Portfolio choice with permanent and temporary transaction costs

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Optimal portfolio selection

- Agent invests in a riskless asset and \( d \) risky assets with objective
  - optimising the present value of the future stream of expected excess returns,
  - penalising for risk, trading costs, and market impact.
- The market is driven by Brownian motions.
- Markovian Factors and affecting the prices process.
- Temporary quadratic transaction costs and permanent mean reverting type frictions.
- Asymptotics for small costs and fast mean reversion.
The frictionless Problem

The frictionless problem

- We assume the market is driven by a Markovian factor $Y$ that satisfies

  $$dY_t = \mu_Y(Y_t)dt + \sigma_Y(Y_t)dW_t$$

- and the price process evolves according

  $$dS_t = \mu(Y_t, S_t)dt + \sigma(Y_t, S_t)dW_t$$

- For simplicity of notation we denote

  $$\mu_t := \mu(Y_t, S_t) \quad \text{and} \quad \Sigma_t := \sigma(Y_t, S_t)^2.$$  

- For notational simplicity we will consider $S$ as an additional dimension in $Y$. 

Frictionless problem

- We assume that the agent can trade continuously without any cost.
- For a given strategy $x_t$ that represents the position in each risky asset, the self-financing wealth process evolves according to
  \[ dV_t = x_t(\mu_t dt + \sigma_t dW_t) \]
- The agent optimises the present value of the future stream of expected excess returns penalised with risk of the portfolio with risk aversion $\gamma$
  \[
  \max_{(x_t)} \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \left( x_t\mu_t - \frac{\gamma}{2} \Sigma_t x_t^2 \right) dt \right]
  \]
  The pointwise maximum is achieved at $x_t^* = M_t = \frac{\mu_t}{\gamma \Sigma_t}$ (Markowitz portfolio).
There are different types of frictions in the market.
Markowitz portfolio cannot be used in practice.
Proportional and fixed transaction costs literature for portfolio selection problem:
Magill and Constantinides 1976
Davis and Norman 1990
Shreve and Soner 1994 (asymptotics)
The objective is to find the non-trading region
Temporary price impact: Quadratic cost

- We treat two different types of frictions.
- For every trade of size $\Delta x$ the agent pays $\Lambda(\Delta x)^2$ as transaction costs.
- Equivalent to requiring that the limit order book is constant depth $\frac{1}{2\Lambda}$.
Temporary price impact : Quadratic cost

- We treat two different types of frictions.
- For every trade of size $\Delta x$ the agent pays $\Lambda(\Delta x)^2$ as transaction costs.
- Equivalent to requiring that the limit order book is constant depth $\frac{1}{2\Lambda}$.
- An order of size $\Delta x$ ”consumes” the order book from the unaffected prices $S_t$ to $S_t + 2\Lambda\Delta x$.
- The total cost of transaction is $\Lambda(\Delta x)^2 + S_t\Delta x$.
- Unless $\frac{\mu_t}{\gamma\Sigma_t}$ is absolutely continuous the Markowitz portfolio has infinite cost.

$$dx_t = \tau_t dt.$$

- The natural question is for a given cost $\varepsilon$ small what kind of control $\tau$ one should use.
This gives rise to a nonlinear PDE in the Markovian case.

There are no explicit formulas.

For small $\varepsilon > 0$, the quantities are too large or too small. Any computation error might lead to instability of algorithms.

Find the asymptotics for small $\varepsilon > 0$ when the transaction costs is $\varepsilon^2 \Lambda$.

The value of the problem with friction is an expansion around the value of the Markowitz problem.

We want an asymptotically optimal strategy.
Review of the Literature for asymptotics

- Soner and Touzi 2012 and Possamai, Soner, and Touzi 2013, proportional costs, viscosity approach.
- Guasoni and Weber 2015 nonlinear impact.
- Moreau, Muhle-Karbe and Soner 2015, quadratic costs, viscosity approach. They also provide with an asymptotically optimal policy.
- Altarovici, Muhle-Karbe, and Soner 2013, fixed costs, viscosity approach.
- Kallsen and Muhle-Karbe 2014, high resilience limit.
- Cai, Rosenbaum and Tankov, Bichuch and Shreve, Janecek and Shreve, Korn, Feodoria,. Survey of Muhle-Karbe, Reppen and Soner.

