Optimal Insurance: Belief Heterogeneity, Ambiguity, and Arrow’s Theorem

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Outline

(1) The Classical Theory of Insurance Demand

(2) Insurance Demand with Heterogeneous Beliefs

(3) Insurance Demand with Ambiguity on the Insurer’s Side
The Classical Theory of Insurance Demand (Arrow)
The Classical Theory of Insurance Demand

- Two parties: insurer and insured (DM)
- Both are EU-maximizers (no ambiguity)
- The DM seeks an insurance coverage against a random loss $X$ to maximize her EU of terminal wealth
- Both assign the same distribution to $X$ (belief homogeneity)
- DM risk-averse, insurer risk-neutral, premium set by insurer

What is an optimal indemnity schedule for the DM?

A linear, or straight deductible (Arrow’s Theorem): $I_p x q^\text{max} p x d, 0 q
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$\Rightarrow$ What is an optimal indemnity schedule for the DM?
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What is an optimal indemnity schedule for the DM?

A linear, or straight deductible (Arrow’s Theorem):

$$I(x) = \max(x - d, 0)$$
The Classical Theory of Insurance Demand: Some Extensions

Effects on the optimal deductible level of:
- Distributional shifts of the loss
- Changes in the DM's risk-aversion
- Changes in DM's initial wealth
- General forms of market incompleteness (e.g., background risk)
- Information asymmetry
- More general premium principles
- Non-EU preferences

However, beliefs are still perfectly homogeneous (in the sense that the loss $X$ has an objective distribution).
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The Classical Theory of Insurance Demand

Setting:
The Classical Theory of Insurance Demand

Setting:

- $S$ is the set of states of the world
- $X : S \rightarrow [0, M] \text{ is a given random loss}$
- The DM seeks an indemnity schedule $I(X)$, as coverage against the loss $X$, at a premium $\Pi$
- Insurance coverage $I$ is a Borel-measurable map
  $I : X(S) \rightarrow \mathbb{R}^+$
- The premium $\Pi$ is set by the insurer
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- $\Sigma = \sigma\{X\}$, and $P$ is a probability measure on $(S, \Sigma)$
The Classical Theory of Insurance Demand

Setting: The DM
The Classical Theory of Insurance Demand

Setting: The DM

- Risk-averse EU-maximizer (concave utility function)
- Initial wealth: $W_0$
- Wealth in state $s \in S$:

$$W(s) := W_0 - \Pi - X(s) + I(X(s))$$
The Classical Theory of Insurance Demand

Setting: The Insurer

Risk-neutral EU-maximizer (linear utility function $V$)

Initial wealth: $W_{ins}$

Wealth in state $s$: $W_{ins|s}$

$W_{ins|0} - \Pi_{p} X_{p} s_{q} - c_{p} X_{p} s_{q} \bar{\alpha}$

where $c_{p} X_{p} s_{q} \bar{\alpha}$, a cost function $\alpha \geq 0$, a "loading factor"
The Classical Theory of Insurance Demand

Setting: The Insurer

- Risk-neutral EU-maximizer (linear utility function $\mathcal{V}$)
- Initial wealth: $W_{ins}^0$
- Wealth in state $s \in S$:

$$W_{ins}^1 (s) := W_{ins}^0 + \Pi - I (X (s)) - c\left(I (X (s))\right)$$

where

- $c\left(I (X (s))\right) = \alpha \cdot I (X (s))$, a cost function
- $\alpha > 0$, a "loading factor"
The Classical Theory of Insurance Demand

Setting: The Insurer

Therefore,

\[ \int \mathcal{N}(W^{ins}) \, dP \geq W_0^{ins} \iff \Pi \geq (1 + \alpha) \int l(X) \, dP \]

Participation Constraint

Premium Constraint
The Classical Theory of Insurance Demand

Setting: The Insurer

Therefore,

\[
\int \mathcal{V}(W_{ins}) \, dP \geq W_{0ins} \quad \iff \quad \Pi \geq (1 + \alpha) \int l(X) \, dP
\]

