Weak Solution for Fully Nonlinear Stochastic Hamilton-Jacobi-Bellman Equations

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Outline

1. Classical Optimal Control Problems
2. DPP $\Rightarrow$ Hamilton-Jacobi-Bellman equation
3. A Heuristic Idea for solvability
4. Regular Potential
   - Quasi-continuity
   - Regular Potential
   - Regular Measure
5. Well-posedness of the Stochastic HJB Equation
   - Existence and Uniqueness
   - Regularity
   - On Generalization
• Consider the following control problem

\[
\inf_{\xi \in \mathcal{U}} E \left[ \int_0^T f(s, X_s^0, x; \xi, \xi_s) \, ds + G(X_T^0, x; \xi) \right]
\]

subject to

\[
X_t^{0, x; \xi} = x + \int_0^t b_s(X_s^{0, x; \xi}, \xi_s) \, ds + \int_0^t \sigma_s(X_s^{0, x; \xi}, \xi_s) \, dW_s, \quad t \in [0, T].
\]

• optimal consumption /trading / portfolio allocation /investment /utility maximization, · · ·
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• Define the dynamic cost functional

\[ J(t, x; \xi) = \mathbb{E}_{\mathcal{F}_t} \left[ \int_t^T f(s, X_s^t, x; \xi, \xi_s) \, ds + G(X_T^t, x; \xi) \right], \quad t \in [0, T] \]

and the value function

\[ u(t, x) = \text{essinf}_{\xi \in \mathcal{U}} J(t, x; \xi), \quad t \in [0, T]. \]
• Markovian case: all the coefficients $b, \sigma, f$ and $G$ are deterministic $\Rightarrow$ deterministic value function

$$
- \partial_t u(t, x) = \operatorname{essinf}_{v \in U} \left\{ \operatorname{tr} \left( \frac{1}{2} \sigma \sigma'(t, x, v) D^2 u(t, x) \right) + b'(t, x, v) Du(t, x) + f(t, x, v) \right\};
$$

$$
u(T, x) = G(x), \quad x \in \mathbb{R}^d. $$

• viscosity solution:
  
  Crandall-Ishii-Lions, BAMS, 1992; Flemming-Soner, BOOK, 2006; . . .

• Path-dependent PDE (Ekren-Keller-Touzi-Zhang, AOP, 2014)
• **General** case (Peng, SICON, 1992; Musiela-Zariphopoulou-2009):

\[
- du(t, x) = \operatorname{essinf}_{v \in U} \left\{ \begin{aligned}
&\text{tr} \left( \frac{1}{2} \sigma \sigma'(t, x, v) D^2 u(t, x) + \sigma(t, x, v) D\psi(t, x) \right) \\
&\quad + b'(t, x, v) Du(t, x) + f(t, x, v) \right\} dt - \psi(t, x) dW_t; \\
&u(T, x) = G(x), \quad x \in \mathbb{R}^d.
\end{aligned} \right.
\]  

(1)

• **Typical Examples:**

- Utility function/Hedging error with \( G(X^0_T; x; \xi - H) \) (Horst-Hu-Imkeller-Rèveillac-Zhang, SPA, 2014)

- habit-forming utility maximization (Englezos-Karatzas, SICON, 2009) with

\[
f(t, x, \xi) = f(t, \xi_t - a^{\xi}_t), \quad a^{\xi}_t = a_0 e^{- \int_0^t \alpha_s \, ds} + \int_0^t \beta_s e^{- \int_s^t \alpha_r \, dr} \xi_s \, ds.
\]

- optimal portfolio choices with stochastic factors volatility (Mania-Tevzadze, JMS, 2008; Musiela-Zariphopoulou-2009)
General case (Peng, SICON, 1992; Musiela-Zariphopoulou-2009):

\[
-du(t, x) = \text{essinf}_{v \in U} \left\{ \text{tr} \left( \frac{1}{2} \sigma \sigma'(t, x, v) D^2 u(t, x) + \sigma(t, x, v) D \psi(t, x) \right) \\
+ b'(t, x, v) Du(t, x) + f(t, x, v) \right\} dt - \psi(t, x) dW_t;
\]

\[
u(T, x) = G(x), \quad x \in \mathbb{R}^d.
\]

