Smart TWAP and VWAP trading in continuous-time equilibria

Jin Hyuk Choi†, Kasper Larsen‡, and Duane J. Seppi*

†Ulsan National Institute of Science and Technology (South Korea)
‡Rutgers University
*Carnegie Mellon University

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TWAP trading (time-weighted average price)

- How to optimally buy (sell) $\tilde{a}_i$ shares dynamically over $t \in [0,1]$?
  - New feature is the use of equilibrium theory (i.e., demand = supply)
- Trader $i$ is given a trading target curve so that at time $t \in [0,1]$, the trader is incentivized to hold

$$\gamma(t)\tilde{a}_i, \quad i = 1, \ldots, M$$

- Examples of $\gamma(t)$ with $[0,1]$ split into 100 trading rounds:

![Graph showing examples of different curves](image)

affine (—–), piecewise affine (- - -), square root (- · - · -)
TWAP trading (2)

- Trader $i$ has the terminal target $\tilde{a}_i$ for $i \in \{1,\ldots,M\}$ with $M < \infty$
  - $\tilde{a}_i$ is private/non-public information (model input)
- There is a given monotone path $\gamma(t)$ with $\gamma(0) \in [0,1]$ and $\gamma(1) = 1$
  - $\gamma(t)$ is non-random (model input — stochastic $\gamma$ for VWAP comes later)
  - When $\gamma(t)$ is smooth, $\gamma'(t)$ is the target execution trading rate
- Simple TWAP: At time $t \in [0,1]$, the trader has reached $\gamma(t)\%$ of his terminal target $\tilde{a}_i$
- Smart TWAP: The trader is allowed to deviate from the trading target path $\gamma(t)\tilde{a}_i$
  - Deviations from $\gamma(t)\tilde{a}_i$ are penalized but could be profitable to make
  - In our equilibrium model, the optimal smart TWAP strategy $\neq \gamma(t)\tilde{a}_i$
TWAP trading (3)

- The deviation penalty is defined by (quadratic costs)

\[ L_{i,t} := \int_0^t \kappa(s) (\gamma(s)(\tilde{a}_i - \theta_{i,-}) - (\theta_{i,s} - \theta_{i,-}))^2 ds, \quad t \in [0,1] \]

- Trader \(i\)'s initial stock position is \(\theta_{i,-}\) (model output)
- \(\theta_{i,t}\) denotes trader \(i\)'s stock position at time \(t \in [0,1]\) (model output)
- \(\kappa(t)\) controls the penalty severity (model input)

\[
\kappa_1(t) := 1 \quad (---), \quad \kappa_2(t) := 1 + t \quad (- - -), \quad \kappa_3(t) := (1 - t)^{-1/4} \quad (\cdots \cdots)
\]
Trader $i$ solves the optimization problem

$$\max_{\theta_i \in A_i} \mathbb{E} [U(X_{i,1} - L_{i,1})]$$ \hspace{1cm} (1)

- $U(x) := x$ (model input - exponential utilities are also possible)
- $A_i$ is the set of admissible controls (model input)
- $X_{i,t}$ is trader $i$’s wealth at time $t \in [0,1]$ (more about $X_{i,t}$ later - model output)

In the above objective (1), we have two competing drivers:

(i) **Wealth $X_{i,1}$**: Concavity of $U$ gives incentive to use Merton’s utility-maximizing strategy

(ii) **Penalty $L_{i,1}$**: The terminal penalty gives incentive to use the simple TWAP strategy $\theta_{i,-} + \gamma(t) (\tilde{a}_i - \theta_{i,-})$
Questions (qualitative)

- How does the widespread presence of smart-TWAP traders affect equilibrium price dynamics?
  - How does the equilibrium return change over the trading day in order to clear the markets (stock and bank)?
  - What is the resulting equilibrium price-impact of orders?