There is only temporary costs and there is no permanent.
Microstructure foundation for permanent impact

- Permanent price impact is commonly used in optimal execution literature.
- This is in addition to some temporary costs.
- Jaimungal et al, Obizawa and Wang, Almgren and Chris, Alfonsi et al.
- In Garleanu and Pedersen 2016 the authors gives a microstructural foundation to these frictions.
  - Three different kind of market participants.
  - Optimising traders, market makers and end users.
  - Frictions as a consequence of inventory risk of market makers.
  - The resilience is related to the inventory depletion rate of the market makers.
- The continuous time model with friction can be obtained as a limit of discrete time model.
 Permanent impact

The trades of the agent have a lasting impact on the price process that he faces. Besides the unaffected price $S_t$ the agent trades with a price $S_t + D_t$ where $D_t$ is the price distortion. Starting from the state $(D_0, S_0, Y_0)$, a trade $\tau_t \Delta t$ during $\Delta t$
- increases $D$ from $D_0$ to $D_0 + C\tau_t\Delta t$ where $C$ is the distortion impact parameter
- $D$ reverts back to 0 at a rate $R$ (mean reversion parameter).

The equation for $D$ is

$$\frac{dD_t}{dt} = -RD_t + C\tau_t.$$
Admissible strategies

- The agent should not have the possibility to delay the risk to future times.
- We need to have an admissibility condition on the controls. $A_{\rho}(D, x, Y)$ is the set of controls $\tau$ such that
  \[
  \lim_{t \to \infty} e^{-\rho t} (|x_t|^2 + |D_t|^2) = 0.
  \]
- This condition allows integration by parts formulas.
- Recall our convention that $Y$ also contains the elements of $S$. 

\[ \]
In the market described above, an agent optimises the present value of the future stream of expected excess returns.

The optimisation criterion penalises the risk of the portfolio and transaction costs.

The value $J(D, x, Y, \tau)$ associated to a strategy $\tau$

$$
\mathbb{E} \left[ \int_0^\infty e^{-\rho t} \left( x_t (\mu_t - RD_t + C\tau_t) - \frac{\gamma}{2} \sum x_t^2 - \frac{1}{2} \Lambda \tau_t^2 \right) dt \right]
$$

where $(D_t, x_t, Y_t)$ starts at $(D, x, Y)$ and evolves as above.

Our value function of interest is

$$
V(D, x, Y) := \sup_{\tau \in \mathcal{A}_\rho(D, x, y)} J(D, x, Y, \tau)
$$
Viscosity property of the Value function

Theorem

$V$ is a viscosity solution of the PDE

$$
\rho V = -\frac{\gamma}{2} \Sigma x^2 + \mu x - RD(\partial_D V + x) + \mathcal{L}^Y(V)
$$

$$
+ \sup_{\tau} \left\{ -\frac{1}{2} \Lambda \tau^2 + \tau (\partial_x V + C(x + \partial_D V)) \right\}
$$

- Elliptic degenerate PDE.
- The statement even holds for path-dependent problems where the value functional is a viscosity solution to the correspondant path-dependent PDE.
The control can be obtained by the solution of the following FBSDE

\[ dX_t = b(X_t, Y_t) dt \in \mathbb{R}^2 \]
\[ dY_t = -F(t, X_t, Y_t) dt + ZdW_t \in \mathbb{R}^2 \]
\[ Y_T = G(X_T) \]

where \( X = (D, x) \) and \( Y \) allows us to obtain \( \tau^* \).

This is a linear coupled multidimensional FBSDE.

