On the Market-Neutrality of Optimal Convergence Trading

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Layout

- Cointegration (background knowledge)
- Convergence trading (Motivation)
- Optimal investment with cointegrated assets (problem formulation and Maths)
- Market-Neutrality (result)
- Conclusion and references
Layout

- Cointegration
- Convergence trading
- Optimal investment with cointegrated assets
- Market-Neutrality
- Conclusion and references
Cointegration

- A property of multivariate time series


- Two or more stochastic processes are **cointegrated** if a linear combination of them is “mean reverting”

- Intuitively, two cointegrated processes are tied together and will never get too far from each other
Cointegration vs. Correlation

- Cointegration $\rightarrow$ long term equilibrium / common trend
  “A drunk man and his dog on leash”

- Correlation $\rightarrow$ co-movement
  “Two drunk men going the same way”

$(X_t)$ and $(Y_t)$ are cointegrated

$(X_t)$ and $(Y_t)$ are correlated
Formal Definition of Cointegration

Definition (Stationary process)

\((X_t)\) is stationary if \((X_{t_0}, \ldots, X_{t_n})\) and \((X_{t_0+\tau}, \ldots, X_{t_n+\tau})\) have the same law for any \(n \in \mathbb{N}\) and \(t_0, \ldots, t_n, \tau\).

Definition (Cointegration)

Non-stationary processes \((X_t^1), \ldots, (X_t^n)\) are cointegrated if \(\exists \beta \in \mathbb{R}^n\) and some function \(f(.)\) such that \(\beta \cdot X_t + f(t)\) is stationary

- \(\beta\) is a cointegrating vector and \(Z_t := \beta \cdot X_t + f(t)\) is a cointegrating relationship
- The (vector) space of all cointegrating vectors is cointegration space; its dimension is cointegration rank
Cointegration Analysis and Estimation

• Assume \((X_t, Y_t)\) is Vector Autoregressive (VAR)

\[
\begin{pmatrix}
X_t \\
Y_t
\end{pmatrix} = \sum_{i=1}^{l} \Pi_i \begin{pmatrix} X_{t-i} \\ Y_{t-i} \end{pmatrix} + \text{error}
\]

Theorem (The Granger representation Theorem)

\((X_t)\) and \((Y_t)\) are cointegrated if and only if there exists

\[
Z_t = Y_t - cX_t
\]

such that \((X_t, Y_t)\) satisfies the Error Correction Model

\[
\begin{pmatrix}
\Delta X_t \\
\Delta Y_t
\end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} Z_{t-1} + \sum_{i=1}^{l} \Gamma_i \begin{pmatrix} \Delta X_{t-i} \\ \Delta Y_{t-i} \end{pmatrix} + \text{error}
\]
Cointegration Analysis and Estimation

Engle-Granger two-step procedure

\[ Z_t = Y_t - cX_t \]

\[ \text{ECM: } \begin{pmatrix} \Delta X_t \\ \Delta Y_t \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} Z_{t-1} + \sum_{i=1}^{l} \Gamma_i \begin{pmatrix} \Delta X_{t-i} \\ \Delta Y_{t-i} \end{pmatrix} + \text{error} \]

- Suggestion:
  1. Regress \( Y \) on \( X \) to obtain \( c \) and \( Z_t \), i.e. \( Y_t = cX_t + Z_t \)
  2. Regress \( X \) and \( Y \) on \( Z \) to find ECM

▶ CTECM
Cointegration Analysis and Estimation

Engle-Granger two-step procedure, cont.

- The usual regression analysis cannot be used
- Let \((X_t)\) and \(Y_t\) be independent random walks

\[
X_t = X_{t-1} + \varepsilon_t, \quad Y_t = Y_{t-1} + \nu_t, \quad \varepsilon_t \perp \nu_t
\]

- Using a usual t-test with \(H_0: c = 0\) on the regression \(Y_t = -cX_t + Z_t\) is quite likely to show significance (i.e. conclude \(c \neq 0\))

Simulation shows a probability of 75%

- This phenomenon is known as the spurious regression problem
Cointegration Analysis and Estimation

Engle-Granger two-step procedure, cont.