Participation Constraint \hspace{2cm} \text{Premium Constraint}

Since \( W_{ins}(s) = W_{0ins} + \Pi - l(X(s)) - \alpha l(X(s)) \)
The Classical Theory of Insurance Demand

The DM’s Problem:
The Classical Theory of Insurance Demand

The DM’s Problem:

Problem

$$
\sup_l \left\{ \int u(W_0 - \Pi - X + l \circ X) \ dP \right\}
$$

Subject to:

$$
0 \leq l \circ X \leq X
$$

and

$$
\Pi \geq (1 + \alpha) \int l \circ X \ dP, \text{ for a given } \alpha > 0
$$
The Classical Theory of Insurance Demand

Optimality of the Deductible Indemnity Schedule:

There exists a deductible indemnity schedule $I_d p t q^p t q^d$, for some $d > 0$, such that $Y_{\hat{I}d p t}$ is optimal for the DM’s problem. Therefore, the optimal indemnity is:

(i) Increasing in the loss
(ii) Fully characterized analytically
(iii) Fully characterized in terms of its distribution

How much of this can we retain in the presence of belief heterogeneity and/or ambiguity?
The Classical Theory of Insurance Demand

Optimality of the Deductible Indemnity Schedule:

Theorem (Arrow (Belief Homogeneity - No Ambiguity))

There exists a deductible indemnity schedule \( I_d(t) = (t - d)^+ \), for some \( d > 0 \), such that \( Y^* := I_d \circ X \) is optimal for the DM’s problem.
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\[ \Rightarrow \text{How much of this can we retain in the presence of belief heterogeneity and/or ambiguity?} \]
The Main Results: Belief Heterogeneity

First, we consider a setting with heterogeneous beliefs. If the insurer's beliefs are compatible with the DM's beliefs, we show the existence and monotonicity of optimal indemnity schedules. Optimal indemnity schedules are fully characterized in terms of their distribution (for the DM). Arrow's classical result can be obtained as a special case. Belief Compatibility is a strictly weaker condition than the Monotone Likelihood Ratio (MLR) condition. In the special case of a MLR, the optimal indemnity schedule is a variable deductible, with a state-contingent deductible that depends on the likelihood ratio.
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- If the insurer is either ambiguity-seeking or ambiguity-averse in the sense of Schmeidler (1989), then the optimal indemnity schedule coincides with an optimal indemnity obtained from a problem with no ambiguity but only belief heterogeneity.

- If the insurer is ambiguity-seeking with a capacity that is a (concave) distortion of the DM’s probability measure, or if the insurer is ambiguity-averse with a capacity whose core has the MLR property w.r.t. the DM’s probability measure, then the optimal indemnity schedule takes the form of a variable deductible contract.
Insurance Demand
with Heterogeneous Beliefs
Why Heterogeneous Beliefs?

Practical reasons:
Some of the observable contracts do not correspond to the predictions of the information asymmetry literature.
Policy implications: heterogeneity of beliefs cannot be mitigated by disclosure requirements.

Advancements in insurance analytics / predictive modeling.

Theoretical reasons:
Subjective uncertainty (Savage).
Rationality of "agreeing to disagree" / Aumann's Agreement Theorem / Critique of the CPA.
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Related Literature

"Agreeing-to-disagree"-type models in the broad literature:
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- Arrow-Debreu model and the fundamental theorems of welfare economics allow for belief heterogeneity


DM assigns a pdf to the loss and the insurer assigns the pdf to the loss. Marshall argues that at that level of generality, the optimal indemnity could take any shape. Special case: DM assigns a higher probability to the no-loss event than the insurer. Conditioning on the loss being non-zero, the two parties assign the same conditional distribution to the loss. In this case, the optimal insurance indemnity is a deductible. Rather restrictive approach to belief heterogeneity. Heterogeneity is reduced only to the likelihood that each party attaches to the event of a zero loss.

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- $f$ and $g$ follow a second-order stochastic dominance ordering

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**Special case:** DM assigns a higher probability to the no-loss event than the insurer. Conditioning on the loss being non-zero, the two parties assign the same conditional distribution to the loss.

