From Martingale Optimal Transport to McKean-Vlasov Control Problems

Xiaolu Tan

University of Paris-Dauphine, PSL University

26 September 2018
Financial/Actuarial Mathematics Seminar @U. Michigan
Outline

1 Introduction
   - Mathematical Finance Modelling
   - Stochastic Optimal Control

2 From MOT to McKean-Vlasov Control Problems
   - The McKean-Vlasov Control Problem
   - Martingale Optimal Transport (MOT)
   - From MOT to McKean-Vlasov Control

3 Solving the McKean-Vlasov Control Problems
   - Solving the McKean-Vlasov Control Problem
   - Back to the MOT-MKV Control Problem
Financial market

- **Financial market modelling**: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{T}}$, prices of some financial assets is given by $S = (S_t)_{t \in \mathbb{T}}$, the payoff at maturity $T$ of a derivative option is a random variable $\xi : \Omega \to \mathbb{R}$ (e.g. $\xi = (S_T - K)_+$).
  - Discrete time market: $\mathbb{T} = \{0, 1, \cdots, T\}$.
  - Continuous time market: $\mathbb{T} = [0, T]$.

- **Trading**: A trading strategy is predictable process $H = (H_t)_{0 \leq t \leq T}$, whose P&L given by

\[
(H \circ S)_T := \sum_{t=1}^{T} H_t(S_t - S_{t-1}).
\]
Fundamental Theorem of Asset Pricing (FTAP)

For simplicity $\mathbb{T} = \{0, 1\}$, interest rate $r = 0$.

- No arbitrage condition (NA): there is no $H$ s.t.
  \[ H(S_1 - S_0) \geq 0, \quad \text{and} \quad \mathbb{P}[H(S_1 - S_0) > 0] > 0. \]
- Equiv. martingale measures: $\mathcal{M} := \{ \mathbb{Q} \sim \mathbb{P} : \mathbb{E}^\mathbb{Q}[S_1|\mathcal{F}_0] = S_0 \}$.
- The market is complete if
  \[ \forall \xi \in \mathcal{F}, \ \exists (y, H) \text{ s.t. } y + H(S_1 - S_0) = \xi, \text{ a.s.} \]

**Theorem (FTAP (discrete time))**

(i) (NA) is equivalent to the existence of equivalent martingale measure, i.e.

\[ (\text{NA}) \iff \mathcal{M} \neq \emptyset. \]

(ii) The market is complete iff $\mathcal{M}$ is a singleton.
Basic problems in mathematical finance

- **Pricing and hedging** for a derivative option $\xi : \Omega \to \mathbb{R}$:
  - **Complete market**: Price of derivative option equals to its replication cost.
  - **Incomplete market**: Every equivalent martingale measure $Q \in \mathcal{M}$ provides a no-arbitrage price: $E^Q[\xi]$.

  A pricing-hedging duality:
  \[
  \sup_{Q \in \mathcal{M}} E^Q[\xi] = \inf \left\{ y \in \mathbb{R} : y + H(S_1 - S_0) \geq \xi, \text{ a.s.} \right\}.
  \]

- **Utility maximization** Let $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ be a utility function, we consider
  \[
  \max_H E \left[ U(x + (H \circ S)_T) \right].
  \]
Concrete market models

- The simplest martingales:
  - Random walk: \( \Delta X_t := X_t - X_{t-1} \perp (X_0, \ldots, X_{t-1}) \) and
    \[
    \mathbb{P}[\Delta X_t = 1] = \mathbb{P}[\Delta X_t = -1] = \frac{1}{2}.
    \]
  - Brownian motion: \( W = (W_t)_{t \geq 0} \) is a continuous process with independent and centred stationary increment.