There is no wellposedness result in the general case.
Reasons for Asymptotics

- There is no closed form formula of the value function for general dynamics.
- The quadratic transaction cost and the permanent price impact should be small.
- Obtain a simple tractable formulas that allows us to understand the structure of the solution.
- $V$ should be an expansion around the frictionless problem.

$$V^\varepsilon(D, x, Y) = V^0(Y) + \varepsilon^\alpha V^1(D, x, y) + o(\varepsilon^\alpha)$$

- $V^0$ is the frictionless problem.
- $V^1$ simple and allows us to understand the effect of different parameters.
- Numerical methods are problematic if some parameters are small.
- Additionally we want an asymptotically optimal control.
Correct scaling of the parameters

- We need to find the correct critical regime where we obtain a formula that exposes the effect of different parameters on the value and the dynamics.
- We assume that the quadratic transaction costs scale in $\varepsilon$ as follows
  \[ \Lambda_\varepsilon = \varepsilon^2 \Lambda. \]
- We can write formal asymptotics for a simple linear quadratic model in Garleanu and Pedersen (2016).
- The only scaling where both quadratic costs and the permanent price impact are both non negligible.
  - $R_\varepsilon = R\varepsilon^{-1}$.
  - $C_\varepsilon = C\varepsilon$. 
There are two limit order books that are being used in parallel.

- One book of constant height $\frac{1}{2\Lambda\varepsilon^2}$.
- It mean reverts instantaneously to the unaffected price $S_t$.
- Another book of constant depth $\frac{1}{C\varepsilon}$.
- That mean reverts with a mean-reversion speed $R\varepsilon^{-1}$.

We want the high mean-reversion and small impact asymptotics.
According to the formal asymptotics of Garleanu and Pedersen 2016 to have non-trivial limits we need to plug $D\varepsilon^{-1}$.

If $R_\varepsilon = R\varepsilon^{-1}$ then for a given $\tau$ and $D_0 = 0$ we have

$$D_t\varepsilon^{-1} = C \int_0^t e^{-R_\varepsilon(t-s)} \tau_s d\varepsilon_s \sim \frac{C\varepsilon}{R} \tau_t.$$ 

Optimal portfolio is of order $\varepsilon^{-1}(x_t - M_t)$.

$$\rightarrow D_t\varepsilon^{-1} \text{ is of the order of } (x_t - M_t).$$
Asymptotics without permanent impact

In Moreau, Muhle-Karbe and Moreau 2016, the problem without permanent price impact is studied. The expansion

\[ V^\varepsilon(x, Y) = V^0(Y) + \varepsilon(u(y) + w(x - \mathcal{M}(y))) + o(\varepsilon) \]

is proven. \( \mathcal{M}(y) \) is the Markowitz portfolio corresponding to the state \( y \). An asymptotically optimal portfolio is

\[ \tau^\varepsilon = -\frac{\alpha}{\varepsilon} (x - \mathcal{M}(y)) \]

This is a portfolio tracking the frictionless position at an optimal rate \( \alpha \).
We need to study the function

\[ \tilde{V}^\varepsilon(D, x, Y) = V^\varepsilon(D^\varepsilon, x, Y) \]

which solves the PDE

\[
\rho \tilde{V}^\varepsilon = -\frac{\gamma}{2} \sum x^2 + \mu x - RD(\varepsilon^{-1} \partial_D \tilde{V}^\varepsilon + x) + \mathcal{L}^Y(\tilde{V}^\varepsilon) \\
+ \sup_{\tau} \left\{ -\frac{1}{2} \Lambda_\varepsilon \tau^2 + \tau (\partial_x \tilde{V} + C_\varepsilon(x + \varepsilon^{-1} \partial_D \tilde{V}^\varepsilon)) \right\}
\]
A condition for wellposedness

- In Garleanu and Pedersen 2016, the authors require an inequality between $\gamma$, $\Sigma$ and $C$ to write the problem in a linear-quadratic form.
- In the asymptotic regime, the condition of Garleanu and Pedersen 2016 is satisfied.
- We don’t need this assumption. The value function is finite.
- One can find a finite supersolution to the PDE.
- By comparison principle, we obtain that value function is finite.
Theorem (Main Result)

We have the following expansion for the value function

\[ \tilde{V}^{\varepsilon}(D, x, Y) = V^0(Y) + \varepsilon u(Y) + \varepsilon w(D - M_1(Y), x - M_2(Y)) + o(\varepsilon) \]

where \( w \) is a quadratic form solving the first order corrector equation and \( u \) solves the second corrector equations.

