Dynamic Programming for Mean Field Control with Numerical Applications

Mathieu LAURIÈRE

joint work with Olivier Pironneau

University of Michigan, January 25, 2017
Mean Field Control

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Outline

1. Mean field control and mean field games
2. Dynamic programming for MFC
3. Numerical example 1: oil production
4. Numerical example 2: Bertrand equilibrium
5. Conclusion
Outline

1. Mean field control and mean field games
   - Mean field type control problems
   - Comparison with mean field games

2. Dynamic programming for MFC

3. Numerical example 1: oil production

4. Numerical example 2: Bertrand equilibrium

5. Conclusion
Mean field control and mean field games
  Mean field type control problems
  Comparison with mean field games

Dynamic programming for MFC

Numerical example 1: oil production

Numerical example 2: Bertrand equilibrium

Conclusion
A stochastic control problem is typically defined by:

**Cost function** (running cost $L$, final cost $h$, control $v$, time horizon $T$)

$$\mathcal{J}(v) = \mathbb{E} \left[ \int_0^T L(t, X_t^v, v_t) dt + h(X_T^v) \right]$$

**Dynamics** (drift $g$, volatility $\sigma$, Brownian motion $W$)

Let $X^v$ be a solution of

$$dX_t^v = g(t, X_t^v, v_t) dt + \sigma dW_t,$$
Optimal Control (formal)

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Control Problem: Minimise $J(v)$

i.e., find $\hat{v}$ such that $J(\hat{v}) \leq J(v)$, for all control $v$
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\[i.e., \text{find } \hat{v} \text{ such that } J(\hat{v}) \leq J(v), \text{ for all control } v\]

**Remark:** the state is given by $X^v$
Example: Min-variance portfolio selection

Let $X_t$ be the value of a self-financing portfolio, with dynamics

$$dX_t = (r_tX_t + (\alpha_t - r_t)v_t)dt + v_t dW_t, \quad X_0 = x_0 \text{ given},$$

investing $v_t$ in a risky asset $S_t$ and the rest in a non-risky asset $B_t$:

$$\begin{cases}
    dS_t = \alpha_t S_t dt + S_t dW_t, & S_0 \text{ given}, \\
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\end{cases}$$

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---

1 [Andersson-Djehiche]
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$$= \mathbb{E}[X_T] - \mathbb{E}[(X_T)^2] + (\mathbb{E}[X_T])^2$$

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$$\quad \text{non-linear in } \mathbb{E}$$

$$J(v) = \mathbb{E} \left[ X_T - (X_T)^2 + (\mathbb{E}[X_T])^2 \right]$$

[1][Andersson-Djehiche]
Mean Field Control: definition (formal)

A problem of mean field control (MFC) or control of McKean-Vlasov (MKV) dynamics consists in:

**Cost function** (running cost $L$, final cost $h$, control $v$, time horizon $T$)

$$
J(v) = \mathbb{E} \left[ \int_0^T L[m_{X^v}(t, X^v_t, v_t)] dt + h[m_{X^v_T}(X^v_T)] \right]
$$

**Dynamics** (drift $g$, volatility $\sigma$, Brownian motion $W$)

Let $X^v$ be a solution of the *controlled MKV equation*

$$
\begin{aligned}
    dX^v_t &= g[m_{X^v_t}(t, X^v_t, v_t)] dt + \sigma dW_t, \\
    m_{X^0} &= m_0 \text{ given},
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$$

where $m_{X^v_t}$ is the distribution of $X^v_t$.

---

2 [Bensoussan-Frehse-Yam]
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**MFTC Problem: Minimise $J(v)$**

i.e., find $\hat{v}$ such that $J(\hat{v}) \leq J(v)$, for all control $v$
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MFC vs MFG: motivations

Mean field control (MFC) problem

(1) typical agent optimizing a cost depending on the state distribution
    ⇒ risk management, . . .

(2) collaborative equilibrium with a continuum of agents
    ⇒ distributed robotics, . . .
MFC vs MFG: motivations

Mean field control (MFC) problem

1. typical agent optimizing a cost depending on the state distribution
   ⇒ risk management, . . .

2. collaborative equilibrium with a continuum of agents
   ⇒ distributed robotics, . . .

Mean field game (MFG)

Nash equilibrium in a game with a continuum of agents
   ⇒ economy, sociology, . . .
MFC vs MFG: frameworks

Minimise $J(v, \mu) = \mathbb{E} \left[ \int_0^T L[\mu_t](t, X_t^v, \nu_t) dt + h[\mu_T](X_T^v) \right]$
MFC vs MFG: frameworks

Minimise \( J(v, \mu) = \mathbb{E} \left[ \int_0^T L[\mu_t](t, X^v_t, v_t)\,dt + h[\mu_T](X^v_T) \right] \)

MFC problem

Find \( \hat{v} \) such that

\[ J \left( \hat{v}, m_{X^\hat{v}} \right) \leq J \left( v, m_{X^v} \right), \quad \forall v \]

where \( X^v \) satisfies

\[ dX_t = g \left[ m_{X^v_t} \right] (t, X_t, v_t)\,dt + \sigma dW_t, \quad m_{X_0} = m_0, \]

and \( m_{X^v_t} \) is the distribution of \( X^v_t \).

