Long Forward Probabilities, Recovery and the Term Structure of Bond Risk Premiums

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Supported in part by NSF grants CMMI 1536503 and DMS 1514698.
Based on:

Long Term Factorization

- Stochastic discount factor (pricing kernel) $S$ assigns prices to risky future payoffs:

$$\mathbb{P}_{t,T}(Y) = \mathbb{E}^P \left[ \frac{S_T Y}{S_t} \mid \mathcal{F}_t \right].$$

- Long-term factorization of PK:

$$S_t = \frac{1}{B_t} M_t,$$

where $B_t$ is the long bond so that $1/B_t$ discounts at the rate of return on the long bond and $M_t$ is a martingale.

- Alvarez and Jermann (Econometrica, 2005) first introduced this factorization in a discrete-time ergodic setting.

- Hansen and Scheinkman, *Long Term Risk: An Operator Approach*, Econometrica 2009, gave a study in continuous-time ergodic Markovian environments and identified the factors in terms of the principal eigenfunction and eigenvalue of the Markovian pricing operator.
Ross’ question: can we uniquely recover *physical* probabilities $\mathbb{P}$ from currently observed asset prices?

Ross’ Recovery Theorem (J of Finance, 2015):
- All uncertainty is generated by a finite-state, discrete time irreducible Markov chain
- Transition-independent pricing kernel
- Then there is a unique recovery of transition probabilities from Arrow-Debreu prices (via Perron-Frobenius Theorem)

Carr and Yu (2012)
- 1D diffusions on bounded intervals with regular boundaries

Walden (2013)
- 1D diffusions on $\mathbb{R}$

Q & L (2014, PE)
- Detailed analysis for general Markov processes (Borel right processes)
Connection between Long Term Factorization and Ross’ Recovery

- Hansen and Scheinkman (2014), Borovicka, Hansen and Scheinkman (Misspecified Recovery, 2014, to appear in J of Finance), Q & L (2014, LT and PE) for general continuous-time Markov models (also closely related results for discrete-time, finite state Markov chains in Martin and Ross (2013)):
  
  ▶ Connect Ross’ recovery to the long-term factorization of the pricing kernel
  ▶ Identify Ross’ transition independence assumption with setting the martingale component in the long-term factorization to unity:

\[
M_t = 1, \quad S_t = \frac{1}{B_t}.
\]
Q & L (2014) Long Term Risk: A Martingale Approach

- General (non-Markovian) **semimartingale environment**
- Give a sufficient condition for convergence in semimartingale topology of trading strategies investing in zero-coupon bonds and rolling over to the **long bond**
- Show convergence in total variation of $T$-forward measures $Q^T$ to the **long forward measure** $L = Q^\infty$
- Obtain long-term factorization of HJM models
- Restricting to ergodic Markovian environments, recover Hansen & Scheinkman factorization in terms of the principal eigenvalue and eigenfunction of the Markovian pricing operator
- Show that Ross’ recovery identifies $P = L$ and implies **growth optimality of the long bond**
- Start with \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) with the usual hypothesis.
- A semimartingale pricing kernel \((S_t)_{t \geq 0}\) is assumed to satisfy:
  - Strict positivity: \(S\) and \(S_-\) are strictly positive,
  - Normalization: \(S_0 = 1\),
  - Integrability: \(\mathbb{E}^\mathbb{P}\left[\frac{S_T}{S_t}\right] < \infty\) for all \(T \geq t \geq 0\).
- Price at time \(t\) of a payoff \(Y \in \mathcal{F}_T\) at time \(T > t\):

  \[
  \mathbb{P}_{t,T}(Y) = \mathbb{E}^\mathbb{P}\left[\frac{S_T Y}{S_t} \mid \mathcal{F}_t\right].
  \]
Bonds and Forward Measures $Q^T$