$\implies$ In this case, the optimal insurance indemnity is a deductible.

- Rather restrictive approach to belief heterogeneity.
- Heterogeneity is reduced only to the likelihood that each party attaches to the event of a zero loss.
Insurance Demand with Heterogeneous Beliefs

Consider the previous setting: $S, X$: $S \subseteq \mathbb{R}^n$, $M$, $\Sigma$.

The DM is a risk-averse EU-maximizer, with probability measure $P$ on $\mathbb{P}(S, \Sigma)$.

The insurer, is a risk-neutral EU-maximizer, with probability measure $Q$ on $\mathbb{P}(S, \Sigma)$.

The insurer has initial wealth $W_{ins}^0$, and utility of wealth $V(W_{ins}) = W_{ins}^{\alpha}$.
Insurance Demand with Heterogeneous Beliefs

Setting:

- Consider the previous setting:

\[ S, \ X : S \to [0, M], \ \Sigma = \sigma\{X\} \]

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- The insurer has initial wealth \( W_{0}^{ins} \), and utility of wealth 

\[ \mathcal{V}(W^{ins}) = W_{0}^{ins} + \Pi - I(X) - \alpha I(X) \]
Setting:

- Therefore,

\[ \int \mathcal{V} \left( W^{ins} \right) \, dQ \geq W_0^{ins} \quad \iff \quad \Pi \geq (1 + \alpha) \int l(X) \, dQ \]

Participation Constraint

Premium Constraint
Insurance Demand with Heterogeneous Beliefs

Setting:

Therefore,

$$\int \nu (W_{ins}) \ dQ \geq W_{0}^{ins} \iff \Pi \geq (1 + \alpha) \int l(X) \ dQ$$

Participation Constraint \hspace{2cm} Premium Constraint

⇒ Nothing else is assumed about \( Q \) and \( P, a priori \)
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Participation Constraint  
Premium Constraint

⇒ Nothing else is assumed about \( Q \) and \( P \), \textit{a priori}

(In particular, they can disagree on zero-probability events)
The DM’s problem
The DM’s problem becomes:
Insurance Demand with Heterogeneous Beliefs

The DM’s problem becomes:

**Problem**

For a given loading factor $\alpha > 0$,

$$\sup_l \left\{ \int u \left( W_0 - \Pi - X + l(X) \right) dP \right\}$$

Subject to:

$$0 \leq l(X) \leq X$$

and

$$\Pi \geq (1 + \alpha) \int l(X) dQ$$
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**and**

$$\Pi \geq (1 + \alpha) \int I(X) \, dQ$$
Dealing with Heterogeneity: Belief Compatibility

Definition (Belief Compatibility)

The probability measure \( Q \) is said to be \( P \)-compatible, if for any two indemnity schedules \( I_1 \) and \( I_2 \) such that (i) \( I_1 \) and \( I_2 \) have the same distribution under \( P \), and, (ii) \( I_2 \) is a nondecreasing function of \( X \), the following holds:

\[
\int I_2 dQ - \int I_1 dQ \geq \Pi^\alpha \sum_{i=0}^{\Pi^\alpha} W_i \cdot \Pi^\alpha
\]

In other words, \( I_2 \) is \( \geq \) ins \( I_1 \).
Dealing with Heterogeneity: Belief Compatibility

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Dealing with Heterogeneity: Belief Compatibility

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(i) $l_1(X)$ and $l_2(X)$ have the same distribution under $P$, and,

(ii) $l_2$ is a nondecreasing function of $X$. 