- This is an expansion around \( V_0 \).
The function $w$

- The function $w(\xi_1, \xi_2, y)$ is a quadratic form in $(\xi_1, \xi_2)$ that solves the following PDE

\[
0 = -\frac{\gamma \xi_2^T \Sigma \xi_2}{2} - R \xi_1^T (\xi_2 + \partial_1 w) \\
+ \frac{1}{2} (C(\xi_2 + \partial_1 w) + \partial_2 w)^T \Lambda^{-1} (C(\xi_2 + \partial_1 w) + \partial_2 w)
\]

- It can be written under the form

\[
w(\xi, y) = \xi^T A(y) \xi
\]

where $A(y)$ solves an algebraic Riccati equation that can be completely solved in $1d$. 
Let $Q_x = -\sqrt{\bar{\Lambda}\gamma\Sigma}$ and $Q_D$ be the unique negative solution of

$$0 = -1 - Q_D \left( 1 - \frac{Q_x}{\bar{\Lambda}R} \right) + \frac{CQ_D^2}{2\bar{\Lambda}R}$$

Then $A$ can be written as

$$A = \begin{pmatrix}
\frac{Q_D^2}{4\bar{\Lambda}R} & \frac{1}{2} \left( 1 + \frac{Q_DQ_x}{R} \right) \\
\frac{1}{2} \left( 1 + \frac{Q_DQ_x}{R} \right) & \left( \bar{\Lambda} - \frac{CQ_D}{R} \right) \frac{Q_x}{4}
\end{pmatrix}$$

- Positive in $\xi_1$
- Negative in $\xi_2$
Expansion around which controls?

The expansion is around some $M_i$,

$$\xi_1 = \frac{D - M_1(y)}{\varepsilon^{1/2}} \text{ and } \xi_2 = \frac{x - M_2(y)}{\varepsilon^{1/2}}$$

where $M_i$ solves the linear system

$$\frac{Q^2_D}{2\Lambda} M_1 + \left( R - \frac{CQ_D}{\Lambda} \right) M_2 = 0$$

$$Q_D M_1 + (Q_x - C) M_2 = \frac{\Lambda \mu}{Q_x}$$

The determinant is always negative. The solution exists.

The definition of $M_i$ is needed to obtain that the value function converges $V^0$. 
The function $u$

Theorem

The function $u$ solves the linear PDE

$$\rho u(y) = \mathcal{L}^Y u(Y) + a(Y)$$

where $a$ depends on $M_i$.

- Linear PDE.
- Feynmann Kac representation.
- $a$ can be written as a covariation using $M_i$.
- After simplification it can be shown only depend on $d\langle M \rangle$.
- The gamma of the Markowitz portfolio is the main statistics for losses due to frictions.
The final expansion is

$$\tilde{V}^\varepsilon(D, x, Y) = V^0(Y) + \varepsilon u(Y)$$

$$+ \varepsilon w(D - M_1(Y), x - M_2(Y)) + o(\varepsilon)$$

- $M_i$ solves a linear system,
- $w$ is quadratic in $\xi$,
- $w$ solves a Ricatti equation,
- $u$ solves linear PDE.
Qualitative properties

- In Moreau, Muhle-Karbe and Soner 2015, the term $D$ does not appear and $w$ is negative.
- In our case the quadratic $w$ is negative in $\xi_2$ and positive in $\xi_1$.
- The sign in $\xi_2$ express the fact that this deviation is a friction and the agent needs to pay extra to achieve the target position.
- The sign in $\xi_1$ expresses the fact $\xi_1$ gives an anticipation on the future evolution of the prices.
Qualitative properties

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- The sign in $\xi_2$ express the fact that this deviation is a friction and the agent needs to pay extra to achieve the target position.
- The sign in $\xi_1$ expresses the fact $\xi_1$ gives an anticipation on the future evolution of the prices.
- The initial conjecture would be an expansion of the value function around $(M, 0)$ for the value function where $M$ is the Markowitz portfolio.
- The expansion for the value is around $(M_1, M_2)$ where the couple satisfies a linear equation given by the data of the problem and $w$. 
An asymptotically optimal policy for the investment problem is given by

\[ \tau^\varepsilon \ := \frac{\Lambda^{-1}}{\varepsilon} \left( Q_D(D - M_1) + Q_x(x - M_2) + C \Lambda M_2 \right). \]