- Two basic models:
  - Binomial tree model: for \( 0 < d < 1 < u \),
    \[
    \mathbb{P}[S_t = dS_{t-1}] = p, \quad \mathbb{P}[S_t = uS_{t-1}] = 1 - p.
    \]
  - Black-Scholes model: \( S_t = S_0 \exp(-\frac{1}{2}\sigma^2 t + \sigma W_t) \), or equivalently
    \[
    dS_t = \sigma S_t dB_t.
    \]
Diffusion process, SDE and PDE

- **Diffusion process** defined by SDE (stochastic differential equation)

\[ dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t. \]

- **Kolmogorov forward equation** (Fokker-Planck equation) on the density \( p(t, x) \) of \( X_t \) :

\[ \partial_t p(t, x) + \partial_x (b(t, x)p(t, x)) - \frac{1}{2} \partial_{xx}^2 (\sigma^2(t, x)p(t, x)) = 0. \]

- **Kolmogorov backward equation** (Feynman-Kac formula) on \( u(t, x) := \mathbb{E}[g(X_{T}^{t, x})] \) :

\[ \partial_t u(t, x) + b(t, x)\partial_x u(t, x) + \frac{1}{2} \sigma^2(t, x)\partial_{xx}^2 u(t, x) = 0. \]
Calibration, local volatility model

- Assume that the price $C(T, K) = \mathbb{E}[(S_T - K)_+]$ is known for all $K \in \mathbb{R}$, then one can obtain the distribution $p(T, K)$ of $S_T$ by

$$\partial_K C(T, K) = -\mathbb{E}[\mathbb{1}_{S_T \geq K}], \quad \partial_{KK}^2 C(T, K) = \mathbb{E}[\delta_K(S_T)] = p(T, K).$$

- Dupire’s local volatility model : $dS_t = \sigma_{loc}(t, S_t)dW_t$, then the Fokker-Planck equation leads to

$$\partial_T C(T, K) - \frac{1}{2} \sigma_{loc}^2(T, K) \partial_{KK}^2 C(T, K) = 0.$$

- Markovian projection (Gyöngy, 1986) : let $dS_t = \sigma_t dW_t$ has the marginal distribution $p(t, x)$, then

$$\mathbb{E}[\sigma_t^2 | S_t] = \sigma_{loc}^2(t, S_t), \text{ a.s. } \forall t.$$
Controlled diffusion processes problem

- Let $X_\alpha$ be a controlled process, with the control process $\alpha$, defined by

$$dX_\alpha^t = b(t, X_\alpha^t, \alpha_t)dt + \sigma(t, X_\alpha^t, \alpha_t)dW_t,$$

a standard stochastic control problem is given by

$$V(0, X_0) := \sup_{\alpha} E \left[ \int_0^T f(t, X_\alpha^t, \alpha_t)dt + g(X_T^\alpha) \right].$$

- Applications in finance: utility maximization, pricing under model uncertainty, American option pricing, etc.

- A very large literature on the subject:
  - A big community on deterministic control.
Different approaches

- **Approach 1**: Pontryagin’s maximum principle: if \((\alpha^*, X^*)\) is an optimal control, the first order necessary condition leads to a forward-backward system.

- **Approach 2**: Dynamic Programming (Bellman) Principle:

\[
V(0, X_0) = \sup_{\alpha} \mathbb{E} \left[ \int_0^t L(s, X_s^\alpha, \alpha_s) ds + V(t, X_t^\alpha) \right].
\]

- PDE (HJB equation) characterization for the value function,
  - numerical methods, etc.
- Numerical computation by a backward scheme,
- Supermartingale characterization of \(V(t, X_t^\alpha)\),
- Optional decomposition, duality, etc.
Outline

1. Introduction
   - Mathematical Finance Modelling
   - Stochastic Optimal Control

2. From MOT to McKean-Vlasov Control Problems
   - The McKean-Vlasov Control Problem
   - Martingale Optimal Transport (MOT)
   - From MOT to McKean-Vlasov Control

3. Solving the McKean-Vlasov Control Problems
   - Solving the McKean-Vlasov Control Problem
   - Back to the MOT-MKV Control Problem
The McKean-Vlasov stochastic equation