- The DPP is used informally because of the insufficient study on the regularity of the random value function.
- Well-posedness:
  - Case with uncontrolled leading coefficients ($\sigma$ is not controlled) is extensively studied:
    - Singular case: Graewe-Horst-Q. (SICON, 2014), \ldots
    - \ldots
  - General case with controlled leading coefficients is OPEN.
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Our Heuristic Idea

- Heuristically, we define a family of random measures by putting

\[
\mu^{\tilde{v}}(dt, dx) = \left( \text{tr} \left( \frac{1}{2} \sigma \sigma'(t, x, \tilde{v}) D^2 u(t, x) + \sigma(t, x, \tilde{v}) D \psi(t, x) \right) + b'(t, x, \tilde{v}) Du(t, x) + f(t, x, \tilde{v}) \right)
\]

\[
- \text{essinf}_{v \in U} \left\{ \text{tr} \left( \frac{1}{2} \sigma \sigma'(t, x, v) D^2 u(t, x) + \sigma(t, x, v) D \psi(t, x) \right) + b'(t, x, v) Du(t, x) + f(t, x, v) \right\} \right) dtdx.
\]

Then the stochastic HJB (1) equation writes

\[
\begin{cases}
- du(t, x) + \mu^{\tilde{v}}(dt, x) \\
= \left\{ \text{tr} \left( \frac{1}{2} \sigma \sigma'(t, x, \tilde{v}) D^2 u(t, x) + \sigma(t, x, \tilde{v}) D \psi(t, x) \right) \\
+ b'(t, x, \tilde{v}) Du(t, x) + f(t, x, \tilde{v}) \right\} dt - \psi(t, x) dW_t; \\
u(T, x) = G(x), \quad x \in \mathbb{R}^d,
\end{cases}
\]

(2)

If the family of triples \((u, \psi, \mu^{\tilde{v}})\) satisfying (2) with \(\inf_{v} \mu^{v}(dt, dx) \equiv 0\), the pair \((u, \psi)\) can be expected to be a weak solution for (1) in some sense.

- Characterization of \(\mu^{\tilde{v}}\).
• For the Markovian case, a similar idea is conjectured by Lions (1983) but only realized for the one-dimensional elliptic case by Coron-Lions (1986).

• We want to realize such an idea for the general case by introducing a class of regular parabolic potentials.
To the end, let us consider a simplified version:

\[
\begin{aligned}
(*) \quad & -\frac{du(t, x)}{dt} = \text{essinf}_{\sigma \in U} \left\{ \text{tr} \left( \frac{1}{2} \sigma \sigma' D^2 u + \sigma D \psi \right) (t, x) + f(t, x, \sigma) \right\} \ dt, \\
& - \psi(t, x) \ dW_t, \quad (t, x) \in Q := [0, T] \times \mathbb{R}^d; \\
& u(T, x) = G(x), \quad x \in \mathbb{R}^d.
\end{aligned}
\]

- \( T \in (0, \infty); \)
- \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}), m\)-dimensional Wiener process \( \{W_t\}_{t \geq 0}; \)
- the pair \((u, \psi)\) is unknown;
- \(U\) is a nonempty bounded subset of \(\mathbb{R}^{d \times m}.\)
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Denote by $\mathcal{U}$ the set of all the $U$-valued and $\mathcal{F}_t$-adapted processes and for each $\sigma \in \mathcal{U}$,

$$X^\sigma_t := \int_0^t \sigma_s \, dW_s, \quad t \in [0, T]. \quad (3)$$

One has the "norm equivalence"

$$E \int_0^T \|h(t, \cdot + X^\sigma_t)\|^2 dt = \|h\|_{L^2(L^2)}^2 \leq T \int_{\mathbb{R}^d} E \sup_{t \in [0, T]} |h(t, x + X^\sigma_t)|^2 \, dx. \quad (4)$$