In other words, $l_2 \leq_{ins} l_1$
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$$\int I_2(X) \, dQ \leq \int I_1(X) \, dQ$$
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(ii) $I_2$ is a nondecreasing function of $X$

the following holds:

$$\int l_2(X) \, dQ \leq \int l_1(X) \, dQ$$

$$\int (W_0^{ins} + \Pi - (1 + \alpha)l_2(X)) \, dQ \geq \int (W_0^{ins} + \Pi - (1 + \alpha)l_1(X)) \, dQ$$
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In other words, $l_2 \succeq_{ins} l_1$
Dealing with Heterogeneity: Belief Compatibility

- Clearly, $P$ is $P$-compatible (so Arrow’s model is a special case)
Dealing with Heterogeneity: Belief Compatibility

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- If the DM and the insurer assign pdf-s to $X$, then the assumption of Belief Compatibility is a (strictly) weaker assumption than that of a Monotone Likelihood Ratio (MLR)
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- Amarante, Ghossoub, and Phelps (2015, 2016) give several examples of compatible beliefs, including in a setting of ambiguity
Recall the DM’s problem:

Problem

*For a given loading factor $\alpha > 0$,*

$$\sup \left\{ \int u \left( W_0 - \Pi - X + I(X) \right) \, dP \right\}$$  \hspace{1cm} (1)

*Subject to:*

$$0 \leq I(X) \leq X$$

*and*

$$\Pi \geq (1 + \alpha) \int I(X) \, dQ$$
Theorem H1

If the insurer’s subjective belief $Q$ is compatible with the DM’s subjective belief $P$ then:

- There exists an optimal indemnity schedule $I^*(X)$ which is a nondecreasing function of the loss $X$
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If the insurer’s subjective belief $Q$ is compatible with the DM’s subjective belief $P$ then:

1. There exists an optimal indemnity schedule $I^*(X)$ which is a nondecreasing function of the loss $X$

2. If, moreover, the DM is strictly risk-averse, then optimal indemnity schedules are necessarily nondecreasing in the loss
Theorem H1 (Continued)

Moreover, \( l^*(X) \) has the same distribution (under the DM’s beliefs) as a variable deductible schedule

\[
\Psi^*(X(s)) = \min \left( X(s), \max \left( 0, X(s) - d(\theta^*, h(s)) \right) \right)
\]

where:

- \( d(\theta^*, h) = W_0 - \Pi - (u')^{-1}(\theta^* h) \) is a state-contingent deductible
- \( h \) is entirely characterized from the subjective probabilities of the two parties:
  - Lebesgue Decomposition: \( Q = Q_{ac} + Q_s \), \( Q_s \perp P \), \( Q_{ac} \ll P \)
  - \( h := dQ_{ac}/dP \)
- \( \theta^* \geq 0 \) is chosen appropriately [see paper]
Theorem H1 (Continued)

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  - \( h := dQ_{AC}/dP \)
- \( \theta^* \geq 0 \) is chosen appropriately [see paper]

Belief homogeneity \( \implies h = 1 \), and we recover Arrow’s result
Corollary H1 (Compare with Arrow’s Theorem)

Suppose that the previous assumptions hold, and suppose also that $P \leq Q$. Then there exists a constant deductible $d$ and a solution $I_{dpXq}$ to Problem (1) such that $I_{dpXq} = \min \{X, \max pX, \max qX \}$, $P$-a.s.

Note: We also formalize Marshall’s argument that the optimal indemnity can take “any shape” (see paper).

In particular, it may include a non-zero deductible or a disappearing deductible.
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$$l^*(X) = \min \left[ X, \max (0, X - d^*) \right], \ P\text{-a.s.}$$
Corollary H1 (Compare with Arrow’s Theorem)

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Then there exists a constant deductible \( d^* \) and a solution \( l^*(X) \) to Problem (1) such that

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l^*(X) = \min \left[ X, \max (0, X - d^*) \right], \quad P\text{-a.s.}
\]

Note: We also formalize Marshall’s argument that the optimal indemnity can take "any shape" (see paper).

\[\rightarrow\] In particular, it may include a non-zero deductible or a disappearing deductible.
Belief Compatibility and Monotone Likelihood Ratios

Suppose that densities (pdf-s) for the loss $X$ can be defined:

Let $f_{p,t}^q$ be the pdf that the DM assigns to $X$.

Let $g_{p,t}^q$ be the pdf that the insurer assigns to $X$.

The likelihood ratio is the function $LR_{p,t}^q = \frac{f_{p,t}^q}{g_{p,t}^q}$, for all $t$.