- A large population \( (i = 1, \cdots, N) \) system with interaction:

\[
dX_{t}^{i,N} = b\left(t, X_{t}^{i,N}, \frac{1}{N} \sum \delta_{X_{t}^{i,N}} \right) dt + \sigma\left(t, X_{t}^{i,N}, \frac{1}{N} \sum \delta_{X_{t}^{j,N}} \right) dW_{t}^{i}.
\]

- McKean-Vlasov equation: let \( N \to \infty \), using the limit theory (law of large numbers) and considering a representative agent, it leads to

\[
dX_{t} = b\left(t, X_{t}, \mathcal{L}(X_{t}) \right) dt + \sigma\left(t, X_{t}, \mathcal{L}(X_{t}) \right) dW_{t}.
\]
The McKean-Vlasov control problem

- The McKean-Vlasov control problem:

$$\sup_{\alpha} \mathbb{E}\left[ \int_0^T f(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t) dt + g(X_T^\alpha, \mathcal{L}(X_T^\alpha)) \right],$$

where

$$dX_t^\alpha = b(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t) dt + \sigma(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t) dW_t.$$

- Main questions:
  - The limit theory: Fischer-Livieri(16), Lacker(17), etc.
  - Pontryagin’s maximum principle: Andersson-Djehiche(10), Yong(13), Carmona-Delarue (14), Buckdahn-Li-Ma(16), etc.
  - The dynamic programming principle: Laurière-Pironneau(14), Bensoussan-Frehse-Yam(15), Pham-Wei(16,17), Bayraktar-Cosso-Pham(18), etc.
  - Applications in Economy, Finance, etc.
An optimal control problem under marginal constraint

- An optimal control problem under marginal constraint (Tan-Touzi (AOP, 13)):

\[
dX_t^\alpha = b(t, X_t^\alpha, \alpha_t)dt + \sigma(t, X_t^\alpha, \alpha_t)dW_t,
\]

and for some given marginal distribution $\mu$,

\[
\sup_{\alpha : X_T^\alpha \sim \mu} \mathbb{E}\left[ \int_0^T L(t, X_{t\wedge .}^\alpha, \alpha_t)dt + \Phi(X_T^\alpha) \right].
\]

- Motivation from finance: model-free no-arbitrage pricing for exotic options,
  - When $b \equiv 0$, $X^\alpha$ is a martingale, i.e. no-arbitrage pricing.
  - Marginal law $\mu$ is recovered from the price of call options.
Martingale Optimal Transport (MOT)

- Numerous varied formulations:
  - Discrete time vs Continuous time,
  - Continuous path vs Càdlàg path,
  - One marginal vs Multiple marginals.

- Some pioneering works: Skrokoahod Embedding: Hobson (98); MOT problem: Beiglböck, Henry-Labordère, Penkner (13), Galichon, Henry-Labordère, Touzi (14), Tan-Touzi (13), etc.

Theorem (Kellerer, 1972)

Given a family of marginals $\mu = (\mu_t)_{t \geq 0}$, there is martingale $M$ such that $M_t \sim \mu_t$ for all $t \geq 0$ iff $\mu_t$ has finite first order and $t \mapsto \mu_t(\phi)$ is increasing for all convex functions $\phi$.

Peacocks

Francis Hirsch
Christophe Profeta
Bernard Roynette
Marc Yor

Peacocks and Associated Martingales, with Explicit Constructions

Bocconi University Press
Springer
Peacocks

Haïku by Pf. Y. Takahashi

A proud peacock spreads

Its tail pretending to be

A martingale.
Optimal martingales given full marginals

- **Fake Brownian motion**: Albin (08), Oleszkiewicz (08), etc.
- **Explicit constructions**:
  - **Madan-Yor (02)**: maximizing the expected value of the running maximum.
  - **Hobson (17)**: minimizing the expected total variation.
  - **Henry-Labordère-Tan-Touzi (16, SPA)**, maximizing the expected quadratic variation.
- **Duality and existence of optimal martingales**:
  - **Guo-Tan-Touzi (16, SICON)**: S-topology.
  - **Källblad-Tan-Touzi (17, AAP)**: Skorokhod embedding approach.
A MOT problem given full marginals