- Zero-coupon bonds:
  
  $$P^T_t := \mathbb{E}^P \left[ \frac{S_T}{S_t} \mid \mathcal{F}_t \right].$$

- For each $T > 0$, a roll-over strategy $B^T_t$ invests $1$ at time zero in $P^T_0$, at time $T$ rolls over into $P^T_{2T}$ for $[T, 2T]$, etc.:
  
  $$B^T_t = \frac{P^{(k+1)T}_t}{\prod_{i=0}^k P^{(i+1)T}_{iT}}, \quad t \in [kT, (k+1)T).$$

- $\mathbb{P}$-Martingales and $T$-forward factorizations:
  
  $$M^T_t := S_t B^T_t, \quad S_t = \frac{1}{B^T_t} M^T_t, t \geq 0.$$

- For each $T > 0$, the $T$-forward measure $Q^T$ (Jarrow 1987, Geman 1989, Jamshidian 1989):
  
  $$Q^T \mid \mathcal{F}_t = M^T_t \mathbb{P} \mid \mathcal{F}_t, \quad t \geq 0.$$
The Short-term Limit: Risk-Neutral Measure $\mathbb{Q} = \mathbb{Q}^0$

- If $S$ is a special semimartingale, due to our strict positivity assumption there is a multiplicative decomposition:

$$S_t = e^{-D_t} M_t,$$

$M$ is a local martingale and $D$ is predictable.

- If $M$ is a martingale, then $e^{D_t}$ can be interpreted as the risk-free asset ("implied savings account" D"oberlein and Schweizer (2001)), and $M$ defines the RN measure:

$$\mathbb{Q}_\mathbb{F}_t = M_t \mathbb{P}_\mathbb{F}_t, \quad t \geq 0.$$

- Under technical conditions, D"oberlein and Schweizer (2001) prove that $e^{D_t}$ coincides with the limit $B_t^0$ of roll-over strategies $B^T_t$ as $T \downarrow 0$ ("classical savings account").

- Further, when $S$ is a supermartingale, $D$ is non-decreasing.
Theorem (Q& L (2014, LT))

Assume that for each $t > 0$ there exists a positive random variable $M_t^\infty > 0$ such that

$$\lim_{T \to \infty} \mathbb{E}^P[|M_t^T - M_t^\infty|] = 0.$$ 

(i) Positive $\mathbb{P}$-martingales $(M_t^T)_{t \geq 0}$ converge to a positive $\mathbb{P}$-martingale $(M_t^\infty)_{t \geq 0}$ in Emery’s semimartingale topology.

(ii) Positive semimartingales $(B_t^T)_{t \geq 0}$ converge to a positive semimartingale $(B_t^\infty)_{t \geq 0}$ in semimartingale topology.

(iii) Forward measures $Q^T$ converge in total variation to an equivalent measure $Q^\infty$ on each $\mathcal{F}_t$, and

$$Q^\infty|_{\mathcal{F}_t} = M_t^\infty P|_{\mathcal{F}_t}, \quad t \geq 0.$$
We call \( Q^\infty \) the *long forward measure* and denote it \( \mathbb{L} \).

We call \((B_t^\infty)_{t \geq 0}\) the *long bond*.

Long-term factorization of the semimartingale pricing kernel

\[
S_t = \frac{1}{B_t^\infty} M_t^\infty
\]

into discounting at the rate of return on the long bond and a martingale component.

Extends Alvarez and Jermann (2005) and Hansen and Scheinkman (2009) to general semimartingale environments without Markovian assumption.

Under \( \mathbb{L} \), the long bond \( B_t^\infty \) is the numeraire asset.

The long term factorization is a general semimartingale phenomenon, *not* an artifact of Markovian models.
Theorem (Q & L (2014, LT))

(i) The long bond is growth optimal under \( \mathbb{L} \), that is, it has the highest expected log return under \( \mathbb{L} \) among all assets priced by the pricing kernel \( S \).