- How do traders — high frequency (HF) traders with $\tilde{a}_i := 0$ and smart-TWAP traders with $\tilde{a}_i \neq 0$ supply liquidity and optimally trade?
  - How do traders absorb the inelastic noise-trader orders (they must do this for markets to clear)?
  - How sensitive are the traders’ optimal positions $\hat{\theta}_{i,t}$ relative to their own trading targets $\tilde{a}_i$?
  - How sensitive are the traders’ objective value (i.e., certainty equivalent $CE_i$) relative to their own trading targets $\tilde{a}_i$?
Questions (quantitative)

- As with any other model in applied math we need (Hadamard 1902)
  - Existence
  - Uniqueness (led us to a new calculus of variations problem)
  - Stability

- Issues related to numerical computation
  - Speed (low dimensional linear or quadratic ODEs — this is both fast and stable)
  - Discretization (convergence of discrete-time pre-limits works numerically but is not developed analytically)
  - Online algorithm (filtering components are not developed)
  - Big data issues (how to handle various forms of data irregularities is not developed)
Equilibrium with traders with heterogeneous dynamic random endowments (no asymmetric info related to cash-flows):

- Discrete-time incomplete models: Vayanos (1999) and Calvet (2001)
  - Vayanos (1999) has both the competitive (Radner) equilibrium and a Nash price-impact equilibrium — non-uniqueness


- Continuous-time equilibrium models with targets, linear utilities, and absolutely continuous optimal controls (rates)
  - Brunnermeier and Pedersen (2005): elastic noise-trader supply and hard public targets
  - Gârleanu and Pedersen (2016): one agent, inelastic noise-trader supply, and competitive equilibrium with quadratic costs (zero targets)
  - Bouchard, Fukasawa, Herdegen, and Muhle-Karbe (2018): many agents, inelastic noise-trader supply, and competitive equilibrium with quadratic costs (zero targets)
Literature (2)

- Optimal order-splitting when the price-impact of order flow is model input
  - Find an optimal trading strategy to minimize $\mathbb{E}[\text{cost}]$ of trading a fixed number of shares

- Equilibrium with long-lived asymmetric info

- Equilibrium with asymmetric cash-flow info and trading targets
  - Degryse, de Jong, and van Kervel (2014): short-lived non-public info
TWAP model details

- Continuous-time with $t \in [0,1]$. Stock and bank account with $r := 0$.
- Non-public variables $(\tilde{a}_1,\ldots,\tilde{a}_M)$ and Brownian motions $(W,D)$ on $\Omega$:
  $$(\tilde{a}_1,\ldots,\tilde{a}_M) \perp W = (W_t)_{t\in[0,1]} \perp D = (D_t)_{t\in[0,1]}$$
- The equilibrium stock price $\hat{S} = (\hat{S}_t)_{t\in[0,1]}$ is to be determined. $\hat{S}$
  must satisfy the terminal condition
  $$\hat{S}_1 = D_1$$
- Inelastic supply of shares $w_t$ from noise traders (OU-process)
  $$dw_t := (\alpha - \pi w_t)dt + \eta dW_t, \quad w_0 := 0$$
- $M$ traders with heterogenous terminal targets $(\tilde{a}_i)_{i=1}^M$
  - Zero initial stock positions
  - Smart-TWAP traders have $\tilde{a}_i \neq 0$
  - HF traders (liquidity providers/market makers) have $\tilde{a}_i := 0$
  - The functions $\kappa(t), \gamma(t)$, and $U(x)$ are the same for all traders
  - The only(!) heterogeneity is in the realization $\tilde{a}_1(\omega),\ldots,\tilde{a}_M(\omega), \omega \in \Omega$
TWAP model details (2)

- Trader $i$ solves the optimization problem (linear utilities $U(x) := x$)

$$\max_{\theta_i \in A_i} \mathbb{E}[X_{i,1} - L_{i,1}]$$

- Penalty process

$$L_{i,t} := \int_0^t \kappa(s) \left( \gamma(s)(\tilde{a}_i - \theta_{i,-}) - (\theta_{i,s} - \theta_{i,-}) \right)^2 ds, \quad t \in [0,1]$$

- $\theta_{i,-} =$ initial stock position for trader $i$ (model input)
- $\theta_{i,t} =$ cumulative stock position for trader $i$ at time $t$ (model output)
- $\gamma(t) =$ target path such as $\gamma(t) := t$ for simple TWAP (model input)
- $\kappa(t) =$ penalty severity such as (model input)

$$\kappa_1(t) := 1, \quad \kappa_2(t) := 1 + t, \quad \kappa_3(t) := (1 - t)^{-0.25}$$
Model details (3)