- **Statistical test to separate:**
  - Spurious regressions: Apparently significant relationship between unrelated series
  - Genuine relationships when the time series are cointegrated

- **Fortunately, one may use** unit-root test, such as Augmented Dickey-Fuller (ADF) test

- **Engle-Granger two-step procedure:**
  1. Regress $Y$ on $X$ to obtain $c$ and $Z_t$, i.e. $Y_t = cX_t + Z_t$. Apply ADF test to $Z_t$ to determine cointegration
  2. If cointegration exist, regress $X$ and $Y$ on $Z$ to find $ECM$
Overview of related statistical theory

Cointegration Analysis and Estimation

Johansen procedure

There are well established statistical methods for structural
vector error correction models (SVECM):

\[
\Delta X_t = \Pi X_{t-1} + \sum_{i=1}^{p-1} \Xi_i \left( \Delta X_{t-i} / \Delta I_{t-i} \right) + \Xi_p D_t + \varepsilon_t
\]

- If \( \text{rank}(\Pi) = r \), then \( \Pi = \alpha \beta^\top \) for \( n \times r \) loading matrix \( \alpha \) and \( n \times r \) cointegrating matrix \( \beta \)

- The Johansen cointegration test is utilized to determine the rank
  of coefficient matrix \( \Pi \), which is the number of cointegrating
  relations

- \( \alpha \) and \( \beta \) are not unique (identification issue)
  one must choose a suitable normalization

- The mean-reverting signal for C.T. is \( (Z_t) := (\beta^\top F_t) \)

- See Johansen (1995) and Pesaran et al. (2000)
Empirical evidence

- There are numerous empirical studies in different fields that indicate cointegration relationships, c.f. Alexander (2006) and Juselius (2006)

- Well-documented examples in financial markets includes: interest rates, foreign exchange rates, stock price indices, stock prices, future and spot prices, and commodities

- Cointegration relations between specific asset prices exist and are usually not stable over extended periods
Empirical evidence of cointegration

Example

\[ z = \ln(\text{MSFT}) - 0.73 \ln(\text{IBM}) \]
Empirical evidence of cointegration

**Example**

![Graphs showing cointegration tests](image)

- **P_u Test**
  - Test statistic
  - 10% critical level

- **P_z Test**
  - Test statistic
  - 10% critical level

- **\( \hat{c} \)**
  - Trend
  - Values from 2005 to 2010
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Convergence Trading (C.T.)

- **Convergence trading** is perhaps the oldest form of statistical arbitrage strategies used by hedge funds
  - Tandem-Trading strategies used in early 20th century
  - At least 3 decades of history in Wall Street

Cf. Ehrman (2006) for historical insights

- The underlying principle
  1. Identifying assets that are affected by common economic forces, such that their prices have an equilibrium state
  2. Take positions when prices deviate from equilibrium, such that profit is made when the prices converge to the equilibrium
  3. Maintaining market neutrality by taking offsetting long/short positions, to hedge against other market movements
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Major themes in convergence trading literature

(1) Empirical studies on profitability of C.T.

(2) Theoretical studies on optimal C.T.

(3) Destabilizing effect of C.T. on the market

- The first extensive academic study:
  - Gatev, Goetzmann and Rouwenhorst (1999) and Gatev et al. (2006)
  - steady profitability of a simple pairs-trading rule in the US equity market over an extended period
Gatev, Goetzmann and Rouwenhorst (GGR) strategy

- A pool of equities (e.g. components of S&P500)

- The mean reverting signal is **assumed** to be the spread (price difference)

- Over any period of time, each pair of stocks is assigned a **distance measure** which is the sum of squared spread
Gatev, Goetzmann and Rouwenhorst (GGR) strategy

- **Pair-selection rule**: At the beginning of 6 months intervals, all possible pairs are ranked by their distance measure over the previous year, the top 5, 20, etc. are selected.