- Let $\mu = (\mu_t)_{0\leq t\leq T}$ a peacock, we consider the MOT problem
  \[ V_{\text{MOT}} := \sup_{\sigma} \mathbb{E}[\Phi(X^\sigma)], \quad dX_t^{\sigma} = \sigma_t dW_t, \quad X_t^{\sigma} \sim \mu_t, \quad \forall t \in [0, T]. \]

- **Remark 1**: There exists a unique Markovian diffusion process $X$, defined by
  \[ dX_t = \sigma_{\text{loc}}(t, X_t) dW_t, \]
  satisfying the marginal constraints.

- **Remark 2**: Let $X_t^{\sigma} = X_0^{\sigma} + \int_0^t \sigma_s dW_s$ be a diffusion process such that $X_t^{\sigma} \sim \mu_t$, for all $t \in [0, 1]$, then, by Markovian projection,
  \[ \mathbb{E}[\sigma_t^2 | X_t^{\sigma}] = \sigma_{\text{loc}}^2(t, X_t^{\sigma}), \quad \text{a.s.} \quad \forall t \in [0, T], \]
  and it follows that
  \[ dX_t^{\sigma} = \frac{\sigma_t}{\sqrt{\mathbb{E}[\sigma_t^2 | X_t^{\sigma}]}} \sigma_{\text{loc}}(t, X_t^{\sigma}) dW_t. \]
A McKean-Vlasov control problem

- Let us consider all processes \((\sigma, \hat{X}^\sigma)\) satisfying

\[
d\hat{X}_t^\sigma = \frac{\sigma_t}{\sqrt{\mathbb{E}[\sigma_t^2|\hat{X}_t^\sigma]}} \sigma_{\text{loc}}(t, \hat{X}_t^\sigma) dW_t,
\]

and let

\[
V_{\text{MKV}} := \sup_{(\sigma, \hat{X}^\sigma)} \mathbb{E}[\Phi(\hat{X}^\sigma)].
\]


\[
\hat{\sigma}_t := \frac{\sigma_t}{\sqrt{\mathbb{E}[\sigma_t^2|\hat{X}_t^\sigma]}} \sigma_{\text{loc}}(t, \hat{X}_t^\sigma) = \hat{\sigma}(t, \hat{X}_t^\sigma, \mathcal{L}(\sigma_t, \hat{X}_t^\sigma), \sigma_t)
\]

satisfies

\[
\mathbb{E}[\hat{\sigma}_t^2|\hat{X}_t^\sigma] = \sigma_{\text{loc}}^2(t, \hat{X}_t^\sigma).
\]
A McKean-Vlasov control problem

Theorem

Under technical conditions, one has

\[ V_{\text{MOT}} = V_{\text{MKV}}. \]

An optimal solution to \( V_{\text{MOT}} \) induces an optimal solution to \( V_{\text{MKV}} \), and vice versa.
Outline

1. Introduction
   • Mathematical Finance Modelling
   • Stochastic Optimal Control

2. From MOT to McKean-Vlasov Control Problems
   • The McKean-Vlasov Control Problem
   • Martingale Optimal Transport (MOT)
   • From MOT to McKean-Vlasov Control

3. Solving the McKean-Vlasov Control Problems
   • Solving the McKean-Vlasov Control Problem
   • Back to the MOT-MKV Control Problem
The dynamic programming principle