- The aggregate target is \( \tilde{a}_\Sigma := \sum_{i=1}^{M} \tilde{a}_i \) turns out to be public info
- Wealth process for trader \( i \) is

\[
dx_{i,t} := \theta_{i,t} dS_{t}^{\theta_{i}}
\]

\( \theta_{i,t} \in \sigma(\tilde{a}_i, \tilde{a}_\Sigma, w_u, D_u)_{u \in [0,t]} \) is trader \( i \)'s stock position at \( t \in [0,1] \)
- The stock price dynamics faced by investor \( i \) are

\[
dS_{t}^{\theta_{i}} := \mu_{i,t} dt + \sigma_w(t) \eta dW_t + dD_t. \tag{2}
\]

In (2), the drift is defined by

\[
\mu_{i,t} := \mu_0(t) \tilde{a}_\Sigma + \mu_1(t) \theta_{i,t} + \mu_2(t) \tilde{a}_i + \mu_3(t) w_t
\]

where \( \mu_0(t), \ldots, \mu_3(t) \), and \( \sigma_w(t) \) are deterministic (model output)
- \( \mu_1(t) \) controls the drift-impact of trader \( i \)'s position
- The equilibrium stock price \( \hat{S} = (\hat{S}_t)_{t \in [0,1]} \) satisfies

\[
\sigma(\tilde{a}_i, \tilde{a}_\Sigma, w_u, D_u)_{u \in [0,t]} = \sigma(\tilde{a}_i, \hat{S}_u, D_u)_{u \in [0,t]}
\]

- There is also an integrability condition placed on \( \theta_{i,t} \) for \( \theta_{i} \in \mathcal{A}_i \)
  (needed to rule out doubling strategies)
Equilibrium definition (reduced-form notion)

- The deterministic functions \((\mu_0, \ldots, \mu_3, \sigma_w)\) form an equilibrium if the associated optimal stock positions \( (\hat{\theta}_{i,t})_{i=1}^M \) satisfy
  
  (i) The market-clearing condition \( w_t = \sum_{i=1}^M \hat{\theta}_{i,t} \) holds
  
  (ii) The terminal stock-price condition \( \hat{S}_1 = D_1 \) holds at time \( t = 1 \)
  
  (iii) The equilibrium price drift \( \hat{\mu}_t \) does not depend on any trader-specific variables \( (\tilde{a}_i, \hat{\theta}_{i,t}) \)

- Our equilibrium is not unique:
  
  ▶ Uncountably many equilibria
  
  ▶ One degree of freedom: Each equilibrium is pinned down by a different choice of the function \( \mu_1(t) \)
  
  ▶ The market-clearing and equilibrium conditions (i)-(iii) give one too few restrictions
What makes the equilibrium model non-trivial?

- For market-clearing, the traders must absorb the inelastic noise-trader orders (driven by the OU process $w_t$)
  - Requires an equilibrium risk premium (or discount) to induce this behavior
- As $t \uparrow 1$, the terminal price condition $\hat{S}_1 = D_1$ constrains prices. The “wiggle room” for prices to induce market-clearing inventory shrinks
- To induce the terminal trading constraint (i.e., forcing the optimizers $\hat{\theta}_{i,1}$ to be close to $\tilde{a}_i$), the penalty severity needs to explode

$$\lim_{t \uparrow 1} \kappa(t) = +\infty$$
Existence theorem

Let $\gamma : [0,1] \rightarrow [0,\infty)$ and $\mu_1, \kappa : [0,1) \rightarrow (0,\infty)$ be continuous functions with $(\mu_1, \kappa)$ integrable such that $\mu_1(t) < \kappa(t)$. Then a unique equilibrium exists in which

(i) Trader optimal holdings $\hat{\theta}_i$ in equilibrium are given by

$$
\hat{\theta}_{i,t} = \frac{w_t}{M} + \frac{2\kappa(t)\gamma(t)}{2\kappa(t) - \mu_1(t)} \left( \tilde{a}_i - \frac{\tilde{a}_\Sigma}{M} \right)
$$