- **Trading rule**: A dollar-neutral position is opened when the spread is bigger than twice its historical standard deviation, the position is closed when the spread converges.
Gatev, Goetzmann and Rouwenhorst (GGR) strategy

Between 1963-2002, the strategy earns an average monthly excess return of 1%, twice as large as S&P500 with half to one third standard deviation, and positive skewness.
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Merton’s problem or optimal portfolio choice problem

- An agent invests in a riskless asset and n stocks with prices 
  \( (S_t) = (S_t^1, \ldots, S_t^n) \) over \([0, T]\)

- Stock prices satisfy:
  \[
  S_t = S_0 + \int_0^t S_r \mu(r, S_r) dr + \int_0^t S_r \sigma(r, S_r) dW_r
  \]

- Agent’s trading strategy \((\pi_t) = (\pi^1, \ldots, \pi^n)\), portfolio 
  weights of stocks

- \((X^\pi_t)\), agent’s wealth following a strategy \((\pi_t)\), solves
  \[
  X^\pi_t = x + \int_0^t X^\pi_r \pi^\top_r \mu(r, S_r) dr + \int_0^t X^\pi_r \pi^\top_r \sigma(r, S_r) dW_r
  \]
Merton’s problem or optimal portfolio choice problem

- $\mathcal{A}$ is the set of admissible strategies such that if $\pi \in \mathcal{A}$ then $(X_t^\pi)$ is defined and $X_t^\pi \geq 0$

- The Merton investment problem

$$(\pi^*_t)_{t \in [0,T]} := \arg \max \left\{ \mathbb{E} \left( U \left( X_T^\pi \right) \right) : \pi \in \mathcal{A} \right\}$$

- $U(.)$ is a utility function, i.e.

$U'(\cdot) > 0, U''(\cdot) < 0, U'(+\infty) = 0, U'(0^+) = +\infty$

logarithmic utility: $U(x) = \log x$

power utility: $U(x) = \frac{x^{1-\gamma}-1}{1-\gamma}, \gamma \in (0, +\infty)/\{1\}$
Theoretical studies related to optimal C.T.

- Literature on mean-reverting risk premia, including Kim and Omberg (1996), Wachter (2002), Liu (2007), and Jurek and Viceira (2011): Similar calculations, but, not necessarily model cointegrating prices / C.T. setting

- Assuming one mean-reverting asset (i.e. imposing market-neutrality, to be explained): Xiong (2001), Liu and Longstaff (2004), Jurek and Yang (2007), and Mudchanatongsuk et al. (2008)

Market setting

- A riskless asset with zero short rate
- Two stocks:

\[
\frac{dS_t^1}{S_t^1} = \alpha_1 Z_t dt + \sigma_1 dW_t^1
\]

\[
\frac{dS_t^2}{S_t^2} = \alpha_2 Z_t dt + \sigma_2 \rho dW_t^1 + \sigma_2 \sqrt{1 - \rho^2} dW_t^2
\]

where the log-spread process is:

\[
Z_t := \log S_t^1 - c \log S_t^2 + \frac{1}{2} \left( \sigma_1^2 - c \sigma_2^2 \right) t
\]

- The model is a continuous-time error correction model (CTECM)
Market setting – Assumptions

Assumption

(i) \( \sigma_1, \sigma_2 > 0 \) and \( |\rho| < 1 \)
(ii) \( \alpha_1 < c\alpha_2 \)
(iii) \( Z_0 \) is a Gaussian random variable with mean zero and variance

\[
\frac{\sigma_1^2 + c^2\sigma_2^2 - 2c\rho\sigma_1\sigma_2}{2(c\alpha_2 - \alpha_1)},
\]

and it is independent of \((W_t)_{t \geq 0}\).

- By Assumption (i), the market price of risk \((Z_t\lambda)\) and the state price density \(dY_t = -Y_tZ_t\lambda \cdot dW_t\) exist, where \(\lambda := \Sigma^{-1}\alpha\), and \(\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sigma_2\sqrt{1-\rho^2} \end{pmatrix}\)
Market setting – Cointegration

Proposition

1. \((Z_t)_{t \geq 0}\) satisfies:

\[ dZ_t = -\kappa Z_t dt + \sigma_Z dW_t^Z \]

where

\[ \kappa := c\alpha_2 - \alpha_1, \quad \sigma_Z^2 := \sigma_1^2 + c^2\sigma_2^2 - 2c\rho\sigma_1\sigma_2 \]

\[ W_t^Z := \frac{1}{\sigma_Z} \left\{ (\sigma_1 - c\sigma_2\rho) W_t^1 - c\sigma_2 \sqrt{1 - \rho^2} W_t^2 \right\} \]

