\alpha) > w(\tau^\alpha, X(\tau^\alpha))\}$, we clearly have that $\{Y(\rho) > w(\rho, X(\rho))\}$. 
Lemma 2.3

1. If $w_1, w_2 \in \mathcal{U}^+$, then $w_1 \land w_2 \in \mathcal{U}^+$;
2. If $w_1, w_2 \in \mathcal{U}^-$, then $w_1 \lor w_2 \in \mathcal{U}^-$.

Proof of (1): for $w_1, w_2 \in \mathcal{U}^+$, let $w = w_1 \land w_2$. For fixed $(t, x, y) \in \mathcal{D}_< T \times \mathbb{R}$ and $\{\tau^\alpha\} \in \mathcal{G}^t$,

$$u[\alpha] = u_1[\alpha] \mathbb{1}_A + u_2[\alpha] \mathbb{1}_{A^c},$$

where $u_1$ and $u_2$ are the strategies starting at $\{\tau^\alpha\}$ for $w_1$ and $w_2$, respectively and $A = \{w_1(\tau^\alpha, X(\tau^\alpha)) < w_2(\tau^\alpha, X(\tau^\alpha))\}$. $u$ works for $w$ in the definition of stochastic super-solutions. □
Lemma 2.4

There exists a non-increasing sequence $U^+ \ni w_n \downarrow v^+$ and an non-decreasing sequence $U^- \ni v_n \uparrow v^-$. 

Lemma 2.5

$f(x, a)$ is defined on $X \times A \in \mathbb{R}^m \times \mathbb{R}^n$ and $f(x, a)$ is uniformly continuous. Assume $F(x) := \sup_{a \in A} f(x, a) < \infty$, then $F(x)$ is continuous.
The viscosity sub-solution

Proposition 3.1

Under all standing assumptions, if \( g \) is USC, the function \( v^+ \) is a bounded upper semi-continuous (USC) viscosity sub-solution of (2.2);

Proof of the proposition:

1.1 The interior sub-solution property: Assume for some \((t_0, x_0) \in \mathcal{D}_{<T},\)

\[
\varphi_t + H(t, x, \varphi, D\varphi, D^2\varphi) < 0 \quad \text{at} \quad (t_0, x_0).
\]

where the test function \( \varphi \) strictly touches \( v^+ \) from above at \((t_0, x_0).\) Then the map \((t, x, y) \mapsto H(t, x, y, D\varphi(t, x), D^2\varphi(t, x))\) is continuous near \((t_0, x_0, \varphi(t_0, x_0))\) from Lemma 2.5.
There exists a $\varepsilon > 0$ and $\delta > 0$ such that

\[
\varphi_t + H(t, x, y, D\varphi, D^2\varphi) < 0, \\
\forall (t, x) \in B(t_0, x_0, \varepsilon) \text{ and } |y - \varphi(t, x)| \leq \delta,
\]

(3.1)

On $\mathcal{T} = B(t_0, x_0, \varepsilon) - B(t_0, x_0, \varepsilon/2)$, $\varphi > v^+ + \eta$ on $\mathcal{T}$ for some $\eta > 0$.

$w_n \downarrow v^+$ and Dini type argument $\Rightarrow$ there exists a $n$ such that $\varphi > w_n + \eta/2$ on $\mathcal{T}$ and $\varphi > w_n - \delta$ on $B(t_0, x_0, \varepsilon/2)$. Let $w = w_n$. Define, for small $\kappa < \frac{\eta}{2} \land \delta$,

\[
w^\kappa \overset{\Delta}{=} \left\{
\begin{array}{ll}
(\varphi - \kappa) \land w & \text{on } B(t_0, x_0, \varepsilon), \\
 w & \text{outside } B(t_0, x_0, \varepsilon).
\end{array}
\right.
\]
Proof

We will obtain a contradiction if we show $w^\kappa \in \mathcal{U}^+$ and $w^\kappa(t_0, x_0) < \nu^+(t_0, x_0)$. Fix $t$ and $\{\tau^\alpha\} \in \mathcal{G}^t$.