(ii) The Sharpe ratio of any asset priced by the PK \( S \) takes the form under \( \mathbb{L} \):

\[
\frac{\mathbb{E}^\mathbb{L}_t \left[ R^V_{t,t+\tau} \right] - R^f_{t,t+\tau}}{\sigma^\mathbb{L}_t \left( R^V_{t,t+\tau} \right)} = -\text{corr}^\mathbb{L}_t \left( R^V_{t,t+\tau}, \frac{1}{R^\infty_{t,t+\tau}} \right) R^f_{t,t+\tau} \sigma^\mathbb{L}_t \left( \frac{1}{R^\infty_{t,t+\tau}} \right)
\]

where \( R^V_{t,t+\tau}, R^f_{t,t+\tau} \) and \( R^\infty_{t,t+\tau} \) is the return from holding asset \( V \), risk-free zero-coupon-bond and long bond from \( t \) to \( t + \tau \).
Proposition (Q & L (2014, LT))

(i) Under diffusion setting,

\[ \lim_{\tau \downarrow 0} \text{corr}_{t}^{\text{L}} \left( R_{t,t+\tau}^{\infty}, 1/R_{t,t+\tau}^{\infty} \right) = -1. \]

(ii) If furthermore the risk free asset exists,

\[ \lim_{\tau \downarrow 0} \text{corr}_{t}^{\text{L}} \left( R_{t,t+\tau}^{0}, 1/R_{t,t+\tau}^{\infty} \right) = 0, \]

where \( R_{t,t+\tau}^{0} = B_{t+\tau}^{0}/B_{t}^{0} \) is the return on the risk free asset from \( t \) to \( t + \tau \).
A Quartet of Measures

\[ Q^T \]

\[ Q = Q^0 \leftrightarrow L = Q^\infty \]

[Diagram showing relationships between measures]
Markovian Pricing Kernels

- $X$ is a conservative Borel right process (Borel topology on the state space, strong Markov, right continuous paths). $(\mathcal{F}_t)_{t \geq 0}$ is generated by $X$.

- Pricing kernel $S$ is a positive semimartingale multiplicative functional of $X$:

\[
S_{t+s}(\omega) = S_t(\omega)S_s(\theta_t(\omega)),
\]

where $\theta_s$ is the shift operator, $\theta_s : \Omega \to \Omega$,

\[
X_s(\theta_t(\omega)) = X_{t+s}(\omega).
\]

- Pricing operators $(\mathcal{P}_t)_{t \geq 0}$:

\[
(\mathcal{P}_tf)(x) = \mathbb{E}^P_x[S_tf(X_t)]
\]

for any Borel payoff $f(x)$ for which expectation is well defined.
Positive Eigenfunctions and Eigen-Measures

- Suppose $\mathcal{P}_t$ has an eigenfunction $0 < \pi(x) < \infty$:

  $$\mathcal{P}_t \pi(x) = e^{-\lambda t} \pi(x)$$

  for each $t > 0$, $x \in E$, and some $\lambda \in \mathbb{R}$.

- The process

  $$M^\pi_t = S_t e^{\lambda t} \frac{\pi(X_t)}{\pi(X_0)}$$

  is a positive $\mathbb{P}$-martingale, and PK admits an eigen-factorization (Hansen-Scheinkman (2009)):

  $$S_t = e^{-\lambda t} \frac{\pi(X_0)}{\pi(X_t)} M^\pi_t.$$ 

- We can define an eigen-measure $Q^\pi$,

  $$Q^\pi_{|\mathcal{F}_t} := M^\pi_t \mathbb{P}_{|\mathcal{F}_t}, \quad (\mathcal{P}_t f)(x) = e^{-\lambda t} \pi(x) \mathbb{E}^Q_x \left[ \frac{f(X_t)}{\pi(X_t)} \right].$$
Suppose the Markovian pricing kernel satisfies the sufficient condition for long term factorization under $P_x$ for each initial state $x \in E$. Then, under some regularity condition, the long bond is identified with a positive multiplicative functional of $X$ in the transition independent form:

$$B_t^\infty = e^{\lambda_L t} \frac{\pi_L(X_t)}{\pi_L(x)},$$

where $\pi_L(x)$ is a positive eigenfunction of the pricing operators $(P_t)_{t \geq 0}$ with the eigenvalues $e^{-\lambda_L t}$ for some $\lambda_L \in \mathbb{R}$. The long forward measure $\mathbb{L}$ is identified with the corresponding eigen-measure $\mathbb{Q}^{\pi_L}$. 
Theorem (Q & L (2014, PE))

There is at most one positive eigenfunction $\pi_R$ such that $X$ is recurrent under the corresponding eigen-measure $Q^\pi_R$.