2. By assumptions(ii) and (iii), \((Z_t)_{t \geq 0}\) is a stationary Gaussian process with

\[ \mathbb{E}(Z_t) = 0 \quad \text{and} \quad \mathbb{E}(Z_t Z_s) = \frac{\sigma_Z^2}{2\kappa} e^{-\kappa|t-s|}, \quad t, s \geq 0 \]
Optimal strategies for CRRA investors

- An agent invests in the riskless asset and stocks $S^1$ and $S^2$ over $[0, T]$

- $\mathcal{A}$ is the set of admissible strategies, i.e adapted processes $(\pi_t^\top) = (\pi_t^1, \pi_t^2)$ (portfolio weights) such that
  \[
  \int_0^T \left( |\pi_t^\top \alpha Z_t| + \pi_t^\top \Sigma \Sigma^\top \pi_t \right) dt < \infty
  \]

- Agent’s wealth
  \[
  X^\pi = x \exp \left( \int_0^t (\pi_r^\top \alpha Z_r - \frac{1}{2} \|\pi_r^\top \Sigma\|^2) dr + \int_0^t \pi_r^\top \Sigma dW_r \right) > 0
  \]

- The Merton problem
  \[
  (\pi_{\gamma,t}^*)_{t \in [0,T]} := \arg\max_{\pi \in \mathcal{A}} \mathbb{E} (U_\gamma (X_T^\pi))
  \]
  \[
  U_\gamma(x) = \frac{x^{1-\gamma} - 1}{1-\gamma}, \quad \gamma \in (0, +\infty) / \{1\} \text{ and } U_1(x) = \log x
  \]
Logarithmic Utility

- $\gamma = 1$

$\left( \pi_{1,t}^* \right)_{t \in [0,T]} := \arg\max_{\pi \in A} \mathbb{E} \left( \log \left( X_T^\pi \right) \right)$

- Value function:

$u(t, x, z; T, 1) := \sup_{\pi \in A} \mathbb{E} \left( \log \left( X_T^{\pi,x,z,t} \right) \right)$

Theorem

1. $\pi_{1,t}^* = (\Sigma \Sigma^\top)^{-1} \alpha Z_t = \frac{1}{\sigma_1 \sigma_2 (1-\rho^2)} \left( \frac{\alpha_1 \sigma_2}{\sigma_1} - \rho \alpha_2 \right) \left( \frac{\alpha_2 \sigma_1}{\sigma_2} - \rho \alpha_1 \right) Z_t$

2. $u(t, x, z; T, 1) = \log x + \frac{1}{2} \|\lambda\|^2 \mathbb{E} \left( \int_t^T (Z_s^{z,t})^2 ds \right) < \infty$
Power Utility

- $\gamma \in (0, 1) \cup (1, +\infty)$

\[
(\pi^*_\gamma,t)_{t\in[0,T]} := \arg \max_{\pi \in A} \mathbb{E} \left( \frac{(X_T^\pi)^{1-\gamma} - 1}{1 - \gamma} \right)
\]

- Value function:

\[
u(t, x, z; T, \gamma) := \sup_{\pi \in A} \mathbb{E} \left( \frac{(X_T^{\pi,x,z,t})^{1-\gamma} - 1}{1 - \gamma} \right)\]
Optimal strategies for CRRA investors

Power Utility, cont.

- Critical relative risk aversion
  \[ \gamma_0 := 1 - \left( \frac{\kappa}{\sigma_Z \| \lambda \|} \right)^2 = 1 - \left( \frac{(1, -c)\alpha}{\|(1, -c)\Sigma\| \|\Sigma^{-1}\alpha\|} \right)^2 \]
  \[ \gamma_0 \in [0, 1): \]
  \[ \kappa := -(1, -c)\alpha \leq \|(1, -c)\Sigma\| \|\Sigma^{-1}\alpha\| =: \sigma_Z \| \lambda \| \]

- The following dichotomy holds:

1. For \( \gamma \geq \gamma_0 \), the Merton problem is well-posed:
   \[ u(t, x, z; T, \gamma) < \infty \text{ for all } T \]

2. For \( \gamma < \gamma_0 \), the Merton problem is ill-posed
   Nirvana strategies exist, i.e.
   \[ \lim_{T \uparrow T_{esc}(\gamma)} u(0, x, z; T, \gamma) = +\infty \]
   for some \( T_{esc}(\gamma) > 0 \)
Optimal strategies for CRRA investors

Power Utility, cont.