(i) if $w^\kappa(\tau^\alpha, X(\tau^\alpha)) = w(\tau^\alpha, X(\tau^\alpha))$: set $\tilde{u} = \tilde{u}_1$, which is the "optimal" strategy in the definition of stochastic super-solutions for $w$ starting at $\{\tau^\alpha\}$.

(ii) if $w^\kappa(\tau^\alpha, X(\tau^\alpha)) < w(\tau^\alpha, X(\tau^\alpha))$: let $\overline{Y}$ be the solution to

$$
\overline{Y}(t) = Y^u_{t,x,y}(\tau^\alpha) + \int_{\tau^\alpha}^{\tau^\alpha \vee t} \mu_{Y}(s, X^\alpha_{t,x}, \overline{Y}(s), \sigma_X(s, X^\alpha_{t,x}, \alpha_s)D\varphi, \alpha_s)ds
+ \int_{\tau^\alpha}^{\tau^\alpha \vee t} \sigma_X(s, X^\alpha_{t,x}(s), \alpha_s)D\varphi(s, X^\alpha_{t,x}(s))dW_s, \quad t \geq \tau^\alpha,
$$
Proof

Let $\theta_1^\alpha = \inf \{ s \geq \tau^\alpha, (s, X(s)) \notin B(t_0, x_0; \varepsilon/2) \}$,
$\theta_2^\alpha = \inf \{ s \geq \tau^\alpha, |Y(s) - \varphi(s, X(s))| \geq \delta \}$ and $\theta^\alpha = \theta_1^\alpha \land \theta_2^\alpha$.

$\{\theta^\alpha \} \in \mathcal{S}^t$ (Example 1, Bayraktar and Huang). We will set $\tilde{u}$ to be

$\tilde{u}_0[\alpha](s) := \hat{u}(s, X_{t,x}^\alpha(s), Y(s), \sigma_X(s, X_{t,x}^\alpha(s), \alpha_s)D\varphi(s, X_{t,x}^\alpha(s), \alpha_s))$

until $\theta^\alpha$. Starting at $\theta^\alpha$, we will then follow the strategy $u^\theta$ which is "optimal" for $w$. In other words, define $\tilde{u}$ by

$\tilde{u}[\alpha] := 1_A \tilde{u}_1[\alpha] + 1_A^c (\tilde{u}_0[\alpha] 1_{[t, \theta^\alpha]} + u^\theta[\alpha] 1_{[\theta^\alpha, \tau^\alpha]} ) \in \mathcal{U}(t, \{\tau^\alpha\})$, \n
where $A = \{ w^\kappa(\tau^\alpha, X(\tau^\alpha)) = w(\tau^\alpha, X(\tau^\alpha)) \}$. 

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Proof

We need to show that

\[ Y(\rho) \geq w^\kappa(\rho, X(\rho)) \text{ on } \{ Y(\tau^\alpha) > w^\kappa(\tau^\alpha, X(\tau^\alpha)) \}, \]

where \( X := X_{t,x}^\alpha, Y := Y_{t,x,y}^{u \otimes \tau^\alpha \tilde{u}[\alpha], \alpha} \). Note that \( Y = Y_{t,x,y}^{u \otimes \tau^\alpha \tilde{u}_1[\alpha], \alpha} \)
and

\[ Y = 1_A Y_{t,x,y}^{u \otimes \tau^\alpha \tilde{u}_1[\alpha], \alpha} + 1_{A^c} Y_{t,x,y}^{u \otimes \tau^\alpha \tilde{u}_0[\alpha], \alpha} \text{ for } \tau^\alpha \leq s \leq \theta^\alpha. \quad (3.2) \]