---

\[ [\text{Bensoussan-Frehse-Yam, Carmona-Delarue}] \]
MFC vs MFG: frameworks

Minimise \( \mathcal{J}(v, \mu) = \mathbb{E} \left[ \int_0^T L[\mu_t](t, X^v_t, v_t)dt + h[\mu_T](X^v_T) \right] \)

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Find \( \hat{v} \) such that
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MFG

Find \((\hat{v}, \mu)\) such that
\[
\mathcal{J} \left( \hat{v}, \mu \right) \leq \mathcal{J} \left( v, \mu \right), \quad \forall v
\]
where

(i) \( X_{\mu}^{\hat{v}} \) satisfies
\[
dX_t = g[\mu_t](t, X_t, \hat{v}_t)dt + \sigma dW_t, \quad m_{X_0} = m_0,
\]
(ii) \( \mu \) coincides with \( m_{X_{\mu}^{\hat{v}}} \).

\[\text{[Bensoussan-Frehse-Yam, Carmona-Delarue]}\]
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   - Dynamic programming principle
   - Link with calculus of variations

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Formulation with **McKean-Vlasov dynamics:**

\[
\mathcal{J}(v) = \mathbb{E} \left[ \int_0^T L[m_{X_t^v}](t, X_t, v_t)dt + h[m_{X_T^v}](X_T) \right]
\]

where

\[
\begin{cases} 
  dX_t^v = g[m_{X_t^v}](v_t)dt + \sigma dW_t \\
  X_0 \text{ given}
\end{cases}
\]

and \(m_{X_t^v}\) is the distribution of \(X_t^v\).
MFC rewritten

Formulation with McKean-Vlasov dynamics:

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where

\[ \begin{align*}
  \frac{dX_t^v}{dt} &= g[m_{X_t}^v](v_t) dt + \sigma dW_t \\
  X_0 &\text{ given}
\end{align*} \]

and \( m_{X_t}^v \) is the distribution of \( X_t^v \).

We can see the distribution as part of the state:

Formulation with Fokker-Planck PDE:

\[ \mathcal{J}(v) = \int \int_0^T L[m^v(t, \cdot)](t, x, v_t) m^v(t, x) dt dx + \int h[m^v(T, \cdot)](x) dx \]

where

\[ \begin{align*}
  \partial_t m^v - \frac{\sigma^2}{2} \Delta m^v + \text{div} \left( m^v g[m^v](v) \right) &= 0 \\
  m^v(0, x) &= m_0(x) \quad \text{given.}
\end{align*} \]
Dynamic Programming Principle

Let $V[m_\tau](\tau) = \min_v J(\tau, v)$ (problem starting $\tau$).

Theorem (Dynamic Programming Principle)

For all $\tau \in [0, T]$ and all $m_\tau \geq 0$ on $\mathbb{R}$:

$V[m_\tau^v](\tau) = \min_v \left\{ \int_\tau^{\tau+\delta \tau} \int_\mathbb{R} L[m^v(t, \cdot)](t, x, v_t)m^v(t, x)dxdt + V[m_\tau^v+\delta \tau](\tau + \delta \tau) \right\}$

\[
\begin{align*}
\partial_t m^v - \frac{\sigma^2}{2} \Delta m^v + \text{div} \left( m^v g[m^v](v) \right) &= 0 \\
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$$\begin{cases} 
\partial_t m^\nu - \frac{\sigma^2}{2} \Delta m^\nu + \text{div} \left( m^\nu g[m^\nu](v) \right) = 0 \\
m^\nu(0, x) = m_0(x) \quad \text{given.}
\end{cases}$$

Assume that $V$ and $L$ are Fréchet differentiable in $m$.