Theorem (well-posed case, $\gamma \geq \gamma_0$)

1. $\pi_{\gamma,t}^* = \frac{1}{\gamma} \pi_{1,t}^* + h_{w.p.}(T - t, \gamma) Z_t \begin{pmatrix} 1 \\ -c \end{pmatrix}$

2. $u(t, x, z; T, \gamma) = \frac{x^{1-\gamma} \left(e^{g_{w.p.}(T-t, \gamma)} + \frac{1}{2} h_{w.p.}(T-t, \gamma) z^2 \right)^\gamma - 1}{1-\gamma} < \infty$

Theorem (ill-posed case, $\gamma < \gamma_0$)

$T < T_{esc}(\gamma) := \frac{\gamma}{\sigma_Z \| \lambda \| \sqrt{\gamma_0 - \gamma}} \left( \frac{\pi}{2} + \arctan \left( \frac{\kappa \gamma}{\sigma_Z \| \lambda \| \sqrt{\gamma_0 - \gamma}} \right) \right)$

1. $h_{w.p.} \to h_{i.p.}$ and $g_{w.p.} \to g_{i.p.}$

2. $\lim_{T \uparrow T_{esc}(\gamma)} h_{i.p.}(T, \gamma) = \lim_{T \uparrow T_{esc}(\gamma)} g_{i.p.}(T, \gamma) = +\infty$
Comparison with Existing Results

1. Mean-reverting risk premia, Kim and Omberg (1996), etc.
   - Similar calculations, but, specialized for cointegrated assets and convergence trading
   - Closed forms for nirvana strategies and the associated escape time
   - Clearer interpretation and possibility of statistical test for existence of nirvana

2. Studies that assume one mean-reverting asset, i.e. assuming market-neutrality, e.g. Jurek and Yang (2007)
   - By not assuming market-neutrality per se, it is possible to investigate its relevance / implications

3. Liu and Timmermann (2013) and Chiu and Wong (2012c)
   - The same general forms for the optimal strategy (which can be traced back to Kim and Omberg (1996))
   - They did not recognize the ill-posed case, nor did they highlighted the role of $\gamma_0$
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An auxiliary second order linear PDE

- Aux. PDE for the bivariate CRRA case:

\[ \varphi_t + z(a \cdot b)\varphi_z + \frac{1}{2}\|b\|^2\varphi_{zz} + \frac{\xi}{2}z^2\|a\|^2\varphi = 0 \]

- \( \varphi(t, z) = e^{g(T-t)+\frac{1}{2}z^2h(T-t)} \) yields the scalar Riccati differential equation

\[ h'(t) = 2(a \cdot b)h(t) + \|b\|^2h^2(t) + \xi\|a\|^2 \]

which can be solved in closed form

Lemma (1.13, 1.14)

Let \( \mathcal{D} := (a \cdot b)^2 - \xi\|a\|^2\|b\|^2 \). Then:

1. If \( \mathcal{D} \geq 0 \) RDE is well-posed and stable (i.e. \( h(\infty) < \infty \))
2. If \( \mathcal{D} < 0 \), RDE has a escape time
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Pairs-trading in investment practice

1. Find appropriate $S^1$ and $S^2$, identify the “mean-reverting signal” $Z_t := \log S^1_t - c \log S^2_t$

2. Assume a model for $Z_t$

$$dZ_t = -\kappa Z_t dt + \sigma_Z dW^Z_t$$

3. Only consider market-neutral (M-N) strategies

$$\pi^2_t = -c \pi^1_t \quad \text{or} \quad \pi_t = \begin{pmatrix} \pi^1_t \\ \pi^2_t \end{pmatrix} = \begin{pmatrix} 1 \\ -c \end{pmatrix} \xi_t$$

$\pi^i$ is the portfolio weight of stock $i$ and $\xi_t$ is some (scalar) process

4. Choose a market-neutral strategy based on some criterion (mean-variance etc.) and other considerations
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Pairs-trading in investment practice, cont.