(i) On the set \( A \cap \{ Y(\tau^\alpha) > w^\kappa(\tau^\alpha, X(\tau^\alpha)) \} \), from (3.2) and the "optimality" of \( \tilde{u}_1 \) (for \( w \)),

\[ Y(\rho) = Y_{t,x,y}^{u \otimes \tau^\alpha \tilde{u}_1[\alpha], \alpha}(\rho) \geq w(\rho, X(\rho)) \geq w^\kappa(\rho, X(\rho)) \quad \mathbb{P}\text{-a.s.} \]
Proof

(ii) On the set $A^c \cap \{ Y(\tau^\alpha) > w^\kappa(\tau^\alpha, X(\tau^\alpha)) \}$, by the definition of $\tilde{u}_0$, (3.1) and (3.2), using Itô’s Formula, 

$$Y(\cdot \wedge \theta^\alpha) - \varphi(\cdot \wedge \theta^\alpha, X(\cdot \wedge \theta^\alpha))$$

is increasing on $[\tau^\alpha, \theta^\alpha]$ and

$$Y(\theta^\alpha) - \varphi(\theta^\alpha, X(\theta^\alpha)) + \kappa \geq Y(\tau^\alpha) - \varphi(\tau^\alpha, X(\tau^\alpha)) + \kappa > 0. \quad (3.3)$$

From (3.3), we can prove

$$Y(\theta^\alpha) - w(\theta^\alpha, X(\theta^\alpha)) > 0 \quad \text{on} \quad A^c \cap \{ Y(\tau^\alpha) > w^\kappa(\tau^\alpha, X^\alpha) \}. \quad (3.4)$$

It follows from this conclusion and the fact the "optimality" of $u^\theta$ starting at $\{\theta^\alpha\}$ that on $A^c \cap \{ Y(\tau^\alpha) > w^\kappa(\tau^\alpha, X^\alpha) \}$

$$Y(\rho \lor \theta^\alpha) - w^\kappa(\rho \lor \theta^\alpha, X(\rho \lor \theta^\alpha)) \geq Y(\rho \lor \theta^\alpha) - w(\rho \lor \theta^\alpha, X(\rho \lor \theta^\alpha)) \geq 0. \quad (3.4)$$
Proof

Also, since \( Y(\cdot \wedge \theta^\alpha) - \varphi(\cdot \wedge \theta^\alpha, X(\cdot \wedge \theta^\alpha)) \) is increasing on \([\tau^\alpha, \theta^\alpha]\), then

\[
( Y(\rho \wedge \theta^\alpha) - \varphi(\rho \wedge \theta^\alpha, X(\rho \wedge \theta^\alpha)) + \kappa ) > 0,
\]

which further gives

\[
( Y(\rho \wedge \theta^\alpha) - w^\kappa(\rho \wedge \theta^\alpha, X(\rho \wedge \theta^\alpha))) > 0, \text{ on } A_c \cap \{ Y(\tau^\alpha) > w^\kappa(\tau^\alpha, X^\alpha) \}.
\]

From (3.4) and (3.5) we have

\[
Y(\rho) - w^\kappa(\rho, X(\rho)) \geq 0, \text{ on } A_c \cap \{ Y(\tau^\alpha) > w^\kappa(\tau^\alpha, X^\alpha) \}.
\]
Proof

1.2 The boundary condition:
**Step A:** Let $\hat{\mu}_Y$ be non-decreasing in its y-variable. Assume that for some $x_o \in \mathbb{R}^d$, we have $v^+(T, x_o) > g(x_o)$. There exists $\varepsilon > 0$ such that $v^+(T, x_o) > g(x) + \varepsilon$ for $|x - x_o| \leq \varepsilon$. Choose $\beta$, such that $v^+(T, x_o) + \frac{\varepsilon^2}{4\beta} > \varepsilon + \sup_T v^+(t, x)$. There exists a $w \in \mathcal{U}^+$, such that

$$v^+(T, x_o) + \frac{\varepsilon^2}{4\beta} > \varepsilon + \sup_T w(t, x).$$

Define for $C > 0$,

$$\varphi^{\beta, \varepsilon, C} = v^+(T, x_o) + \left\frac{|x - x_o|^2}{\beta} + C(T - t)\right.$$
There exists for large enough $C > 0$, such that

$$- \varphi^\beta,\epsilon, C - H(\cdot, y, D\varphi^\beta,\epsilon, C, D^2\varphi^\beta,\epsilon, C)(t, x) > 0,$$

$$\forall (t, x, y) \in B(T, x_0; \epsilon) \times \mathbb{R} \text{ s.t. } y \geq \varphi^\beta,\epsilon, C(t, x) - \epsilon.$$