- Proof is essentially based on the fact that for a recurrent Markov process excessive functions are constant.
- Q & L (2014, PE) give several sets of sufficient conditions for existence
- and explicitly verify existence in many financial models (incl. affine, quadratic).
Ergodicity Identifies $\mathbb{Q}^{\pi_R} = \mathbb{L}$

**Theorem (Q & L (2014, LT))**

Assume $X$ has a stationary distribution $\mu$ under $\mathbb{Q}^{\pi_R}$ and there exist $c > 0$, $\alpha > 0$ and $T_0 > 0$ s.t. for each $T \geq T_0$

$$|\mathbb{E}_X^{\mathbb{Q}^{\pi_R}}[f(X_T)] - \mathbb{E}^{\mu}[f(X)]| \leq \frac{c}{\pi_R(x)} e^{-\alpha T}$$

for each $f$ s.t. $|f(x)| \leq \frac{1}{\pi_R(x)}$. Then $(B_t^T)_{t \geq 0}$ converge in semimartingale topology to the positive semimartingale

$$B_t^\infty = e^{\lambda t} \frac{\pi_R(X_t)}{\pi_R(X_0)},$$

$M_t^{\pi_R} = M_t^\infty$, $\mathbb{Q}^{\pi_R} = \mathbb{L}$, and, assuming $\int_E (1/\pi_R) d\mu < \infty$,

$$P_t^T = C e^{-\lambda(T-t)} \pi_R(X_t) + O(e^{-(\lambda+\alpha)(T-t)})$$

$$C = \mathbb{E}^{\mu}[1/\pi_R(X)].$$
Ross’ Recovery under Recurrence: $\mathbb{P} = \mathbb{Q}^{\pi R}$

- Ross’ assumption of transition independence of PK:
  \[ S_t = e^{-\lambda t} \frac{h(X_t)}{h(X_0)} \]
  for some positive $h$.
- $\frac{1}{h}$ is a positive eigenfunction, and PK has the factorization
  \[ S_t = e^{-\lambda t} \frac{\pi(X_0)}{\pi(X_t)} M_t^\pi, \]
  where $\pi = \frac{1}{h}$ and $M_t^\pi = 1$.
- $\mathbb{P} = \mathbb{Q}^\pi$ is an eigen-measure.
- If we further assume $X$ is recurrent under $\mathbb{P}$, then
  \[ \mathbb{P} = \mathbb{Q}^{\pi R}. \]
Combining Ross’ transition independence assumption with recurrence yields a unique recovery

\[ \mathbb{P} = \mathbb{Q}^{\pi R} \].

Strengthening to ergodicity we arrive at the further identification

\[ \mathbb{P} = \mathbb{Q}^{\pi R} = \mathbb{L} \].
Ross’ assumption can be extended to non-Markovian semimartingale environments by directly assuming that

\[ M_t^\infty = 1. \]

This leads to

\[ S_t = \frac{1}{B_t^\infty} \]

and identification

\[ \mathbb{P} = \mathbb{L}. \]

Under this assumption, the long bond is the numeraire asset under \( \mathbb{P} \) and is growth optimal by Jensen’s inequality.
Empirical Exploration


Figure: Bootstrapped zero-coupon yield curves
The zero interest rate regime Dec 2008-2015 is a challenge to conventional interest rate models.