Theorem (Hamilton-Jacobi-Bellman minimum principle)

Let $\partial_m V$ be the Fréchet derivative of $V$ and $V'$ be its Riesz representation:

$$\int_{\mathbb{R}^d} V'[m](\tau)(x)\nu(x)dx = \partial_m V[m](\tau) \cdot \nu, \quad \forall \nu \in L^2.$$

If $V'$ is smooth enough,

$$\min_v \int_\mathbb{R} \left( L[m^\nu_\tau](x, \tau, v) + \partial_m L[m^\nu_\tau](x, \tau, v) \cdot m^\nu_\tau + \partial_\tau V' + \frac{\sigma^2}{2} \partial_{xx} V' + v \cdot \partial_x V' \right)m^\nu_\tau dx = 0$$
Proof of HJB min. principle (formal) 1/2

A first order approximation of the time derivative in the FP eq. yields:

$$\delta_\tau m := m_{\tau + \delta_\tau} - m_\tau = \delta_\tau \left[ \frac{\sigma^2}{2} \Delta m_\tau - \text{div}(v_\tau m_\tau) \right] + o(\delta_\tau). \quad (1)$$

As $V$ is assumed to be smooth, we have:

$$V[m_{\tau + \delta_\tau}](\tau + \delta_\tau) = V[m_\tau](\tau) + \partial_\tau V[m_\tau](\tau) \delta_\tau + \partial_m V[m_\tau](\tau) \cdot \delta_\tau m + o(\delta_\tau). \quad (2)$$

Then, by Bellman's principle

$$V[m_\tau](\tau) \simeq \min_v \left\{ \delta_\tau \int_{\mathbb{R}^d} L[m_\tau]m_\tau \, dx + V[m_\tau](\tau) + \partial_\tau V[m_\tau](\tau) \delta_\tau + \partial_m V[m_\tau](\tau) \cdot \delta_\tau m \right\}. \quad (3)$$

Divided by $\delta_\tau$ and combined with (1), letting $\delta_\tau \to 0$ gives

$$0 = \min_v \left\{ \int_{\mathbb{R}^d} L[m_\tau]m_\tau \, dx + \partial_\tau V[m_\tau](\tau) + \partial_m V[m_\tau](\tau) \cdot \left[ \frac{\sigma^2}{2} \Delta m_\tau - \text{div}(v_\tau m_\tau) \right] \right\}. \quad (4)$$

To finalize the proof we need to relate $V$ to $\partial_m V$. 

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DPP for MFC  
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**Proposition**

Let \((\hat{v}, \hat{m})\) denote an optimal solution to the problem starting from \(m_\tau\) at time \(\tau\). Then:

\[
\int_{\mathbb{R}^d} V'[m_\tau](\tau) m_\tau \, dx = V[m_\tau](\tau) + \int_\tau^T \int_{\mathbb{R}^d} \left( \partial_m L[\hat{m}_t](x, t, \hat{v}) \cdot \hat{m}_t \right) \hat{m}_t \, dx \, dt \\
+ \int_{\mathbb{R}^d} \left( \partial_m h[\hat{m}_T](x) \cdot \hat{m}_T \right) \hat{m}_T \, dx.
\]

Differentiating with respect to \(\tau\) leads to

\[
\partial_\tau V[m_\tau](\tau) = \int_{\mathbb{R}^d} \partial_\tau V'[m_\tau](\tau) m_\tau \, dx + \int_{\mathbb{R}^d} \left( \partial_m L[m_\tau](x, \tau, \hat{v}_\tau) \cdot m_\tau \right) m_\tau \, dx,
\]

where \(\hat{v}_\tau\) is the optimal control at time \(\tau\). Now, let us use (4), rewritten as

\[
0 = \min_{u_\tau} \left\{ \int_{\mathbb{R}^d} \left( L[m_\tau](x, \tau, u_\tau(x)) + \partial_m L[m_\tau](x, \tau, u_\tau(x)) \cdot m_\tau \right) m_\tau \, dx \\
+ \int_{\mathbb{R}^d} \left( \partial_\tau V'[m_\tau](\tau) m_\tau + V'[m_\tau](\tau) \left[ \frac{\sigma^2}{2} \Delta m_\tau - \text{div}(u_\tau m_\tau) \right] \right) \, dx \right\}.
\]

Integrating by parts the last term concludes the proof.
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Dynamic Programming in a specific setting

\[
\min_{\nu} \int_{\mathbb{R}} \left( L[m_\tau](x, \tau, \nu) + \partial_m L[m_\tau](x, \tau, \nu) \cdot m_\tau + \partial_\tau V' + \frac{\sigma^2}{2} \partial_{xx} V' + \nu \cdot \partial_x V' \right) m_\tau \, dx = 0
\]

Assume \( L = \tilde{L}(x, t, \nu, m_t(x), \chi(t)) \) with \( \chi(t) = \int_{\mathbb{R}^d} \tilde{h}(x, t, \nu(x, t), m_t(x)) m_t(x) \, dx \).