Proposition MLR (i.e., $LR$ is a nonincreasing) $\implies$ Belief Compatibility (BC)

But BC $\implies$ MLR

BC is a strictly weaker assumption than MLR.

Note: $MLR \implies f \leq g_{sd}$, but this is not the case for BC.
Belief Compatibility and Monotone Likelihood Ratios

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- Let $f(t)$ be the pdf that the DM assigns to $X$
- Let $g(t)$ be the pdf that the insurer assigns to $X$
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**Proposition**

$MLR$ (i.e., $LR$ is a nonincreasing) $\implies$ Belief Compatibility ($BC$)

**BUT**

$BC \iff MLR$

$BC$ is a strictly weaker assumption than $MLR$
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BC $\nRightarrow$ MLR

BC is a strictly weaker assumption than MLR

Note: MLR $\Rightarrow f \succsim_{fsd} g$, but this is not the case for BC
The Special Case of a MLR

Theorem H2

If the MLR assumption holds, then there exists an optimal indemnity schedule which is a nondecreasing function of the loss \( X \), and this optimal indemnity schedule takes the form:

\[
\hat{I} = \min_{0 \leq t \leq d_{\text{LR}}(p, q)} t,
\]

where \( d_{\text{LR}}(p, q) \) is chosen so that the second constraint binds at the optimum.

Note: If \( f = g \), then \( \lambda \) is constant.

We recover Arrow's Theorem.
The Special Case of a MLR

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$$I_{\lambda^*}^*(t) = \min \left[ t, \max \left( 0, t - d(LR(t)) \right) \right]$$

where

- $d(LR(t)) = W_0 - \Pi - (u')^{-1} \left( \lambda^* LR(t) \right)$
- $\lambda^* > 0$ is chosen so that the second constraint binds at the optimum
The Special Case of a MLR

**Theorem H2**

If the MLR assumption holds, then there exists an optimal indemnity schedule which is a nondecreasing function of the loss $X$, and this optimal indemnity schedule takes the form:

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**Note:** If $f = g$, then $LR = 1$ and $d$ is constant  
$\implies$ We recover Arrow’s Theorem
Belief Heterogeneity in a Nutshell...

If the insurer’s beliefs are compatible with the DM’s beliefs, then:

We show the existence and monotonicity of optimal indemnity schedules.

The class of optimal indemnity schedules for the DM is fully characterized in terms of their distribution (for the DM).

Arrow’s classical result can be obtained as a special case.

Belief Compatibility is a strictly weaker condition than the MLR condition.

In the special case of a MLR, the optimal indemnity schedule is a variable deductible, with a state-contingent deductible that depends on the likelihood ratio.
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Ambiguity on the Insurer’s Side
Why Insurer Ambiguity?

Empirical evidence (Hogarth and Kunreuther (1989):
Ambiguity is prevalent in insurance pricing and underwriting.
Insurers often exhibit more ambiguity than the insureds.
Recent empirical research suggests that ambiguity-loving
might be more empirically relevant than ambiguity-aversion:
Ert and Trautmann (2014)
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  - Ert and Trautmann (2014)
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Related Literature (Non-Exhaustive List…)
Alary, Gollier, and Treich (2013): DM is ambiguity-averse in the sense of Klibanoff, Marinacci, and Mukerji

Jeleva (2000): optimal co-insurance level when DM is a CEU maximizer

Young (1999) and Bernard et al. (2015): DM is a RDEU maximizer

Doherty and Eeckhoudt (1995): optimal level of deductible under Yaari’s Dual Theory

Karni (1992) and Machina (1995): DM’s preference have non-EU representation that satisfies certain differentiability criteria

Schlesinger (1997): optimal co-insurance level when the DM’s preferences are not necessarily EU preferences, but they are risk-averse in the sense of disliking mean-preserving increases in risk
Ambiguity on the Insurer’s Side

Amarante, Ghossoub and Phelps (2015):
Optimal insurance with belief heterogeneity and ambiguity on the side of the insurer (in the sense of Schmeidler (1989))