- It is a scaled version of the optimal control for the linear quadratic model with impatience 0.
- Since \( D \) can be inferred from \( \tau \) one could have guessed that
  \[ \tau = -\alpha \varepsilon (x - M). \]
- By definition of \( M_i \), we can write
  \[ \tau^\varepsilon = \Lambda^{-1} \left( \frac{Q_x}{\varepsilon}(x - M) + \frac{Q_D}{\varepsilon} D \right). \]
Asymptotically Optimal Policy cont’d

- $D$ was scaled. For the initial problem an asymptotically optimal control is

$$
\tau^\varepsilon = \frac{\Lambda^{-1} Q_x}{\varepsilon} (x - M_0) + \frac{\Lambda^{-1} Q_D}{\varepsilon^2} D.
$$

where

$$
Q_x = -\sqrt{\gamma \Sigma \Lambda}
$$

and $Q_D < 0$ solves a second order polynomial equation.

- In a multidimensional problem, $Q_x, Q_D$ comes from the solution of a Riccati equation.

- It is also a scaled linear-quadratic problem in the multidimensional case.
Semi-limits

We define the ratio

\[ u^\varepsilon(D, x, y) := \frac{\tilde{V}^\varepsilon(D, x, y) - V^0(y)}{\varepsilon} \]

and the adjusted semi limits

\[ \overline{u}^\varepsilon(D, x, y) := u^\varepsilon(D, x, y) - \varepsilon w(D - M_1(y), x - M_2(y), y) \]

The difficult part of the proof is to show that this function defined after some algebraic computations converges to \( u \).

**Theorem**

" \( \limsup \) \( \overline{u}^\varepsilon(D, x, y) \) and " \( \liminf \) \( \overline{u}^\varepsilon(D, x, y) \) does not depend on \( D, x \).

We use a first order PDE.
Semi-limits, cont’d

Theorem

” lim sup ” $\bar{u}^\varepsilon (M_1(y), M_2(y), y)$ and ” lim inf ” $\bar{u}^\varepsilon (M_1(y), M_2(y), y)$

are respectively viscosity subsolution and supersolution of the second corrector equation.

By definition

” lim sup ” $\bar{u}^\varepsilon (M_1(y), M_2(y), y) \geq ” lim inf ” \bar{u}^\varepsilon (M_1(y), M_2(y), y)$.

By comparison of viscosity solution of the linear equation

” lim sup ” $\bar{u}^\varepsilon (M_1(y), M_2(y), y) \leq ” lim inf ” \bar{u}^\varepsilon (M_1(y), M_2(y), y)$.

Independence in $D, x$ implies the result.
Upper bound for $u^\varepsilon$

- We need $u^\varepsilon$ to be bounded
  - uniformly in $\varepsilon$
  - locally uniformly in $(D, x)$

- In Moreau, Muhle-Karbe and Soner (2015), the following inequality holds

$$u^\varepsilon(D, x, y) := \frac{\tilde{V}^\varepsilon(D, x, y) - V^0(y)}{\varepsilon} \leq 0$$

- In our case this inequality does not hold.

- Integration by parts and admissibility condition implies

$$u^\varepsilon(D, x, y) \leq -xD + \frac{CD^2}{2}.$$

- "lim sup" well-defined.
Obtaining lower bound

- $\tilde{V}$ is defined with a supremum.
- It is sufficient to find $\tau^\varepsilon$ such that
  \[
  \frac{\mathcal{J}(D\varepsilon^{-1}, x, Y, \tau^\varepsilon) - V^0(Y)}{\varepsilon}
  \]
  is uniformly bounded from below.
- We use a general method to obtain this lower.
- It also allow us to obtain an optimal policy in a restricted class policy.
- It is based on the so called Laplace method.
Laplace method

- For $f, g$ smooth function find an expansion for
  \[ \int_0^t e^{\lambda f(s)} g(s) ds \] when $\lambda \to \infty$.

- Only the values of $f$ and $g$ around the maximum of $f$ matters.
Laplace method

- For $f, g$ smooth function find an expansion for

$$\int_0^t e^{\lambda f(s)} g(s) ds$$

when $\lambda \to \infty$.