**Definition (quasi-continuity)**

Random function $u : \Omega \times [0, T] \times \mathbb{R}^d \to \overline{\mathbb{R}}$ is said to be $\sigma$-quasi-continuous provided that for each $\varepsilon > 0$, there exists a predictable random set $D^\varepsilon \subset \Omega \times [0, T] \times \mathbb{R}^d$ such that $\mathbb{P}$-a.s. the section $D^\varepsilon_\omega$ is open and $u(\omega, \cdot, \cdot)$ is continuous on its complement $(D^\varepsilon_\omega)^c$ and

$$\mathbb{P} \otimes dx ((\omega, x) | \exists t \in [0, T] \text{ s.t. } (\omega, t, x + X^\sigma_t(\omega)) \in D^\varepsilon) \leq \varepsilon.$$  

- The quasi-continuity is closed in $L^p(\Omega \times \mathbb{R}^d; C([0, T] : \mathbb{R}))$, for $p > 0$.
- The $\sigma$-quasi-continuity of $u$ implies that $u(t, x + X^\sigma_t)$ is continuous $\mathbb{P} \otimes dx$-a.e.
• $\mathcal{L}^2(H)$ denotes the space of adapted $H$-valued process in $L^2(\Omega \otimes [0, T]; H)$;
• $S^2(H) = \mathcal{L}^2(H) \cap L^2(\Omega; C([0, T]; H))$.

Given $f \in \mathcal{L}^2(L^2)$, $\Psi \in L^2(\Omega, \mathcal{F}_T; L^2)$, the pair $(u, \psi) \in S^2(L^2) \times \mathcal{L}^2((H_2^{-1})^m)$ is called a weak solution of the following Backward SPDE

$$\begin{aligned}
&\left\{ \begin{array}{l}
-du(t, x) = \left[ \text{tr} \left( \frac{1}{2} \sigma\sigma' D^2 u + \sigma D\psi \right) \right] (t, x) + f(t, x) \ dt - \psi(t, x) \ dW_t, \quad (t, x) \in Q; \\
u(T, x) = \Psi(x), \quad x \in \mathbb{R}^d,
\end{array} \right.
\end{aligned}$$

(or writes $(u, \psi) = \mathbb{S}(\sigma, f, \Psi)$), if it holds in the distributional sense, i.e., for any $\varphi \in \mathcal{D}_T := C_c^\infty \otimes C_c^\infty(\mathbb{R}^d)$ and $t \in [0, T]$, there holds

$$\begin{aligned}
&\langle u(t), \varphi(t) \rangle + \int_t^T \langle u(s), \partial_s \varphi(s) \rangle \ ds \\
= &\langle \Psi, \varphi(T) \rangle + \int_t^T \left[ \langle f, \varphi \rangle + \frac{1}{2} \langle u, \text{tr} (\sigma\sigma' D^2 \varphi) \rangle - \langle \sigma \psi, D\varphi \rangle \right] (s) \ ds - \int_t^T \langle \varphi(s), \psi(s) \ dW_s \rangle, \ \text{a.s.}
\end{aligned}$$
Proposition

Given \((u, \psi) = \mathcal{S}(\sigma, f, \Psi)\), one has the following stochastic representations, for \(0 \leq t \leq s \leq T\),

\[
u(t, x + X^\sigma_t) + \int_t^s (\psi + Du \sigma)(\tau, x + X^\sigma_{\tau}) dW_{\tau} = u(s, x + X^\sigma_s) + \int_t^s f(\tau, x + X^\sigma_{\tau}) d\tau, \quad \mathbb{P} \otimes dx\text{-a.e.}
\]

with \(\psi + Du \sigma \in L^2((L^2)^m)\).