(ii) The equilibrium stock price is

$$
\hat{S}_t = g_0(t) + g(t)\tilde{a}_\Sigma + \sigma_w(t)w_t + D_t,
$$

where the deterministic functions $g_0, g, \text{ and } \sigma_w : [0,1] \rightarrow \mathbb{R}$ are the unique solutions of the three linear ODEs:

$$
g'_0(t) = -\alpha\sigma_w(t), \quad g_0(1) = 0,
$$

$$
g'(t) = -\frac{2\gamma(t)\kappa(t)}{M}, \quad g(1) = 0,
$$

$$
\sigma'_w(t) = \frac{2\kappa(t) - \mu_1(t)}{M} + \pi\sigma_w(t), \quad \sigma_w(1) = 0,
$$

(iii) The pricing functions $\mu_0, \mu_2, \mu_3$ are explicitly available in terms of $\mu_1$
Different price-impact functions $\mu_1(t)$

- Recall our drift-impact relation

$$dS_t^{\theta_i} := (\mu_0(t)\bar{a}_\Sigma + \mu_1(t)\theta_{i,t} + \mu_2(t)\bar{a}_i + \mu_3(t)w_t)dt + \sigma_w(t)\eta dW_t + dD_t$$

- Competitive equilibrium (Radner) sets $\mu_1(t) := 0$
- Collusive equilibrium sets $\mu_1(t) := \mu^*_1(t)$ where

$$\mu^*_1(t) \in \arg\max_{\mu_1(t)} \sum_{i=1}^{M} \mathbb{E}[CE_i]$$

- For risk-neutral utilities $U(x) := x$ we have sufficient conditions guaranteeing the existence of $\mu^*_1(t) \in (0, \kappa(t))$
- Finding $\mu^*_1(t)$ requires us to solve a new problem in calculus of variations

- Many refinements of Nash equilibria exist (not developed)
- Empirical calibration (not developed because we don’t have data)
TWAP numerics: $\gamma(t)$ and $\kappa(t)$ functions

- The target-ratio function is

$$\gamma(t) := 0.1 + 0.9t, \quad t \in [0,1] \quad (3)$$

- The penalty functions are

$$\kappa_1(t) := 1, \quad t \in [0,1],$$
$$\kappa_2(t) := 1 + t, \quad t \in [0,1],$$
$$\kappa_3(t) := \frac{9}{8}(1 - t)^{-0.25}, \quad t \in [0,1),$$
$$\kappa_4(t) := \begin{cases} 
.0002 & \text{for } t \in [0,.75], \\
2.3791 + 11.8954(t - 0.95) & \text{for } t \in [.75,.95], \\
\frac{9}{8}(1 - t)^{-0.25} & \text{for } t \in [.95,1) 
\end{cases} \quad (4)$$
\( \mu_1^*(t) \) is larger when penalty severity is larger
TWAP numerics: Price impact of noise trader orders

A: [Collusive] $\kappa_1$ (---), $\kappa_2$ (---), $\kappa_3$ (−·−), $\kappa_4$ (−··−).

B: [Radner] $\kappa_1$ (---), $\kappa_2$ (---), $\kappa_3$ (−·−), $\kappa_4$ (−··−).

■ Price-level loading $\sigma_w(t)$ on $w_t$ is unique
■ The terminal condition $\hat{S}_1 = D_1$ forces $\sigma_w(t) \to 0$ as $t \uparrow 1$
TWAP numerics: Liquidity premium standard deviation

\[ \text{SD}[\hat{S}_t - D_t | \sigma(\tilde{\alpha}_t)] \]

A: [Collusive] $\kappa_1$ (—), $\kappa_2$ (—), $\kappa_3$ (— · —), $\kappa_4$ (— · · —).

B: [Radner] $\kappa_1$ (—), $\kappa_2$ (—), $\kappa_3$ (— · —), $\kappa_4$ (— · · —).