Making sure that $C \geq \epsilon / 2\beta$, we obtain that

$$\varphi^\beta,\epsilon, C \geq \epsilon + w \text{ on } T.$$

Also,

$$\varphi^\beta,\epsilon, C(T, x) \geq \nu^+(T, x_0) > g(x) + \epsilon \text{ for } |x - x_0| \leq \epsilon. \quad (3.6)$$
Now we can choose $\kappa < \varepsilon$ and define

$$w^{\beta, \varepsilon, C, \kappa} \triangleq \begin{cases} 
(\phi^{\beta, \varepsilon, C} - \kappa) \land w & \text{on } B(T, x_0, \varepsilon), \\
w & \text{outside } B(T, x_0, \varepsilon).
\end{cases} \quad (3.7)$$

Note that $w^{\beta, \varepsilon, C, \kappa}(T, x) \geq g(x)$ and $w^{\beta, \varepsilon, C, \kappa}(T, x_0) < v^+(T, x_0)$. Similar to Step 1.1, $w^{\beta, \varepsilon, C, \kappa}$ is a stochastic super-solution.
**Proof**

**Step B:** Fix $c > 0$ and define $\tilde{Y}_{u, \alpha}^{t, x, y}$ as the strong solution of

$$d\tilde{Y}(s) = \tilde{\mu}_Y(s, X_{t, x}^\alpha(s), \tilde{Y}(s), u[\alpha]_s, \alpha_s)ds + \tilde{\sigma}_Y(s, X_{t, x}^\alpha(s), \tilde{Y}(s), u[\alpha]_s, \alpha_s)dW_s \quad \text{with} \quad \tilde{Y}(t) = y,$$

where

$$\tilde{\mu}_Y(t, x, y, u, a) : = cy + e^{ct} \mu_Y(t, x, e^{-ct}y, u, a),$$

$$\tilde{\sigma}_Y(t, x, y, u, a) : = e^{ct} \sigma_Y(t, x, e^{-ct}y, u, a).$$

Then, $\tilde{Y}_{u, \alpha}^{t, x, y}(s)e^{-cs} = Y_{u, \alpha}^{t, x, ye^{-cs}}(s)$ for any $s \in [t, T]$. Set

$$\tilde{g}(x) : = e^{cT}g(x)$$

and define $\tilde{v}(t, x) : = \inf\{y \in \mathbb{R} : \exists u \in \mathcal{U}^{t} \text{ s.t. } \tilde{Y}_{u, \alpha}^{t, x, y}(T) \geq \tilde{g}(X_{t, x}(T)) \text{ -a.s. } \forall \alpha \in \mathcal{A}^{t}\}$. 
Proof

Therefore, \( \tilde{v}(t, x) = e^{ct} v(t, x) \) and for large \( c > 0 \) the map

\[
\tilde{\mu}_Y : (t, x, y, z, a) \mapsto cy + e^{ct} \tilde{\mu}_Y(t, x, e^{-ct}y, e^{-ct}z, a)
\]

is non-decreasing in its \( y \)-variable. \( \tilde{v}^+ \) is a sub-solution of the new PDE with

\[
\hat{H}(t, x, y, p, M) := -cy - \sup_{a \in A} \left\{ e^{ct} \tilde{\mu}_Y(t, x, e^{-ct}y, e^{-ct}\sigma_X(t, x, a),
\right. \\
\left. + \mu_X(t, x, a)^\top p + \frac{1}{2} \left[ \sigma_X \sigma_X^\top(t, x, a)M \right] \right\},
\]

\( w(t, x) \) is a stochastic super-solution of (2.2) if and only if \( \tilde{w}(t, x) = e^{ct} w(t, x) \) is a stochastic super-solution of the new HJB equation with \( \hat{H} \Rightarrow \tilde{v}^+(t, x) = e^{ct} v^+(t, x) \). This concludes the proof.
Proposition 4.1

Under all standing assumptions, if \( g \) is LSC, the function \( v^- \) is a bounded lower semi-continuous (LSC) viscosity super-solution of (2.2).