Moreover, \(u\) is \(\sigma\)-quasi-continuous.
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For any \((t, s) \in [0, T] \times (0, \infty)\)

\[
P^\sigma_s u(t) = E\mathcal{F}_t [u(t + s, \cdot + X^\sigma_{t+s} - X^\sigma_t)1_{t+s \in [0, T]}], \quad \forall u \in \mathcal{L}^2(\mathcal{L}^2).
\]

Then, \((P^\sigma_t)_{t \geq 0}\) is a strongly continuous one-parameter contraction semigroup on \(\mathcal{L}^2(\mathcal{L}^2)\).

**Definition (Regular Potential)**

\(u \in S^2(\mathcal{L}^2)\) is called a regular \(\sigma\)-potential, provided that \(u\) is \(\sigma\)-quasi-continuous, 
\[
\lim_{t \to T} u(t, \cdot) = 0 \text{ in } L^2(\mathbb{R}^d) \text{ a.s.,}
\]

\[
E \int_{\mathbb{R}^d} \sup_{t \in [0, T]} |u(t, x + X^\sigma_t)|^2 dx < \infty,
\]

and for any \((t, s) \in [0, T] \times (0, \infty)\)

\[
P^\sigma_s u(t) \leq u(t), \quad \mathbb{P} \otimes dx\text{-a.e. (supermedian)}
\]

- When \(\sigma = I^{d \times d}, m = d\), it coincides with the classical case.
A variational characterization

Proposition

Let $u \in S^2(L^2)$ be $\sigma$-quasi-continuous. Then $u$ is a regular $\sigma$-potential if and only if there exist a random Radon measure $\mu$ and random field $\psi \in L^2(H_2^{-1})$ such that there holds in the distributional sense

$$
\begin{align*}
-du(t, x) + \mu(dt, x) &= \left\{ \text{tr} \left( \frac{1}{2} \sigma \sigma' D^2 u + \sigma D\psi \right) (t, x) \right\} dt - \psi(t, x) \, dW_t, \\
u(T, x) &= 0.
\end{align*}
$$

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Theorem

Let $u \in S^2(L^2)$. Then $u$ is a regular $\sigma$-potential if and only if there exist random field $\psi \in L^2((H^2_2)^{-1})^m$ and a continuous increasing process $K = \{K_t\}_{t \in [0,T]}$ such that $K_0 = 0$, $K_t$ is $F_t \otimes B(\mathbb{R}^d)$-measurable for each $t \in [0,T]$, $K_T \in L^2(\Omega, F_T; L^2)$, $\psi + Du\sigma \in L^2((L^2)^m)$ and

(i) 

$$
u(t, x + X_t^\sigma) = K_T(x) - K_t(x) - \int_t^T (\psi + Du\sigma)(s, x + X_s^\sigma) \, dW_s, \quad P \otimes dx \text{-a.e.}$$

for each $t \in [0, T]$. The processes $K$ and $\psi$ are uniquely determined by those properties.

Moreover, there hold the following relations:

(ii) 

$$E \left[ \|u(t)\|^2 + \int_t^T \|(\psi + Du\sigma)(s)\|^2 \, ds \right] = E \int_{\mathbb{R}^d} (K_T(x) - K_t(x))^2 \, dx, \quad \forall t \in [0, T];$$
Theorem (continued)

(iii) there holds in the distributional sense

\[
\begin{aligned}
-du(t, x) + \mu(dt, x) &= \left\{ \text{tr} \left( \frac{1}{2} \sigma^\prime D^2 u + \sigma D \psi \right) (t, x) \right\} dt - \psi(t, x) dW_t, \\
u(T, x) &= 0.
\end{aligned}
\]

where \( \mu \) is the random measure \( \mu : \Omega \to \mathcal{M}([0, T] \times \mathbb{R}^d) \)

(iv)

\[
\mu(\varphi 1_{[t, T]}) = \int_{\mathbb{R}^d} \int_t^T \varphi(s, x + X_s^\sigma) dK_s(x) dx, \quad \varphi \in \mathcal{D}_T, \text{ a.s.},
\]

with \( \mathcal{M}([0, T] \times \mathbb{R}^d) \) denoting the set of all the Radon measures on \([0, T] \times \mathbb{R}^d\).
Corollary