Insurer's beliefs: "non-additive" measure (capacity) $\nu$

If the insurer is ambiguity-seeking, then there is a collection $\mathcal{A}$ of (non-ambiguous) priors such that for any event $E$:

$$\max_{Q \in \mathcal{A}} \mathbb{P}^Q_E$$

If the insurer is ambiguity-averse, then there is a collection $\mathcal{C}$ of (non-ambiguous) priors such that for any event $E$:

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$$\nu (E) = \max_{Q \in \mathcal{AC}} Q (E)$$
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- If the insurer’s beliefs are not ambiguous, i.e. represented by some probability measure $Q$, then we are back in the previous setting of belief heterogeneity.
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- **Natural question:** When does the solution to the problem with ambiguity coincide with a solution to a problem with no ambiguity but only belief heterogeneity?
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  ⇒ If this can be done, we say that the problem (or its solution) is **non-ambiguously implementable**.
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- **Natural question:** When does the solution to the problem with ambiguity coincide with a solution to a problem with no ambiguity but only belief heterogeneity?

  $\Rightarrow$ If this can be done, we say that the problem (or its solution) is non-ambiguously implementable

  $\Rightarrow$ If moreover, the non-ambiguous implementation is done via a problem where the insurer’s belief is $Q \neq P$, we then say that the problem (or its solution) is non-ambiguously implementable via $Q
Theorem A1 (Non-Ambiguous Implementation)

*If the insurer is either ambiguity-seeking or ambiguity-averse, then the problem admits a solution and the solution is non-ambiguously implementable:*
Theorem A1 (Non-Ambiguous Implementation)

If the insurer is either ambiguity-seeking or ambiguity-averse, then the problem admits a solution and the solution is non-ambiguously implementable:

- In the ambiguity-seeking case, there exists $Q^* \in \mathcal{AC}$ such that the solution is non-ambiguously implementable via $Q^*$

- In the ambiguity-averse case, there exists $Q^{**} \in \mathcal{C}$ such that the solution is non-ambiguously implementable via $Q^{**}$
Ambiguity on the Insurer’s Side

Theorem A2 (State-Contingent Deductible)

If the insurer is ambiguity-seeking such that his capacity is a distortion of the DM’s probability measure, then a solution to the DM’s problem is a state-contingent deductible indemnity schedule of the form

\[ Y^* = \min \left[ X, \max \left( 0, X - d(T) \right) \right], \]

where

- \( T \) is the insurer’s distortion function
- The state-contingent deductible \( d(T) \) depends on the state of the world only through the distortion function \( T \)
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Hence, Arrow’s theorem is a special case
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- The state-contingent deductible \( d(T) \) depends on the state 
of the world only through the distortion function \( T \)
- Hence, Arrow’s theorem is a special case
- We completely characterize \( d(T) \)
Ambiguity on the Insurer’s Side

Theorem A3 (State-Contingent Deductible)

In the case of an ambiguity-averse insurer whose capacity has a core consisting of probability measures with the monotone likelihood ratio (MLR) property, the optimal indemnity schedule is a state-contingent deductible of the form

\[ Y^* = \min \left[ X, \max \left( 0, X - d(LR) \right) \right], \]

where LR denotes a function of the likelihood ratios.
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where \( LR \) denotes a function of the likelihood ratios.

\[ \implies \text{We completely characterize } d(LR) \]

\[ \implies \text{We obtain Arrow’s result as a limiting case} \]
Ambiguity on the Insurer’s Side: Related Work

Amarante and Ghossoub (2016): Optimal insurance design given a minimum "expected" retention (expectation in the sense of Choquet) belief heterogeneity and ambiguity on the side of the insurer. Optimal indemnity is a state-contingent deductible, and Arrow's result is a special case.

Ambiguity on the Insurer’s Side: Related Work

- Amarante and Ghossoub (2016):
  - Optimal insurance design given a minimum "expected" retention (expectation in the sense of Choquet)
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- **Ghossoub (2015):**
  - Abstract problem of demand for continent claims given a non-law-invariant constraint
  - As a special case, problem of optimal insurance with general non-law-invariant premium principles
  - Existence and monotonicity of optimal indemnities for a wide class of non-law-invariant premium principles
Thank You