Then for all \( \nu \in L^2 \):

\[
\partial_m L[m_t](x, t, u) \cdot \nu = \partial_m \tilde{L} \nu + \left( \int_{\mathbb{R}^d} \partial_\chi \tilde{L} \nu \, dx \right) \tilde{h} + \left( \int_{\mathbb{R}^d} \partial_\chi \tilde{L} m_t \, dx \right) \nu \partial_m \tilde{h}.
\]

In particular, for \( \nu = m_t \) we have:

\[
\partial_m L[m_t](x, t, u) \cdot m_t = \partial_m \tilde{L} m_t + \left( \int_{\mathbb{R}^d} \partial_\chi \tilde{L} m_t \, dx \right) (\tilde{h} + m_t \partial_m \tilde{h}).
\]

Thus, for optimal \( \hat{\nu} \) and \( \hat{m} \),

\[
\partial_t V' + \frac{\sigma^2}{2} \partial_{xx} V' + \hat{\nu} \cdot \partial_x V' = - \left[ \tilde{L} + \hat{m} \partial_m \tilde{L} + (\tilde{h} + \hat{m} \partial_m \tilde{h}) \int_{\mathbb{R}^d} \partial_\chi \tilde{L} \hat{m} \, dx \right]
\]

where \( \partial_m \tilde{L}, \partial_\chi \tilde{L}, \) and \( \partial_m \tilde{h} \) are partial derivatives in the classical sense.
Recall: $L = \tilde{L}(x, t, v, m_t(x), \chi(t))$ with $\chi(t) = \int_{\mathbb{R}^d} \tilde{h}(x, t, v(x, t), m_t(x))m_t(x)dx$.

**Theorem (calculus of variations)**

$\hat{v}$ and $\hat{m}$ are optimal only if for all $t$ and $v$,

$$
\int_{\mathbb{R}^d} \left( \partial_v \tilde{L} + \partial_v \tilde{h} \int_{\mathbb{R}^d} \partial_x \tilde{L} \hat{m} dy + \partial_x m^* \right) (v - \hat{v})\hat{m} dx \geq 0
$$

where $m^*$ satisfies

$$
\partial_t m^* + \frac{\sigma^2}{2} \partial_{xx} m^* + \hat{v} \cdot \partial_x m^* = - \left[ \tilde{L} + \hat{m} \partial_m \tilde{L} + (\tilde{h} + \hat{m} \partial_m \tilde{h}) \int_{\mathbb{R}^d} \partial_x \tilde{L} \hat{m} dy \right]
$$

**Link with dynamic programming**

$V'$ coincides with $m^*$, the adjoint state of $m$. 

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DPP for MFC

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A (toy) model of oil production

Setting: \textit{continuum of producers exploiting an oil field (limited resource)}
A (toy) model of oil production\textsuperscript{5}

Setting: \textit{continuum of producers exploiting an oil field (limited resource)}

Remaining quantity dynamics

\[
dX_t = -a_t dt + \sigma X_t dW_t, \quad X_0 \text{ given by its PDF,}
\]

- \(X_t\) = \text{quantity} of oil left in the field at time \(t\) (seen by a producer)
- \(a_t dt\) = \text{quantity extracted} by the producer during \((t, t + dt)\)
- \(W\) = standard Brownian motion (incertitude), \(\sigma > 0\) = volatility (constant)
- \(a_t = a(X_t, t)\) = feedback law to \text{control} the production.

\textbf{Price}

\[
C = \text{cost of extraction} = C(a) = \alpha a + \beta a^2
\]

- \(\alpha > 0\), \(\beta > 0\).

\[
p_t = \kappa e^{-bt} \left(\mathbb{E}(a_t)\right) - c = \text{price of oil, where:} \quad \kappa > 0, \quad b > 0, \quad c > 0.
\]

\textbf{Intuition:}

- \(p\) decreases with mean production and time because
  - scarcity of oil increases its price and conversely.
  - future oil will be cheaper because it will be replaced by renewable energy.

\textsuperscript{5}[Guéant-Lasry-Lions]
A (toy) model of oil production\textsuperscript{5}

Setting: continuum of producers exploiting an oil field (limited resource)

Remaining quantity dynamics

\[ dX_t = -a_t dt + \sigma X_t dW_t, \quad X_0 \text{ given by its PDF}, \]

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Price

- \( C = \text{cost of extraction} = C(a) = \alpha a + \beta a^2 \) where: \( \alpha > 0 \) and \( \beta > 0 \).
- \( p_t = \kappa e^{-bt(\mathbb{E}(a_t))^{-c}} \) = price of oil, where: \( \kappa > 0 \), \( b > 0 \) and \( c > 0 \).

\textsuperscript{5}[Guéant-Lasry-Lions]

M. Laurière
DPP for MFC
A (toy) model of oil production

Setting: continuum of producers exploiting an oil field (limited resource)

Remaining quantity dynamics

\[ dX_t = -a_t dt + \sigma X_t dW_t, \quad X_0 \text{ given by its PDF,} \]

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- \( p_t = \kappa e^{-bt (\mathbb{E}(a_t))^{-c}} = \text{price of oil, where:} \ \kappa > 0, b > 0 \) and \( c > 0 \).