Let $u$ be a regular $\sigma$-potential and $\mu : \Omega \rightarrow \mathcal{M}([0, T] \times \mathbb{R}^d)$ a random Radon measure such that relation (iii) holds. Then one has

$$\langle \phi, u(t) \rangle = E\mathbb{F}_t \int_t^T \int_{\mathbb{R}^d} \phi(y - X_s^\sigma + X_t^\sigma) \mu(dy, ds),$$

for each $\phi \in L^2(\mathbb{R}^d)$ and $t \in [0, T]$. 

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Definition (Regular σ-measure)

A nonnegative random Radon measure $\mu : \Omega \to \mathcal{M}([0, T] \times \mathbb{R}^d)$ is called regular σ-measure provided that there exists a regular σ-potential $u$ such that relation (iii) is satisfied.

We say the infimum of $\{\mu^\sigma\}_{\sigma \in \mathcal{U}}$ vanishes (or $\inf_{\sigma \in \mathcal{U}} \mu^\sigma = 0$), if for each nonnegative $\tilde{\varphi} \in C_c^\infty(\mathbb{R}^d)$ and any $(t, \varepsilon, \sigma) \in [0, T] \times (0, \infty) \times \mathcal{U}$, there exist a sequence $\{\sigma^i\} \subset \mathcal{U}$ and a standard partition $\{\zeta^i\}_{i \in \mathbb{N}^+}$ of unity in $\mathbb{R}^d$ such that

$$E \left[ \sum_{i \in \mathbb{N}^+} \int_{[t,T] \times \mathbb{R}^d} (\tilde{\varphi} \zeta^i)(x - X_t^\sigma + X_s^{\sigma^i} - X_t^{\sigma^i}) \mu^{\sigma[0,t] \lor \sigma^i}(ds, dx) \right] < \varepsilon,$$

where $\sigma^{[0,t]} \lor \sigma^i \in \mathcal{U}$ with $(\sigma^{[0,t]} \lor \sigma^i)(s) = \sigma_s^1[0,t](s) + \sigma^i_s 1_{(t,T]}(s)$ for $s \in [0, T]$.

- In particular, if $\mu^\lambda(dt, dx) = \rho(t, x, \lambda) dt dx$, then

$$\inf_{\lambda} \mu^\lambda(dt, dx) = 0 \iff \inf_{\lambda} \rho(t, x, \lambda) = 0.$$
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Recall the stochastic HJB equation:

\[
\begin{cases}
- du(t, x) = \underset{\sigma \in U}{\text{essinf}} \left\{ \text{tr} \left( \frac{1}{2} \sigma \sigma' D^2 u + \sigma D \psi \right) (t, x) + f(t, x, \sigma) \right\} dt, \\
- \psi(t, x) dW_t, \quad (t, x) \in Q := [0, T] \times \mathbb{R}^d; \\
u(T, x) = G(x), \quad x \in \mathbb{R}^d.
\end{cases}
\]

**Definition (weak solution)**

A couple \((u, \psi) \in S^2(L^2) \times L^2((H_2^{-1})^m)\) is said to be a weak solution of SHJB (*), if

a) for each \(\sigma \in \mathcal{U}\), \(u\) is \(\sigma\)-quasi-continuous and there exists a random Radon measure \(\mu^\sigma\), such that

\[
\begin{cases}
- du(t, x) + \mu^\sigma(dt, x) = \left\{ \text{tr} \left( \frac{1}{2} \sigma \sigma' D^2 u + \sigma D \psi \right) (t, x) + f(t, x, \sigma_t) \right\} dt - \psi(t, x) dW_t, \\
u(T, x) = G;
\end{cases}
\]

b) the infimum of \(\{\mu^\sigma\}_{\sigma \in \mathcal{U}}\) vanishes.
Assumptions (integrability + Hölder continuity)