- **Q1:** Why ignoring the original prices $S^1$ and $S^2$?
  
  **A:** They are non-stationary, hard to calibrate
  
  $Z$ is stationary by design

- **Q2:** How is it possible to do portfolio optimization with only $Z$? Why imposing M-N?
  
  **A:** P/L of M-N strategies only depend on the change in $Z = \ln S^1 - c \ln S^2$

$$
\frac{\Delta X}{X} = \pi^1 \frac{\Delta S^1}{S^1} + \pi^2 \frac{\Delta S^2}{S^2} \approx \pi^1 \Delta \ln S^1 + \pi^2 \Delta \ln S^2 \\
= \pi^1 \Delta \ln S^1 + \pi^2 \Delta \ln S^2 - c \pi^1 \Delta \ln S^2 + c \pi^1 \Delta \ln S^2 \\
= \pi^1 (\Delta \ln S^1 - c \Delta \ln S^2) + (\pi^2 + c \pi^1) \Delta \ln S^2 \\
= \pi^1 \Delta Z
$$
Pairs-trading in investment practice, cont.

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  $$= \pi^1 \Delta \ln S^1 + \pi^2 \Delta \ln S^2 - c \pi^1 \Delta \ln S^2 + c \pi^1 \Delta \ln S^2$$

  $$= \pi^1 (\Delta \ln S^1 - c \Delta \ln S^2) + (\pi^2 + c \pi^1) \Delta \ln S^2$$

  $$= \pi^1 \Delta Z$$
Pairs-trading in investment practice, cont.

- Q3: Is this approach “optimal”? Can one “benefit” from using non M-N pairs trading strategies (at least in theory)?

Q3’: Is there a theoretical justification for M-N pairs trading?

- These questions are widely ignored in the literature

- The only study is Liu and Timmermann (2013)
  - Two stocks following a CTECM, and a CRRA agent
  - The optimal strategy is not M-N
Pairs-trading in investment practice, cont.

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Market-Neutrality for Logarithmic Utility

- $\pi_{1,t}^*$ is not M-N! $\rightarrow \pi_{1,t}^* \neq -c\pi_{1,t}^*$

- Liu and Timmermann (2013) pointed out that $\pi_{1,t}^*$ is M-N for uncorrelated Siamese twin, i.e. when $c = 1$, $\sigma_1 = \sigma_2$, and $\rho = 0$
- They then proceed to calculate the disutility of imposing M-N if $\rho \neq 0$ (but $c = 1$, $\sigma_1 = \sigma_2$)

Our approach is different:
- There are other cases where $\pi_{1,t}^*$ is M-N
- $\pi_{1,t}^* \neq -c\pi_{1,t}^*$ $\iff \pi_{1,t}^* \propto (\frac{1}{1-c}) \iff \alpha \propto \Sigma \Sigma^\top \left( \frac{1}{-c} \right)$

$\iff \alpha = \frac{-\kappa}{\sigma^2_z} \Sigma \Sigma^\top \left( \frac{1}{-c} \right)$
Market-Neutrality for Logarithmic Utility

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    $\iff$ $\alpha = \frac{-\kappa}{\sigma^2_z} \Sigma\Sigma^\top \left( \frac{1}{-c} \right)$
Market-Neutrality for Logarithmic Utility

Condition (M-N)

(i) \( \alpha_1/\alpha_2 = (\sigma_1^2 - c\sigma_1\sigma_2\rho)/(\sigma_1\sigma_2\rho - c\sigma_2^2) \), or

(ii) There exists \( \xi \in \mathbb{R} \) such that \( \alpha = \Sigma\Sigma^\top (1, -c)^\top \xi \), or

(iii) \( \alpha = \Sigma\Sigma^\top (1, -c)^\top (-\kappa/\sigma_Z^2) \).