Intuition: \( p \) decreases with mean production and time because
- \( \text{scarcity} \) of oil increases its price and conversely.
- future oil will be cheaper because it will be replaced by renewable energy.

---

5 [Guéant-Lasry-Lions]
Optimisation

Goal:

Maximise over $a(\cdot, \cdot) \geq 0$ the profit:

$$J(a) = \mathbb{E} \left[ \int_0^T (p_t a_t - C(a_t))e^{-rt} \, dt \right] - \gamma \mathbb{E}[|X_T|^\eta]$$

subj to: $dX_t = -a_t \, dt + \sigma X_t \, dW_t$, $X_0$ given

with $\gamma$ and $\eta = \text{penalisation}$ parameters (encouraging to consume before $T$).
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subject to: $dX_t = -a_t dt + \sigma X_t dW_t$, $X_0$ given

with $\gamma$ and $\eta$ = penalisation parameters (encouraging to consume before $T$).

Replacing $p$ and $C$ by their expressions gives

$$J(a) = \mathbb{E} \left[ \int_0^T (\kappa e^{-bt} (\mathbb{E}[a_t])^{-\alpha} a_t - \alpha_a - \beta(a_t)^2) e^{-rt} dt \right] - \gamma \mathbb{E}[|X_T|^\eta]$$

Remark: $J =$ mean of a function of $\mathbb{E}[a_t]$ so it is a MFC problem
Remarks on Existence of Solutions

**Sufficient condition**

If $c < 1$ and $\bar{a}_t$ upper bounded on $[0, T]$, $J(a) \leq \int_0^T \left( c\alpha + (1 + c)\beta\bar{a}_t \right) \frac{\bar{a}_t e^{-rt}}{1 - c} dt \leq C$. 

A counter example:

If $c > 1$ and $a_t = |\tau - t|$ for some $\tau \in (0, T)$, then the problem is not well posed (nobody extract oil $\Rightarrow$ infinite price).
Remarks on Existence of Solutions

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A counter example: if $c > 1$ and $a_t = |\tau - t|$ for some $\tau \in (0, T)$, then the problem is not well posed (nobody extract oil $\Rightarrow$ infinite price).

Linear feedback case

Assume $a(x, t) = w(t)x$. Then there is an analytical solution:

$$X_t = X_0 \exp \left( - \int_0^t w(\tau) d\tau - \frac{\sigma^2}{2} t + \sigma (W_t - W_0) \right).$$

For $\eta = 2$, the problem reduces to maximizing over $\tilde{w}_t = w(t) e^{-\int_0^t w(\tau) d\tau} \geq 0$

$$J(\tilde{w}_t) = \int_0^T \left( \kappa e^{-bt} \mathbb{E}[X_0]^{1-c} \tilde{w}_t^{1-c} - \alpha \mathbb{E}[X_0] \tilde{w}_t - \beta \mathbb{E}[X_0^2] \tilde{w}_t^2 e^{\sigma^2 t} \right) e^{-rt} dt$$

$$- \gamma \mathbb{E}[X_0^2] e^{\sigma^2 T - 2 \int_0^T w(\tau) d\tau}$$
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Dynamic Programming

Let \( u = -a = \text{depletion rate (control)} \).

Ignore the constraints on \( X_t \in [0, L] \) and \( u \leq 0 \): let \( X_t, u_t \in \mathbb{R} \) (see the numerical results).

Fokker-Planck eq. for \( \rho(\cdot, t) = \text{density of } X_t \)

\[
\partial_t \rho - \frac{\sigma^2}{2} \partial_{xx}(x^2 \rho) + \partial_x (\rho u) = 0 \quad (x, t) \in \mathbb{R} \times (0, T), \quad \rho|_{t=0} = \rho_0 \quad \text{(FP)}
\]

Minimise, subject to (FP) with \( \rho|_{t=\tau} = \rho_\tau \),

\[
\tilde{J}(\tau, \rho_\tau, u) = \int_{\tau}^{T} \int_{\mathbb{R}} \left( \kappa e^{-\beta t} (-\bar{u}_t)^{-c} u_t - \alpha u_t + \beta \bar{u}_t^2 \right) e^{-rt} \rho_t dx dt + \int_{\mathbb{R}} \gamma |x|^{\eta} \rho|_T(x) dx
\]
Dynamic Programming

Let $u = -a = \text{depletion rate (control)}$.

Ignore the constraints on $X_t \in [0, L]$ and $u \leq 0$: let $X_t, u_t \in \mathbb{R}$ (see the numerical results).