Proof of the proposition:

2.1 The interior super-solution property: Assume for some \((t_0, x_0) \in D_{<T}, \varphi + H(t, x, \varphi, D\varphi, D^2\varphi) > 0\) at \((t_0, x_0)\), where the test function \( \varphi \) strictly touches \( v^- \) from below at \((t_0, x_0)\). Similar to Step 1.1, we get

\[
\varphi_t + H^{u,a_0}(\cdot, y, D\varphi, D^2\varphi) > 0 \quad \forall (t, x) \in \overline{B}_\varepsilon \quad \text{and} \quad (y, u) \in R \times U \quad \text{s.t.} \quad |y - \varphi(t, x)| \leq \delta \quad \text{and} \quad |\sigma_Y(t, x, y, u, a_0) - \sigma_X(t, x, a_0)D\varphi(t, x)| \leq \delta,
\]
Proof

where \( u_0 = \hat{u}(t_0, x_0, \varphi(t_0, x_0), \sigma_X(\cdot, a_0)D\varphi(t_0, x_0), D^2\varphi(t_0, x_0)) \)
and \( H^{u,a}(t, x, y, p, M) := -\mu_Y(t, x, y, u, a) + \mu_X(t, x, a)\top p \
+ \frac{1}{2}\text{Tr} [\sigma_X\sigma_X\top(t, x, a)M] \).

Similarly, there exits a \( w \in \mathcal{U}^- \), such that \( \varphi + \eta/2 < w \) for some \( \eta > 0 \) on \( \mathbb{T} \) and \( \varphi < w + \delta \) on \( B(t_0, x_0, \varepsilon/2) \). Define, for small \( \kappa << \eta/2 \land \delta \),

\[
\begin{aligned}
    w^{\kappa} &\triangleq \begin{cases} 
        (\varphi + \kappa) \lor w \text{ on } \overline{B(t_0, x_0, \varepsilon)}, \\
        w \text{ outside } \overline{B(t_0, x_0, \varepsilon)}. 
    \end{cases}
\end{aligned}
\]

and we want to show \( w^{\kappa} \in \mathcal{U}^- \) with \( w^{\kappa}(t_0, x_0) > \nu^-(t_0, x_0) \). For a given \( u \in \mathcal{U}^t \) and \( \alpha \in \mathcal{A}^t \), we need to construct a process \( \tilde{\alpha} \in \mathcal{A}^t \) in the definition of stochastic sub-solutions for \( w^{\kappa} \).
Proof

(i) if $w(\tau^\alpha, X(\tau^\alpha)) = w^K(\tau^\alpha, X(\tau^\alpha))$: set $\tilde{\alpha} = \tilde{\alpha}_1$, which is optimal" for the nature given $u$, $\alpha$ and $\tau^\alpha$.

(ii) if $w(\tau^\alpha, X(\tau^\alpha)) < w^K(\tau^\alpha, X(\tau^\alpha))$: define

$\theta_1^\alpha = \inf\{s \geq \tau^\alpha, (s, X^\alpha \otimes \tau^\alpha a_0) \not\in B(t_0, x_0; \varepsilon/2)\}$,

$\theta_2^\alpha = \inf\{s \geq \tau^\alpha, |Y_{t, x, y}^{u, \alpha \otimes \tau^\alpha a_0} - \varphi(s, X^\alpha \otimes \tau^\alpha a_0)| \geq \delta\}$ and