- We have shown: \( \pi_{1,t}^* \) is M-N \( \iff \) Condition M-N

- Is there any significance in Condition M-N? Does it play any role in the CTECM market setting?
Market-Neutrality for Power Utility

\[ \pi^*_\gamma,t = \frac{1}{\gamma} \pi^*_1,t + h_{w.p./i.p.}(T - t, \gamma) Z_t \left( \frac{1}{-c} \right) \]

- \( \pi^*_\gamma,t \) is not M-N only because \( \pi^*_1,t \) is not M-N, i.e. if \( \pi^*_1,t \) is M-N, then so is \( \pi^*_\gamma,t \)

- Hence: \( \pi^*_\gamma,t \) is M-N \( \iff \) Condition M-N
Well posedness condition

- In which “economy”, no agent achieves nirvana?

Is there a market condition for absence of nirvana strategies?

- The Merton problem is well-posed for all $\gamma \iff 
\gamma_0 := 1 - \left( \frac{\kappa}{\sigma_Z \|\lambda\|} \right)^2 = 0 \iff \kappa = \sigma_Z \|\lambda\|

- $\kappa := -(1, -c)\alpha \leq \|(1, -c)\Sigma\| \|\Sigma^{-1}\alpha\| =: \sigma_Z \|\lambda\|

By C-S, the equality holds iff

$$
\Sigma^{-1}\alpha \propto \Sigma^T \begin{pmatrix} 1 \\ -c \end{pmatrix} \iff \text{Condition M-N!}
$$

- There is no nirvana strategy if and only if Condition M-N holds
No arbitrage condition

- **Absence of arbitrage** is a fundamental consideration in market models, yet, there is no study on no-arbitrage conditions for CTECM market setting.

- A popular sufficient condition is the Novikov condition:

  \[ \mathbb{E} \left[ \exp \left( \frac{1}{2} \int_0^T \| \lambda \|^2 Z_s^2 ds \right) \right] < \infty \]

- By F-K, \( \varphi(t, z) := \mathbb{E} \left[ \exp \left( \frac{1}{2} \int_t^T \| \lambda \|^2 (Z_{s,t}^z)^2 ds \right) \right] \) satisfies

  \[ \varphi_t - \kappa z \varphi_z + \frac{1}{2} \sigma_Z^2 \varphi_{zz} + \frac{1}{2} z^2 \| \lambda \|^2 \varphi = 0, \quad \varphi(T, z) = 1 \]

- This PDE has a solution for all \( T \) iff

  \[ \kappa^2 - \sigma_Z^2 \| \lambda \|^2 \geq 0 \iff \text{Condition M-N!} \]
Main result

Condition (M-N)

(i) \( \frac{\alpha_1}{\alpha_2} = \frac{\sigma_1^2 - c\sigma_1\sigma_2\rho}{(\sigma_1\sigma_2\rho - c\sigma_2^2)}, \) or
(ii) There exists \( \xi \in \mathbb{R} \) such that \( \alpha = \Sigma \Sigma^\top (1, -c)^\top \xi, \) or
(iii) \( \alpha = \Sigma \Sigma^\top (1, -c)^\top (-\kappa/\sigma_Z^2). \)

Theorem (M-N)

The following statements are equivalent:

(i) Condition M-N
(ii) The Novikov condition holds for all \( T > 0 \)
(iii) The Merton problem is well-posed for all \( \gamma > 0, \) there is no nirvana strategy
(iv) The optimal strategy is market-neutral
General Utility Functions

The Merton problem with a general utility function:

\[ (\pi_t^*)_{t\in[0,T]} := \arg\max_{\pi \in \mathcal{A}} \mathbb{E} (U (X^\pi_T)) \]

with \( U(.) \) satisfying the usual conditions and

\[ \text{A.E.} := \limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1. \]

Theorem (1.15)

The following statements are equivalent:

(i) Condition M-N

(ii) For any \( T > 0 \) and any utility function \( U(.) \) (satisfying the assumptions), the Merton problem is well-posed.

(iii) For any \( T > 0 \) and any utility function \( U(.) \) (satisfying the assumptions), the optimal strategy is market-neutral.
Layout

- Cointegration
- Convergence trading
- Optimal investment with cointegrated assets
- Market-Neutrality
- Conclusion and references
Future research

1. Considering transaction costs (numerical)
2. Connection to statistical arbitrage
3. Extending the model (currently working on regime-switching extension)
4. Controlling drawdowns
5. Statistical test for well-posedness condition
6. Empirical study on market-neutrality
7. Equilibrium models and investigating destabilizing effect of C.T.
Thanks for your attention!