Fokker-Planck eq. for $\rho(\cdot, t) = \text{density of } X_t$

$$\partial_t \rho - \frac{\sigma^2}{2} \partial_{xx}(x^2 \rho) + \partial_x (\rho u) = 0 \quad (x, t) \in \mathbb{R} \times (0, T), \quad \rho|_{t=0} = \rho_0$$ (FP)

Minimise, subject to (FP) with $\rho|_{t=\tau} = \rho_\tau$,

$$\tilde{J}(\tau, \rho_\tau, u) = \int_{\tau}^{T} \int_{\mathbb{R}} \left( \kappa e^{-bt}(\overline{u}_t)^{-c} u_t - \alpha u_t + \beta u_t^2 \right) e^{-rt} \rho_t dx dt + \int_{\mathbb{R}} \gamma |x|^\eta \rho|_T(x) dx$$

DPP for $V[\rho_\tau](\tau) = \min_u \tilde{J}(\tau, \rho_\tau, u)$

$$u(x, t) = \frac{1}{2\beta} \left[ \alpha - e^{rt} \partial_x V' - \kappa (1 - c) e^{-bt} (-\overline{u})^{-c} \right]$$ (EU)

$$\partial_t V' + \frac{\sigma^2 x^2}{2} \partial_{xx} V' = \frac{e^{-rt}}{4\beta} \left( \alpha - e^{rt} \partial_x V' - \kappa (1 - c) e^{-bt} (-\overline{u})^{-c} \right)^2$$ (DV)

(depends only on $\overline{u}$ and not on $u$)
Algorithm 1: Fixed point iteration (parameter $\omega \in (0, 1)$)

**INITIALIZE:** set $u = u_0$, $i = 0$

**REPEAT:**

- Compute $\rho_i$ by solving (FP)
- Compute $\bar{u}_i = \int_{\mathbb{R}} u_i \rho_i$
- Compute $V'_i$ by (DV)
- Compute $\tilde{u}_{i+1}$ by (EU) and set $u_{i+1} = u_i + \omega(\tilde{u}_{i+1} - u_i)$
- Set $i = i + 1$

**WHILE** not converged.
**Algorithm 1: Fixed point iteration (parameter \( \omega \in (0, 1) \))**

**INITIALIZE:** set \( u = u_0, \ i = 0 \)

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- Set \( i = i + 1 \)

**WHILE** not converged.

**Open questions:**

- (FP) eq.: existence of solution ?
- relevant stopping criteria ? *(compare with Riccati, see later)*
- \( 2^{nd} \) order term vanishes at \( x = 0 \). Model does not impose \( u(0, t) = 0 \Rightarrow singularity \)
Introduce an adjoint $\rho^*$ satisfying: $\rho^* |_{T} = \gamma|x|^n$, and in $\mathbb{R} \times (0, T)$

$$
\partial_t \rho^* + \frac{\sigma^2 x^2}{2}\partial_{xx} \rho^* + u\partial_x \rho^* = e^{-rt}(\alpha - \beta u - \kappa(1 - c)e^{-bt}(-\bar{u})^{-c})u
$$

(Adj)

Then

$$
\text{Grad}_u J = -\left(e^{-rt}(\alpha - 2\beta u - \kappa(1 - c)e^{-bt}(-\bar{u})^{-c}) - \partial_x \rho^*\right)\rho
$$

(DJ)
Calculus of Variations on the Deterministic Ctrl Pb

Introduce an adjoint $\rho^*$ satisfying: $\rho^*|_T = \gamma|x|^n$, and in $\mathbb{R} \times (0, T)$

$$\partial_t \rho^* + \frac{\sigma^2 x^2}{2} \partial_{xx} \rho^* + u \partial_x \rho^* = e^{-rt}(\alpha - \beta u - \kappa(1 - c)e^{-bt}(-\bar{u})^{-c})u$$  

(Adj)

Then

$$\text{Grad}_u J = -\left( e^{-rt}(\alpha - 2\beta u - \kappa(1 - c)e^{-bt}(-\bar{u})^{-c}) - \partial_x \rho^* \right) \rho$$  

(DJ)

Algorithm 2: Steepest descent (parameter $0 < \epsilon \ll 1$)

INITIALIZE: $a = a_0$ and $i = 0$

REPEAT:

- Compute $\rho_i$ by (FP) with $\rho_i|_{t=0}$ given
- Compute $\bar{u}_i = \int_{\mathbb{R}} u_i \rho_i \, dx$
- Compute $\rho^*_i$ by (Adj)
- Compute $\text{Grad}_u J$ by (DJ)
- Compute a feasible descent step $\mu_i \in \mathbb{R}$ by Armijo rule
- Set $u_{i+1} = u_i - \mu_i \text{Grad}_u J$, $i = i + 1$

WHILE ($\|\text{Grad}_u J\| > \epsilon$)

Remark: the asymptotic behaviour of $u$ as $x \to \infty$ can be an issue
Riccati Equation when $\eta = 2$