- Let us consider a general McKean-Vlasov control problem

\[ V(0, \mathcal{L}(X_0)) := \sup_{\alpha} \mathbb{E} \left[ \int_0^T f(\cdots, \alpha_t) \, dt + g(X_\tau^\alpha, \mathcal{L}(X_\tau^\alpha | \mathcal{F}_T)) \right], \]

where

\[ dX_t^\alpha = b(\cdot) \, dt + \sigma(\cdot) \, dW_t + \sigma_0(t, X_t^\alpha, \mathcal{L}((\alpha_t, X_t^\alpha) | \mathcal{F}_t^B), \alpha_t) \, dB_t. \]

Theorem (Dejete-Possamaï-Tan, 18)

Assume that \( f, g \) are Borel measurable, then one has the DPP:

\[ V(0, \mathcal{L}(X_0)) := \sup_{\alpha} \mathbb{E} \left[ \int_0^T f(\cdot) \, ds + V(\tau, \mathcal{L}(X_\tau^\alpha | \mathcal{F}_\tau^B)_{t=\tau}) \right]. \]

- Using the measurable selection arguments, no requirement on continuity of coefficients.
  - El Karoui-Tan, Possamaï-Tan-Zhou (18, AOP).
The limit thoery

- A large population control problem, \( i = 1, \ldots, N \),

\[
V_N := \sup_{\alpha} \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \left[ \int_0^T f(\cdot) ds + g(X_T^{i,N,\alpha}, \varphi_T^{N,\alpha}) \right],
\]

where \( \varphi_t^{N,\alpha} := \frac{1}{N} \sum_{j=1}^{N} \delta_{X_t^{N,j,\alpha}} \) and

\[
dX_t^{N,i,\alpha} = b(\cdot) dt + \sigma(\cdot) dW_t^i + \sigma_0(t, X_t^{N,i,\alpha}, \varphi_t^{N,\alpha}, \alpha_t^i) dB_t.
\]

**Theorem (Dejete-Possamaï-Tan, 18)**

*Under some continuity conditions, one has the convergence result:*

\[
V_N \to V \text{ as } N \to \infty.
\]
Numerical approximation

- Let $\Delta = (0 = t_0 < \cdots < t_n = T)$ be a discrete time grid, we consider

$$V^\Delta_N := \sup_{\alpha} \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \left[ \sum_{k=1}^{n} f(\cdot) \Delta t + g(X_{t_n}^{i,N,\Delta,\alpha}, \varphi_{t_n}^{N,\Delta,\alpha}) \right],$$

where $\varphi_{t_k}^{N,\Delta,\alpha} := \frac{1}{N} \sum_{j=1}^{N} \delta X_{t_k}^{N,i,\Delta,\alpha}$ and

$$X_{t_{k+1}}^{N,i,\alpha} = X_{t_k}^{N,i,\alpha} + H_\Delta(t, X_{t_{k+1}}^{N,i,\Delta,\alpha}, \varphi_{t_k}^{N,\Delta,\alpha}, \alpha_{t_k}).$$

**Theorem (Dejete-Possamaï-Tan, 2019?)**

*With good choice of functions $H_\Delta$, one has*

$$V^\Delta_N \to V \quad \text{as } N \to \infty \quad \text{and} \quad \Delta \to 0.$$ 

- Kushner-Dupuis’s weak convergence technique.
  - Tan (AAP, 14), Possamaï-Tan (AAP, 15), Ren-Tan (SPA, 17).
Back to the MOT-MKV problem

- Recall that

\[ V_{MKV} := \sup_{(\sigma, \hat{X}^\sigma)} \mathbb{E} [\Phi(\hat{X}^\sigma)] , \]

where

\[ d\hat{X}_t^\sigma = \frac{\sigma_t}{\sqrt{\mathbb{E}[\sigma_t^2|\hat{X}_t^\sigma]}} \sigma_{loc}(t, \hat{X}_t^\sigma) dW_t. \]

- Remarks:
  - One can obtain the dynamic programming principle from our previous result, but the numerical approximation falls because of lack of continuity.
  - Resolution by Pontryagin’s maximum principle for special reward functionals.
  - Discrete time counter-party (with random walk) on the equivalence of \( V_{MOT} = V_{MKV} \).