- Affine models cannot deal with ZLB.
- Gaussian models admit negative rates, while CIR-type have vanishing volatilities for bonds of all maturities at the ZLB.
- **Shadow rate** idea due to F. Black (J. of Finance, 1995) “Interest Rates as Options”: nominal short rate is a positive part (due to the option to convert to currency) of a shadow rate that can get negative.


- Adopted by the Bank of Japan in 2005 (Baba et al. 2005).

- Kim and Singleton (J of Econometrics, 2012): extended and estimated 2-factor shadow rate models on JGB data.

- Shadow models in use by central banks post-crisis.

- We are working on a more general class based on SDEs with **sticky boundaries** that nest shadow rate models.
Estimate the same specification for 2-factor shadow rate quadratic Gaussian model as Kim and Singleton.

- The state variable $X$ is a 2D Gaussian diffusion under $\mathbb{P}$.
- The market price of Brownian risk is affine in $X$, so that $X$ remains 2D Gaussian diffusion under $\mathbb{Q}$.
- The short rate is a positive part of the shifted quadratic function in $X$.

Estimation: extended Kalman filter, with the bond pricing PDE solved via ADI finite difference scheme, and KNITRO non-linear optimizer.
Estimation Results: $P$ and $Q$ Dynamics

- Short rate
  \[ r(X_t) = (-0.0046 + 0.27X_{1,t}^2 + 0.18X_{1,t}X_{2,t} + 0.05X_{2,t}^2)^+. \]

- State Vector SDEs:
  \[
  dX_t = \begin{bmatrix} 0.65 & 0 \\ 0.22 & 0.04 \end{bmatrix} \left( \begin{bmatrix} -0.05 \\ 0.77 \end{bmatrix} - X_t \right) dt + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} dB_t^P.
  \]
  \[
  dX_t = \begin{bmatrix} 0.32 & 0.04 \\ 0.64 & 0.08 \end{bmatrix} \left( \begin{bmatrix} 0.93 \\ -5.92 \end{bmatrix} - X_t \right) dt + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} dB_t^Q.
  \]

- Market price of risk
  \[
  \lambda^P(X_t) = \begin{bmatrix} -0.89 \\ -0.96 \end{bmatrix} + \begin{bmatrix} -3.33 & 0.42 \\ 4.21 & 0.40 \end{bmatrix} X_t.
  \]
Estimation Results: The Filtered Path of Shadow Rate

<table>
<thead>
<tr>
<th>Year</th>
<th>Yield (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1992</td>
<td>-1</td>
</tr>
<tr>
<td>1994</td>
<td>0</td>
</tr>
<tr>
<td>1996</td>
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</tr>
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<td>1998</td>
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<td>2002</td>
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<tr>
<td>2006</td>
<td>6</td>
</tr>
<tr>
<td>2008</td>
<td>7</td>
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3 Month Rate

<table>
<thead>
<tr>
<th>Year</th>
<th>Shadow Rate</th>
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</thead>
<tbody>
<tr>
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<td>-0.4</td>
</tr>
<tr>
<td>1994</td>
<td>-0.2</td>
</tr>
<tr>
<td>1996</td>
<td>0</td>
</tr>
<tr>
<td>1998</td>
<td>0.2</td>
</tr>
<tr>
<td>2000</td>
<td>0.4</td>
</tr>
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<td>2002</td>
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<td>2004</td>
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<tr>
<td>2006</td>
<td>1</td>
</tr>
<tr>
<td>2008</td>
<td>1.2</td>
</tr>
</tbody>
</table>

Vadim Linetsky
Long Forward Probabilities, Recovery and the Term Structure of Bond Risk Premiums
As time to maturity $T$ increases, the zero-coupon bond price behaves asymptotically as (the long bond asymptotics):

$$P(T, x) \sim Ce^{-\lambda T} \pi(x).$$

Using estimated $\mathbb{Q}$ measure parameters, the principal eigenvalue and eigenfunction are determined numerically by finite differences: $\lambda \approx 2.82\%$