$\theta^\alpha = \theta_1^\alpha \wedge \theta_2^\alpha$. The nature choose $a_0$ until $\theta^\alpha$. Starting from $\theta^\alpha$, choose $\alpha^*$ which is "optimal" for the nature for $w$ given $\alpha$, $u$. In other words, let $A = \{w(\tau^\alpha, X(\tau^\alpha)) = w^K(\tau^\alpha, X(\tau^\alpha))\}$,

$$\tilde{\alpha} = 1_A \tilde{\alpha}_1 + 1_A^c (a_0 1_{[t, \theta^\alpha]} + \alpha^* 1_{[\theta^\alpha, T]}).$$
Proof

we abbreviate \((X, Y) = (X_{t,x}^{\alpha \otimes \tau \alpha \tilde{\alpha}}, Y_{t,x,y}^{u,\alpha \otimes \tau \alpha \tilde{\alpha}})\). Note that

\[
X = \mathbb{1}_A X_{t,x}^{\alpha \otimes \tau \alpha \tilde{\alpha}} + \mathbb{1}_{A^c} X_{t,x}^{\alpha \otimes \tau \alpha a_0} \quad \text{for} \quad \tau \alpha \leq s \leq \theta \alpha, \\
Y = \mathbb{1}_A Y_{t,x,y}^{u,\alpha \otimes \tau \alpha \tilde{\alpha}} + \mathbb{1}_{A^c} Y_{t,x,y}^{u,\alpha \otimes \tau \alpha a_0} \quad \text{for} \quad \tau \alpha \leq s \leq \theta \alpha. 
\]

\hspace{1cm} (4.1)

For simplicity, let

\[
E = \{ Y(\tau \alpha) < w^\kappa(\tau \alpha, X(\tau \alpha)) \} , \quad E_0 = \{ Y(\tau \alpha) < w(\tau \alpha, X(\tau \alpha)) \}, \\
E_1 = \{ w(\tau \alpha, X(\tau \alpha)) \leq Y(\tau \alpha) < w^\kappa(\tau \alpha, X(\tau \alpha)) \}, \\
G = \{ Y(\rho) < w^\kappa(\rho, X(\rho)) \} , \quad G_0 = \{ Y(\rho) < w(\rho, X(\rho)) \}.
\]

Note that \( E = E_0 \cup E_1, \quad E_0 \cap E_1 = \emptyset \) and \( G_0 \subset G \). The goal is to show \( P(G|B) > 0 \) given that \( B \subset E \) and \( P(B) > 0 \). It suffices to show \( \mathbb{P}(G \cap B) = \mathbb{P}(G \cap B \cap E_0) + \mathbb{P}(G \cap B \cap E_1) > 0. \)
Proof

(i) $\mathbb{P}(B \cap E_0) > 0$: Directly from the way $\tilde{\alpha}_1$ is defined and the definition of the stochastic sub-solutions, we get

$$\mathbb{P}(G_0 | B \cap E_0) = \mathbb{P}(Y_{t,x,y}^{u,\alpha \otimes \tau \alpha \tilde{\alpha}_1} (\rho) < w(\rho, X_{t,x}^{\alpha \otimes \tau \alpha \tilde{\alpha}_1} (\rho)) | B \cap E_0) > 0.$$  

This further implies that $\mathbb{P}(G \cap B \cap E_0) \geq \mathbb{P}(G_0 \cap B \cap E_0) > 0$.

(ii) $\mathbb{P}(B \cap E_1) > 0$: from (4.1), we have that

$$\mathbb{P}(Y(\theta^\alpha) < w^\kappa(\theta^\alpha, X(\theta^\alpha)) | B \cap E_1) = \mathbb{P}(Y_{t,x,y}^{u,\alpha \otimes \tau \alpha a_0} (\theta^\alpha) < w^\kappa(\theta^\alpha, X_{t,x}^{\alpha \otimes \tau \alpha a_0} (\theta^\alpha)) | B \cap E_1).$$  