Let $\eta = 2$, look for $V'$ in the form:

$$V'(x, t) = P(t)x^2 + Z(t)x + S(t)$$

Let $Q_t = e^{rt}P_t$ and $\mu = \sigma^2 - r$. For $\beta e^{rt}\mu - Q_t > 0$, (DV) leads to

$$P_t = \frac{4\beta\mu e^{(T-t)\mu}}{\gamma e^{(T-t)\mu} - \gamma + 4\beta\mu}.$$ 

Then:

- $u$ is found by (EU):

$$u(x, t) = \frac{1}{2\beta} \left[ \alpha - e^{rt} \partial_x V' - \kappa(1 - c)e^{-bt}(-\bar{u})^{-c} \right] \quad \text{(EU)}$$

- in particular $\partial_x u = -\frac{1}{8\beta} \partial_{xx} V' = -\frac{1}{4\beta} P_t$

- but the Fokker-Planck eq. must be solved numerically to compute $\bar{u}$.

Remark:

- we can also identify $Z$ and $S$

- $u(\cdot, t) : x \mapsto 2xP(t) + Z(t)$ is not a linear feedback
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Localisation

Fix large $L$ and $T$. Consider $(x, t) \in (0, L) \times (0, T)$ with $\rho(L, t) = 0$, $\forall t$.

The solution is sensitive to the **boundary conditions**. When $\eta = 2$,

$$\frac{1}{2} \sigma^2 x^2 \partial_x V' = \sigma^2 x^3 P_t = \sigma^2 x V'$$

⇒ use this as boundary condition for $V'$
Numerical implementation

**Localisation**

Fix large $L$ and $T$. Consider $(x, t) \in (0, L) \times (0, T)$ with $\rho(L, t) = 0$, $\forall t$.

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$\Rightarrow$ use this as boundary condition for $V'$

**Discretization**

**space-time finite element** method of degree 1 over $(0, L) \times (0, T)$. Using *freefem++*. 
Localisation

Fix large \( L \) and \( T \). Consider \((x, t) \in (0, L) \times (0, T)\) with \( \rho(L, t) = 0, \forall t \).

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\[
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\]

\( \Rightarrow \) use this as boundary condition for \( V' \)

Discretization

**space-time finite element** method of degree 1 over \((0, L) \times (0, T)\). Using freefem++.

Parameters

- 50 points in space and 50 in time, \( L = 10, T = 5 \)
- \( \alpha = 1, \beta = 1, \gamma = 0.5, \kappa = 1, b = 0.1, r = 0.05, \sigma = 0.5 \) and \( c = 0.5 \)
- \( \rho_0 = \) Gaussian curve centred at \( x = 5 \) with volatility 1
- \( u_0 = -\alpha/(2\beta) \)
**Numerical Implementation: Fixed point Algo**

**Non-linearity** of eq. (DV): semi-linearise it using the **iterative loop**

**Stopping criteria:** error $||u - u_e||$, $u_e =$ local min from Ricatti eq. Parameter $\omega = 0.5$.

Optimal $u(x, t)$ and the Ricatti solution slightly below

PDF of resource $X_t$: $\rho(x, t)$
**Numerical Implementation: Fixed Point Algo**

**Non-linearity** of eq. (DV): semi-linearise it using the iterative loop.

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Optimal $u(x, t)$ and the Ricatti solution slightly below.

**Remarks:**
- optimal control is linear
- resource distribution: Gaussian to concentrated around $x = 0.5$

**Convergence:** $\text{error} = \int (\partial_x u - \partial_x u_e)^2 \, dx \, dt$ versus $k = \text{iteration number}$:

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error</td>
<td>1035</td>
<td>661.2</td>
<td>8.605</td>
<td>44.7</td>
<td>3.27</td>
<td>0.755</td>
<td>0.335</td>
<td>0.045</td>
<td>0.015</td>
<td>0.003</td>
</tr>
</tbody>
</table>
Evolution of production $\bar{a}_t = -\bar{u}_t$ and price $p_t = \kappa e^{-bt}(-\bar{u}_t)^{-c}$.
Generates **different solutions** depending on $u_0$:

- $u_0 = u_e$: small **oscillations** to decrease the cost function $\Rightarrow$ mesh dependent
- $u_0 = -0.5$: solution below after 10 iterations

Another solution $u$  
The corresponding $\rho$
Numerical Implementation: Steepest Descent

Generates **different solutions** depending on $u_0$:
- $u_0 = u_e$: small **oscillations** to decrease the cost function $\Rightarrow$ mesh dependent
- $u_0 = -0.5$: solution below after 10 iterations

**Convergence:** Values of $J$ and $\|\text{Grad}_u J\|$ versus iteration number $k$

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>..</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J$</td>
<td>0.7715</td>
<td>0.2834</td>
<td>0.2494</td>
<td>0.1626</td>
<td>..</td>
<td>0.0417</td>
</tr>
<tr>
<td>$|\text{Grad}_u J|$</td>
<td>0.003395</td>
<td>0.001602</td>
<td>0.000700</td>
<td>0.000813</td>
<td>..</td>
<td>0.000794</td>
</tr>
</tbody>
</table>
Linear Feedback Solution

Steepest descent with:
- Automatic differentiation (operator overloading in C++)
- Initializing with the linear part of the Riccati solution
- Gives $w(t)$, very close to the Riccati solution.