\[(A1) \ G \in L^2(\Omega, \mathcal{F}_T; L^2) \text{ is nonnegative and the random function} \]
\[f : \ \Omega \times [0, T] \times \mathbb{R}^d \times U \to [0, \infty) \text{ is } \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(U)\text{-measurable. There exist} \]
\[\alpha \in (0, 1] \text{ and } L > 0 \text{ and some } g \in L^2(L^2) \text{ such that for all } x_1, x_2 \in \mathbb{R}^d, v \in U \text{ and} \]
\[(\omega, t) \in \Omega \times [0, T], \ f(\omega, t, x_1, v) \leq g(\omega, t, x_1) \text{ and} \]
\[|f(\omega, t, x_1, v) - f(\omega, t, x_2, v)| + |G(\omega, x_1) - G(\omega, x_2)| \leq L|x_1 - x_2|^\alpha. \]
Existence and Uniqueness of Weak Solution

**Theorem**

Under assumption \((A1)\), SHJB (*) admits a unique weak solution \((u, \psi)\) with \(u\) coinciding with the value function of the associated optimal control problem. For this solution, \(\psi + Du\sigma \in \mathcal{L}^2((L^2)^m)\) for each \(\sigma \in \mathcal{U}\), and there exists \(L_1 > 0\) such that for any \(x, y \in \mathbb{R}^d\), \(\sup_{t \in [0, T]} |u(t, x) - u(t, y)| \leq L_1 |x - y|^\alpha\) a.s.
Corollary

Under assumption (A1), there holds for any $0 \leq t \leq t + \delta \leq T$ and $\zeta \in L^2(\Omega, \mathcal{F}_t)$,

$$V(t, \zeta) = \operatorname{essinf}_{\sigma \in \mathcal{U}} E_{\mathcal{F}_t} \left[ \int_t^{t+\delta} f(s, \zeta + X_s^\sigma - X_t^\sigma, \sigma_s) \, ds + V(t + \delta, \zeta + X_{t+\delta}^\sigma - X_t^\sigma) \right] , \text{ a.s.}$$
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• We adopt the decomposition \( \sigma = (\tilde{\sigma}, \bar{\sigma}) \) with \( \tilde{\sigma} \) and \( \bar{\sigma} \) valued in \( \mathbb{R}^{n \times m_0} \) and \( \mathbb{R}^{n \times m_1} \) respectively for the control \( \sigma \), i.e.,

\[
\sigma_t \, dW_t = (\tilde{\sigma}_t, \bar{\sigma}_t) \left( \begin{array}{c} d\tilde{W}_t \\ d\bar{W}_t \end{array} \right).
\]

• Denote by \( \{\tilde{\mathcal{F}}_t\}_{t \geq 0} \) the natural filtration generated by \( \tilde{W} \) and augmented by all the \( \mathbb{P} \)-null sets.

\((A2)\) \( G \in L^2(\Omega, \tilde{\mathcal{F}}_T; L^2) \) and for each \((t, \tilde{v}) \in [0, T] \times U\), the random function \( f(\cdot, t, \cdot, \tilde{v}) : \Omega \times \mathbb{R}^d \to \mathbb{R} \) is \( \tilde{\mathcal{F}}_t \otimes \mathcal{B}(\mathbb{R}^d) \)-measurable.

**Lemma**

Under assumptions (A1) and (A2), the value function \( V(t, x) \) is \( \tilde{\mathcal{F}}_t \)-measurable for each \((t, x) \in [0, T] \times \mathbb{R}^d\).
Proposition