Let

$$\Delta(s) = Y(s \wedge \theta^\alpha) - (\varphi(s \wedge \theta^\alpha, X(s \wedge \theta^\alpha)) + \kappa).$$
Proof

We still study $\Delta$. However, the rest of the proof relies on the super-martingale property of a process involving $\Delta$ (in contrast to the increasing property of $\Delta$). In fact, we prove $M\Delta$ is a super-martingale on $[\tau^\alpha, \theta^\alpha]$, where $M$ is given by following formula

$$
\lambda = \sigma_Y(\cdot, X, Y, u[a_0], a_0) - \sigma_X(\cdot, X, a_0)D\varphi(\cdot, X),
$$

$$
\beta = \left(\varphi_t(\cdot, X) + H^u[a_0, a_0](\cdot, Y, \varphi, D\varphi, D^2\varphi)(\cdot, X)\right) \|\lambda\|^{-2}\lambda \mathbb{1}_{\{|\lambda|>\delta\}},
$$

$$
M(\cdot) = 1 + \int_{\tau^\alpha}^{\cdot \wedge \theta^\alpha} M(s)\beta_s^\top dW_s.
$$
Proof

From the definition of $E_1$ and $w_\kappa$, $\Delta(\tau^\alpha) < 0$ on $B \cap E_1$. The super-martingale property of $M\Delta$ implies that there exists a non-null $H \subset B \cap E_1$, $H \in \mathcal{F}_{\tau^\alpha}^t$ such that $\Delta(\theta^\alpha \wedge \rho) < 0$ on $H$. Therefore, we see that

\[
Y(\theta_1^\alpha) - (\varphi(\theta_1^\alpha, X(\theta_1^\alpha)) + \kappa) < 0 \quad \text{on} \quad H \cap \{\theta_1^\alpha < \theta_2^\alpha \wedge \rho\}, \quad (4.2)
\]

\[
Y(\theta_2^\alpha) - (\varphi(\theta_2^\alpha, X(\theta_2^\alpha)) + \kappa) < 0 \quad \text{on} \quad H \cap \{\theta_2^\alpha \leq \theta_1^\alpha \wedge \rho\}, \quad (4.3)
\]

and that

\[
Y(\rho) - (\varphi(\rho, X(\rho)) + \kappa) < 0 \quad \text{on} \quad H \cap \{\rho < \theta^\alpha\}. \quad (4.4)
\]

From (4.2) and (4.3), we can show that
Proof

\[ Y(\theta^\alpha) < \omega(\theta^\alpha, X(\theta^\alpha)) \text{ on } H \cap \{\theta^\alpha \leq \rho\}. \]

Now from the definition of stochastic sub-solutions and of \( \alpha^* \),

\[ \mathbb{P}(G_0 | H \cap \{\theta^\alpha \leq \rho\}) > 0 \text{ if } \mathbb{P}(H \cap \{\theta^\alpha \leq \rho\}) > 0. \quad (4.5) \]

On the other hand, (4.4) implies that

\[ \mathbb{P}(G | H \cap \{\theta^\alpha > \rho\}) > 0 \text{ if } \mathbb{P}(H \cap \{\theta^\alpha > \rho\}) > 0. \quad (4.6) \]

Since \( \mathbb{P}(H) > 0, G_0 \subset G, \) and \( H \subset E_1 \cap B, \) (4.5) and (4.6) imply \( \mathbb{P}(G \cap E_1 \cap B) > 0. \)

2.2 The boundary condition: Similar to that in Step 1.2. \( \square \)
Restate the theorem,

**Theorem 5.1 (Stochastic Perron for stochastic target games)**

*Under all standing assumptions*

1. If $g$ is USC, the function $v^+$ is a bounded upper semi-continuous (USC) viscosity sub-solution of (2.2);
2. If $g$ is LSC, the function $v^-$ is a bounded lower semi-continuous (LSC) viscosity super-solution of (2.2).