**Figure:** Shape of principal eigenfunction
In diffusion models, the market price of risk under $\mathbb{L}$ (Ross’ Recovery) is recovered in terms of the principal eigenfunction ($\sigma$ is the volatility matrix in the SDE):

$$\lambda^\mathbb{L}_i(x) = \sum_j \sigma_{ji}(x) \partial_j \log \pi(x).$$

Numerical result shows that $\lambda^\mathbb{L}$ is well approximated by a linear function in the range $[-0.3, 0.2] \times [-0.1, 1.2]$, which contains the range of filtered state variables. In particular,

$$\lambda^\mathbb{L}(X_t) \approx \begin{bmatrix} 0.16 \\ -0.10 \end{bmatrix} + \begin{bmatrix} -0.38 & 0.17 \\ 0.17 & -0.12 \end{bmatrix} X_t.$$
Comparing $P$, $L$ and $Q$

- By inspection of market prices of risk, we observe that the $L$ measure dynamics is generally closer to $Q$ (from which it is recovered), than to the estimated $P$.

- Recall that the transition independence assumption results in the identification $P = L$. In contrast, in our results we see significant differences between estimated $P$ and recovered $L$.

- This is not surprising. Recall that $P = L$ implies that the long bond is growth optimal.

- In contrast, Frazzini and Pedersen (2014, J of Financial Economics) document that their “Betting Against the Beta” (BAB) factor levering up shorter maturity bonds to risk parity with longer maturity bonds and shorting longer maturity bonds yields Sharpe ratio of 0.81 in the US Treasury bond market during 1952-2012.
How far apart are $P$ vs. $L$ forecasts? Test of $P = L$

- Using estimated $L$ and $P$ dynamics, we can write down the instantaneous volatility of the martingale component:

$$v(x) \approx \begin{bmatrix} -1.055 \\ -0.863 \end{bmatrix} + \begin{bmatrix} -2.946 & 0.246 \\ 4.045 & 0.525 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

- When $P = L$, the martingale component is trivial, i.e. $v(x) = 0$. Thus we can test the hypothesis $P = L$ by testing each of the component of $v(x)$ being equal to 0.

<table>
<thead>
<tr>
<th></th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_{11}$</th>
<th>$v_{21}$</th>
<th>$v_{22}$</th>
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<tbody>
<tr>
<td>p-value</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.08%</td>
<td>0.04%</td>
<td>0.00%</td>
</tr>
</tbody>
</table>

Table: $p$-values for $v_i = 0$ and $v_{ij} = 0$. 
How far apart are $P$ vs. $L$ forecasts? The Timing of Fed Lift-off

We estimate the implied distribution of the first passage time of the short rate above 25 bps as the proxy for the timing of the Fed zero interest rate policy lift-off as of August 19, 2015.

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<th>Median</th>
<th>Mean</th>
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</thead>
<tbody>
<tr>
<td>$P$</td>
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<td>1.07</td>
</tr>
<tr>
<td>$Q$</td>
<td>0.17</td>
<td>0.34</td>
</tr>
<tr>
<td>$L$</td>
<td>0.16</td>
<td>0.32</td>
</tr>
</tbody>
</table>
How far apart are $P$ vs. $L$ forecasts? The Timing of Fed Lift-off

- We estimate the implied distribution of the first passage time of the short rate above 25 bps as the proxy for the timing of the Fed zero interest rate policy lift-off as of Dec. 30, 2011.

<table>
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<tr>
<th></th>
<th>Median</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>2.13</td>
<td>2.83</td>
</tr>
<tr>
<td>$Q$</td>
<td>1.34</td>
<td>1.47</td>
</tr>
<tr>
<td>$L$</td>
<td>1.32</td>
<td>1.46</td>
</tr>
</tbody>
</table>
Figure: Realized, $\mathbb{P}$ forecast and $\mathbb{L}$ forecast Sharpe ratio over 3 month holding period.
Further work

- Refining interest rate modeling at the zero lower bound: general sticky boundary models, stochastic volatility, links with macroeconomic variables.
- Explorations of market-implied forecasts.
- Looking at other markets.