- Initially, SD[$\hat{S}_t - D_t$] grows with the variance of the noise-trader supply but eventually converges to 0 because $\hat{S}_1 = D_1$
VWAP model with multiple sets of different investors

- Stochastic targets: Replace $\gamma(t)$ with a stochastic process $\gamma_t$
  - Gamma bridge model used in Frei and Westray (2015)
  - Volume weighted average price (VWAP) benchmarking

- VWAP penalty process for $i \in \{1, \ldots, M\}$
  \[
  L_{i,t} := \int_0^t \kappa(s) \left( \gamma_s(\tilde{a}_i - \theta_{i,-}) - (\theta_{i,s} - \theta_{i,-}) \right)^2 ds, \quad t \in [0,1]
  \]
  - $\theta_{i,-} =$ initial stock position for trader $i$ (model input)
  - $\theta_{i,t} =$ cumulative stock position for trader $i$ at time $t$ (model output)
  - $\kappa(t) =$ penalty severity such as (model input)

- Liquidity trader penalty process for $i \in \{M + 1, \ldots, M + \bar{M}\}$
  \[
  L_{i,t} := \int_0^t \bar{\kappa}(s) \left( \epsilon B_s - \theta_{i,s} \right)^2 ds, \quad t \in [0,1]
  \]
  - $\theta_{i,t} =$ cumulative stock position for trader $i$ at time $t$ (model output)
  - $\bar{\kappa}(t) =$ penalty severity such as (model input)
  - $\epsilon$ is a constant and $B$ is a Brownian motion (both $\tilde{a}_\Sigma$ and $B$ turn out to become public in equilibrium)
VWAP model with multiple sets of different investors (2)

- We have multiple sets of investors
  - Target traders $i \in \{1, \ldots, M\}$ with private targets $\tilde{a}_i$
    - When $\tilde{a}_i \neq 0$, these are called smart VWAP traders
    - When $\tilde{a}_i = 0$, these are called HF traders
  - Target traders $i \in \{1, \ldots, M\}$ with common public target $\epsilon B_s$
    - When $\epsilon \neq 0$, these traders are similar to those in Sannikov and Skrzypacz (2016)
    - When $\epsilon = 0$, these are HF traders as above (but $\bar{\kappa}$ could differ from $\kappa$)
  - Noise traders suppling $w_t + \rho \gamma \gamma_{t-} + \rho_B B_t$ shares of stock with $w_t$ being the OU process from before and $\rho_\gamma, \rho_B$ are constants

- The market clearing condition becomes

$$w_t + \rho \gamma \gamma_{t-} + \rho_B B_t = \sum_{i=1}^{M} \hat{\theta}_{i,t} + \sum_{i=M+1}^{M+\bar{M}} \hat{\theta}_{i,t}$$

- There is an existence theorem with infinitely many equilibria (parameterized by functions $\mu_3(t)$ for $t \in [0,1]$)
VWAP numerics: Penalty severity functions

- The penalty functions are

\[ \kappa(t) := \frac{(1 - p)}{(1 - t)^p}, \quad t \in [0, 1), \quad p \in [0, 1), \]

\[ \bar{\kappa}(t) := \frac{(1 - \bar{p})}{(1 - t)^\bar{p}}, \quad t \in [0, 1), \quad \bar{p} \in [0, 1). \]
VWAP numerics: Collusive equilibrium

Figur: The collusive equilibrium $\mu_3(t)$. The parameters are $w_t := w_0$ (constant), $\rho_\gamma := -1$, $\rho_B := \epsilon := 0$ (second group consists of HF traders), and the trading day is split into 1000 trading rounds.

$p = \bar{p} = 0$ (——), $p = 0.5, \bar{p} = 0$ (— —),
$p = 0, \bar{p} = 0.5$ (— · —), $p = \bar{p} = 0.5$ (— · · —).
Advertisements

- Summer school and workshop on equilibrium theory in Bielefeld (Germany):
  - Summer school: July 8-12 2019
  - Workshop: July 16-19 2019
  - Contact Frank Riedel for more information

- Summer school and workshop on equilibrium theory at Rutgers in New Brunswick (US):
  - Summer school: June 10-12, 2019
  - Workshop: June 13, 2019
  - Contract me for more information (partial travel funding available).
  - As of yesterday, we have a website

Thanks for your attention.