**Corollary 5.1**

Under all standing assumptions, if $g$ is continuous, then $v$ is continuous and is the unique bounded viscosity solution of (2.2).
Introduction

The Set-up

Sub-solution property for $v^+$

Super-solution property for $v^-$

Future Work
Future work

(1) Add a stopping time in the definition of $\nu$:

$$
\tilde{\nu}(t, x) := \inf\{ y \in \mathbb{R} : \exists u \in \mathcal{U}(t) \text{ s.t. } Y_{t,x,y}^{u,\alpha}(\tau) \geq g(X_{t,x}^{\alpha}(\tau)) \text{ -a.s.} \}
$$

In the super-hedging context in mathematical finance, it is super-hedging a American financial contract (with model uncertainty).

- Realistic model. The holder of the contract may play against the controller since she has the right to exercise at any time $\tau$.
- Controller and stopper game version of stochastic target problem with controlled loss, Bayraktar and Huang.
Future Work

(2) Find a weaker sufficient condition which guarantees that $U^-$ is not empty.

- Under some suitable conditions, Girsanov’s Theorem $\Rightarrow$ $Y$ is a martingale under a measure $Q$ which is equivalent to $P$.
- Assume $\frac{|u_Y|}{\|\sigma_Y\|}$ is bounded on $N = \{(t, x, y, u, a) : \sigma_Y \neq 0\}$ and for fixed $(t, x, y) \in D_T \times \mathbb{R}$, $\{\tau^\alpha\} \in \mathcal{G}^t$, assume that any given $u, \alpha$, there exits a $\tilde{\alpha}$, such that

$$E_P \int_0^T \|\sigma_Y\|^2(s, X(s), Y(s), u[\alpha \otimes_{\tau^\alpha} \tilde{\alpha}]_s, [\alpha \otimes_{\tau^\alpha} \tilde{\alpha}]_s)ds,$$

where $X = X_{t,x}^{\alpha \otimes_{\tau^\alpha} \tilde{\alpha}}$, $Y = Y_{t,x,y}^{u,\alpha \otimes_{\tau^\alpha} \tilde{\alpha}}$. 
Future Work

**Sketch of the proof:** Take \( w(t, x) = c \), where \( c \) is a lower bound of \( g \). For any given \( u, \alpha \), choose the \( \tilde{\alpha} \) in the assumption. Let \( B \subset \{ Y(\tau) < w(\tau, X(\tau)) \} \) and \( \mathbb{P}(B > 0) \). Let

\[
\theta_s \triangleq \begin{cases} 
\frac{u_Y \sigma_Y}{\|\sigma_Y\|^2}(s, X(s), Y(s), u[\alpha \otimes_{\tau}\hat{\alpha}]_s, \alpha \otimes_{\tau}\hat{\alpha}), & \sigma_Y \neq 0 \\
C, & \text{otherwise},
\end{cases}
\]

for any constant \( C \). Therefore, \( \theta_s \) satisfies the Novikov’s condition and \( \hat{W}(s) = W(s) - \int_{\tau}^s \theta_u du \) is a Brownian motion under some probability measure \( \mathbb{Q} \) which is equivalent to \( \mathbb{P} \). Therefore, we have \( \mathbb{Q}(B) > 0 \).

Under \( \mathbb{Q} \), \( dY = \sigma_Y d\hat{W}_s \). Under square integrability condition, \( Y \) is a martingale.
Future Work

From the martingale property of $Y$ under $\mathbb{Q}$, we know that $Y(\rho) \leq Y(\tau)$ on some set $H \subset B$ with $\mathbb{Q}(H) > 0$ ($\mathbb{P}(H) > 0$). In addition,

$$Y(\rho) \leq Y(\tau) < m = w(t, x) \quad \text{on} \quad H.$$ 

This implies $\mathbb{Q}(Y(\rho) < m \mid B) > 0$ and $\mathbb{P}(Y(\rho) < m \mid B) > 0$. Therefore, $w(t, x) = m$ is a stochastic sub-solution.

Remark: the relative growth condition is similar to Assumption 2.1.

(3) Relax the continuity assumption for $\hat{u}$. 
Thank you!