- Only the values of $f$ and $g$ around the maximum of $f$ matters.
- The distortion solves a linear, mean reverting ODE

$$D_t = D_0 e^{-R \varepsilon^{-1} t} + \int_0^t e^{-R \varepsilon^{-1} (t-s)} \tau_s ds$$

- The difference with the classical Laplace method is the fact that $\tau$ is not smooth but a semi-martingale.
An Ansatz

- We only need one policy to have the lower bound. We start with the following family

\[ \tau^\varepsilon = -\alpha \varepsilon (x - M_0(y)) - \beta \varepsilon D. \]

- The couple \((D_t, x_t)\) solves a linear mean-reverting type ODE.
- We need approximation for

\[ \int_0^t \varepsilon^{\alpha(t-s)} M_s ds \]

where \(M\) is a semi martingale satisfying

\[ dM_s = a_s ds + r_s dW_s. \]
Laplace method

Theorem (Laplace Method for Semi-martingales)

Under assumptions we have the following expansion

\[ \int_0^t e^{-\alpha(t-s)} M_s ds \sim_{\alpha \to \infty} \frac{M_t}{\alpha} - \frac{\alpha_t}{\alpha^2} - \frac{r_t G_\alpha(t)}{\alpha \sqrt{2\alpha}} + \text{negligible} \]

- where
  \[ G_\alpha(t) = \sqrt{2\alpha} \int_0^t e^{-\alpha(t-s)} dW_s \sim N(0, 1 - e^{-2\alpha t}) \to N(0, 1). \]
- We also need asymptotics for
  \[ \mathbb{E}[G_\alpha(t) \tilde{M}_t] \sim \mathbb{E} \left[ \frac{d\langle \tilde{M} \rangle}{dt} \right] \sqrt{2\alpha^{-1}} \]

for \( M \) semimartingale with absolutely continuous quadratic variation.
Under regularity assumption on $\mathcal{M}$ we can compute the following expansion for the ansatz

$$\mathcal{J}(D\epsilon^{-1}, x, Y, \tau^\epsilon) = V^0(Y) - \epsilon \frac{\alpha^2 R(\alpha+2C+R)+(\alpha R+(\beta C+R)^2)}{4\alpha R(\alpha+\beta C+R)} \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \gamma \Sigma_t \frac{d\langle \mathcal{M} \rangle_t}{dt} dt \right] + o(\epsilon)$$

One can choose any $\alpha$ and $\beta$ to obtain the boundedness from below of $u^\epsilon$.

"Liminf" well-defined. All assumption for the viscosity approach works.

Thus, there is an asymptotically optimal policy of the form of the ansatz.
Final Remarks

- A posteriori

\[
\frac{\alpha^2 R(\alpha + 2C + R) + (\alpha R + (\beta C + R)^2)}{2\alpha R(\alpha + \beta C + R)}
\]

is indeed the best rate.

- \(\beta = 0\) does not give the maximum.

- The function has a maximum at \(\alpha = -Q_x\) and \(\beta = -Q_D\).

- Without viscosity approach this is only an optimiser within the prescribed class.

- With viscosity approach, we obtained an asymptotically optimal portfolio.

- The viscosity approach is necessary to get the structural results on the shape of the optimal policy.
THANK YOU!
Why does it work?

Let $G^\varepsilon$ be the generator of $E^\varepsilon$, and

$$
\psi^\varepsilon(D, x, y) = v(y) + \varepsilon u\phi(D, x, y) + \varepsilon \omega(X - M(y))
$$

with $v, u, \omega$ smooth. Then the following expansion holds,

$$
G^\varepsilon(D, x, y, \partial D \psi^\varepsilon, \partial_x \psi^\varepsilon, \partial_y \psi^\varepsilon, \partial_{yy} \psi^\varepsilon) = L^Y(v) + \mu T \Sigma^{-1} \frac{\mu}{2\gamma} + E_2(\partial_D \phi(D, x, y), \partial_x \phi(D, x, y), y) + E_1(y, D - M_1, x - M_2, \partial_1 \omega(X - M), \partial_2 \omega(X - M)) + B(X - M, \partial_1 \omega(X - M), \partial_2 \omega(X - M))
$$

where

- $E_1$ is the generator of the equation solved by $w$,
- $E_2$ is the PDE giving us the fact that $u$ does not depend on $D, x$
- $B$ is 0 if $(D, x) = (M_1, M_2)$.
- $\Gamma$ contains the second corrector equation at the highest order.