**Why Riccati solution is not the best solution?**
Plot $J^d(\lambda) = J(w^d_t + \lambda h_t), \lambda \in (-0.5, +0.5)$, $h_t$ is an approximate $w_t - \text{Grad}J(w^d_t)$.

**Left:** $w(t)$ maximizing $J(w)$ VS feedback coef. of the Riccati solution (solid line)
**Center:** $J^d_\lambda$ as a function of $\lambda \in (-0.5, +0.5)$; Ricatti solution at $\lambda = 0$
**Right:** zoom at $\lambda = \pm 0.12$: absolute min of $J^d(.)$ (shallow and mesh dependent).
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Example: Bertrand Equilibrium

Continuum of producers, whose state is the amount of resource $\in \mathbb{R}_+$:

$$dX_t = -q(X_t, t)dt + \sigma dW_t, \ \forall t \in [0, T],$$

if $X_t > 0$ and $X_t$ is absorbed at 0, $X_0$ has density $m_0$,

where $q(x, t) = a(\eta(t))[1 + \epsilon \tilde{p}(t)] - p(x, t)$ is the quantity produced, with

$$\eta(t) = \int_{\mathbb{R}_+} m(x, t)dx : \text{the proportion of remaining producers},$$

$$\tilde{p}(t) = \int_{\mathbb{R}_+} p(x, t)m(x, t)dx : \text{the average price (non local in } p),$$

$$p(t) : \text{the price (control, same for all the agents)}.$$

Last, $a(\eta) = \frac{1}{1+\epsilon \eta}$, and $\epsilon > 0$ reflects the degree of interaction.

---

6[Chan-Sircar]
Example: Bertrand Equilibrium

Continuum of producers, whose state is the amount of resource $\in \mathbb{R}^+$:

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Last, $a(\eta) = \frac{1}{1 + \epsilon \eta}$, and $\epsilon > 0$ reflects the degree of interaction.

The goal of a typical agent is to maximise

$$J(p) = E \left[ \int_0^T e^{-rs} p(s, X_s)q(s, X_s)1_{\{X_s > 0\}}ds \right].$$

[Chan-Sircar]
PDE System

Proposition

The optimal control is given: \( p(x, t) = \frac{1}{2} \left( a(\eta(t))[1 + \epsilon \tilde{p}(t)] + \partial_x u(x, t) \right) \), and the optimal equilibrium is given by:

\[
q(x, t) = \frac{1}{2} \left[ \alpha_{MFTC} + \epsilon \int_{\mathbb{R}^+} \partial_x u(\xi, t)m(\xi, t)d\xi \right] - \partial_x u(x, t)
\]

where \( \alpha_{MFTC} = 1 \), with \((u, m)\) satisfying

\[
\begin{cases}
\partial_t u(x, t) - ru(x, t) + \frac{\sigma^2}{2} \partial_{xx} u(x, t) + \left( \psi(m(\cdot, t), \partial_x u(\cdot, t))(x) \right)^2 = 0, \\
\partial_t m(x, t) - \frac{\sigma^2}{2} \partial_{xx} m(x, t) - \partial_x \left( \psi(m(\cdot, t), \partial_x u(\cdot, t))m(\cdot, t) \right)(x) = 0,
\end{cases}
\]

with \( \psi(m(\cdot, t), \partial_x u(\cdot, t)) : x \mapsto q(x, t) \).

For the corresponding MFG, \( \alpha_{MFTC} \) is replaced by \( \alpha_{MFG} = 2.7 \).\[Bensoussan-Graber\]
Algorithm and Numerical Results

Fixed point algo. (param. $\epsilon > 0$)

INIT.: set $i = 0$, $p = p_0$, compute $\tilde{p}_0$.

REPEAT:
- Compute $u_i$, solution of HJB eq.
- Compute $p_{i+1}, \tilde{p}_{i+1}$ and $q_{i+1}$
- Compute $m_i$, solution of FP eq.
- Set $i = i + 1$

WHILE $||m_{i+1} - m_i|| > \epsilon$

Average price VS time ($\bar{p}$).

Remaining producers VS time ($\eta$).
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Conclusion

Summary:

- dynamic programming for mean field control problems
- two numerical methods
- application to economics
- follow-up articles

Current directions of research:

- proof of existence and uniqueness for the PDE system
- other numerical methods
- other applications

\[\text{Pham-Wei, Pfeiffer, \ldots}\]
Some References (very partial)

Thank you!