Under assumptions (A1) and (A2), for the unique solution \((u, \psi)\) of SHJB (*)\), \(u(t,x)\) is just \(\mathcal{F}_t\)-measurable for each \((t, x) \in [0, T] \times \mathbb{R}^d\) and \(\psi_i = 0\) for \(i = m_0 + 1, \ldots, m_0 + m_1\), and for any \(\sigma = (\tilde{\sigma}, \bar{\sigma}) \in \tilde{\mathcal{U}}\), there holds the gradient estimate \(Du \bar{\sigma} \in L^2((L^2)^{m_1})\). In particular, SHJB (*) can be written equivalently into

\[
\begin{aligned}
-du(t, x) &= \essinf_{\sigma = (\tilde{\sigma}, \bar{\sigma}) \in U} \left\{ \tr \left( \frac{1}{2} (\bar{\sigma} \tilde{\sigma}' + \tilde{\sigma} \bar{\sigma}') D^2 u + \bar{\sigma} D\psi \right) (t, x) + f(t, x, \sigma) \right\} dt \\
&\quad - \psi(t, x) d\tilde{W}_t, \quad (t, x) \in Q; \\
u(T, x) &= G(x), \quad x \in \mathbb{R}^d.
\end{aligned}
\]

If there exists \(\sigma_0 = (\tilde{\sigma}_0, \bar{\sigma}_0) \in U\) such that \(\bar{\sigma}_0 \tilde{\sigma}_0' > 0\), then one has further \(Du \in L^2((L^2)^d)\) and \(\psi \in L^2((L^2)^{m_0})\).
In particular, if $m_0 = 0$, i.e.

$$(A3) \text{ For each } (t, \sigma) \in [0, T] \times U, \ G(\cdot) \text{ and } f(t, \cdot, \sigma) \text{ are deterministic functions on } \mathbb{R}^d,$$

then, it holds that

**Corollary (Markovian case)**

Under assumptions ($A1$) and ($A3$), for the unique weak solution $(u, \psi)$ of SHJB (*), $u$ is deterministic and $\psi \equiv 0$ and for any non-random control $\sigma$, there holds the gradient estimate $Du \sigma \in L^2((L^2)^{m_1})$. In particular, SHJB (*) is equivalent to the following deterministic PDE (HJB)

$$
\begin{cases}
-\partial_t u(t, x) = \text{essinf}_{\sigma \in U} \left\{ \text{tr} \left( \frac{1}{2} \sigma \sigma' D^2 u \right) (t, x) + f(t, x, \sigma) \right\}, \\
\quad (t, x) \in Q; \\
\quad u(T, x) = G(x), \quad x \in \mathbb{R}^d.
\end{cases}
$$

(6)
Outline

1. Classical Optimal Control Problems

2. DPP $\Rightarrow$ Hamilton-Jacobi-Bellman equation

3. A Heuristic Idea for solvability

4. Regular Potential
   - Quasi-continuity
   - Regular Potential
   - Regular Measure

5. Well-posedness of the Stochastic HJB Equation
   - Existence and Uniqueness
   - Regularity
   - On Generalization
A Comment on Generalization

\[
(*) \quad \begin{cases}
-du(t, x) = \essinf_{\sigma \in U} \left\{ \tr \left( \frac{1}{2} \sigma \sigma' D^2 u + \sigma D\psi \right) (t, x) + f(t, x, \sigma, u(t, x)) \right\} \ dt - \psi(t, x) \ dW_t; \\
u(T, x) = G(x), \quad x \in \mathbb{R}^d.
\end{cases}
\]

\[
\Downarrow
\]

\[
\begin{cases}
-du(t, x) = \essinf_{v \in U} \left\{ \tr \left( \frac{1}{2} \sigma \sigma'(t, x, v) D^2 u(t, x) + \sigma(t, x, v) D\psi(t, x) \right) + b'(t, x, v) Du(t, x) \\
+ f(t, x, u, \psi, \sigma) \right\} \ dt - \psi(t, x) \ dW_t, \quad (t, x) \in Q; \\
u(T, x) = G(x), \quad x \in \mathbb{R}^d,
\end{cases}
\]

Two points:

- The solution inherits the regularity of \((f, G)\). This may allow \(f\) depends on \(u\) and \(\psi\);
- For general \((b, \sigma)\), one has to check the associated norm equivalence relationships.


Thank You!