Utility Maximization with Linear Price Impact and Peeking into the Future

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Insider information and price impact

- Bachelier price dynamics:
  \[ S_t = s_0 + \mu t + \sigma W_t, \quad t \in [0, T]. \]

- Insider information modelled by the filtration
  \[ G_t^\Delta := \sigma \{ W_s : s \leq t + \Delta \}. \]

- Egregious arbitrage opportunities curtailed by price impact because execution price is
  \[ S_t^\phi := S_t + \frac{\Lambda}{2} \phi_t \]
  where \( \phi_t = \frac{d\Phi_t}{dt} \) is the investor turnover rate. Namely, we consider a temporary price impact in the spirit of Almgren-Chriss.
Utility Maximization Problem

\[ V_{t}^{\Phi_{0},\phi} := \int_{0}^{t} \Phi_{u}dS_{u} - \frac{\Lambda}{2} \int_{0}^{t} \phi_{v}^{2}dv, \quad t \in [0, T] \]

where, \( \Phi_{t} = \Phi_{0} + \int_{0}^{T} \phi_{v}dv \) in the risky asset that the investor has acquired by time \( t \).

The set of trading strategies is

\[ \mathcal{A}^{\Delta} := \left\{ \phi = (\phi_{t})_{t \in [0, T]} : \phi \text{ is } \mathcal{G}^{\Delta}-\text{adapted with } \int_{0}^{T} \phi_{t}^{2}dt < \infty \right\} . \]

Exponential utility maximization:

\[ \min_{\phi \in \mathcal{A}^{\Delta}} \mathbb{E} \left[ \exp \left( -\alpha V_{T}^{\Phi_{0},\phi} \right) \right] . \]

The initial position \( \Phi_{0} \) is given.
Related Work:

- **Enlargement of filtrations:** Karatzas and Pikovsky 1996, Amendinger, Imkeller and Schweizer 98, Amendinger, Becherer and Schweizer 03, Imkeller 03, Jeanblanc et al 2005 etc.

- **Peeking ahead:** E. Bayraktar and Z. Zhou (2017).

- **Meyer-\(\sigma\)-fields:** Local inside information: Bank et al 2020, 2021.
The optimal strategy

**Theorem:** The optimal solution is given by

\[
\hat{\phi}_t = \frac{1}{\Lambda} \left( \bar{S}_t^\Delta - S_t \right) + \frac{\gamma^\Delta(T - t)}{\Delta} \left( \frac{\mu}{\alpha \sigma^2} - \hat{\phi}_t \right)
\]

where

\[
\bar{S}_t^\Delta := \left( 1 - \gamma^\Delta(T - t) \right) S_{(t+\Delta)\wedge T} + \gamma^\Delta(T - t) \frac{1}{\Delta} \int_0^\Delta S_{t+s} ds
\]

with

\[
\gamma^\Delta(\tau) := \frac{\Delta \sqrt{\rho} \tanh \left( \sqrt{\rho}(\tau - \Delta)^+ \right)}{1 + \Delta \sqrt{\rho} \tanh \left( \sqrt{\rho}(\tau - \Delta)^+ \right)}.
\]

and \( \rho = \frac{\alpha \sigma^2}{\Lambda} \) is the risk/liquidity ratio.
Theorem: The optimal value is given by

\[ \exp \left( \frac{\alpha \sqrt{\rho}}{2 \coth (\sqrt{\rho} T)} \left( \Phi_0 - \frac{\mu}{\alpha \sigma^2} \right)^2 - \frac{1}{2} \frac{\mu^2}{\sigma^2} T \right) \]

\times \exp \left( - \frac{1}{2} \int_0^T \frac{(s \land \Delta) \rho}{1 + (s \land \Delta) \sqrt{\rho} \tanh (\sqrt{\rho} (T - s))} \, ds \right).
Some Intuition

Our feedback description can be interpreted as follows: First, without privileged information, i.e. for $\Delta = 0$, we have $\hat{S}_t^\Delta = S_t$ and, therefore, the optimal policy

$$\hat{\phi}_t = \sqrt{\rho} \tanh(\sqrt{\rho}(T - t)) \left( \frac{\mu}{\alpha \sigma^2} - \Phi_t \right), \quad t \in [0, T].$$

So the uninformed agent will trade towards the optimal position $\mu/(\alpha \sigma^2)$ well known from the frictionless Merton problem with finite urgency $\sqrt{\rho} \tanh(\sqrt{\rho}(T - t))$. 
More Intuition

For the informed agent, i.e. for $\Delta > 0$, the desire to be close to the Merton ratio persists, but the urgency reduces to

$$\gamma^\Delta (T - t) = \frac{\sqrt{\rho} \tanh(\sqrt{\rho}(T - t - \Delta)^+)}{1 + \Delta \sqrt{\rho} \tanh(\sqrt{\rho}(T - t - \Delta)^+)}$$

leaving “some air” to take advantage of the knowledge on future price movements. The weight that this assessment of earnings assigns to the average stock prices is given by $\gamma^\Delta (T - t) \in [0, 1]$; it is about $\Delta \sqrt{\rho}/(1 + \Delta \sqrt{\rho})$ when there is still a lot of time to go, but vanishes completely as soon as $T - t \leq \Delta$, i.e. as soon as full knowledge of stock price movements over the relevant time span $[0, T]$ is attained. In this terminal regime also the ambition to be close to the Merton ratio is wiped out and the investor just chases the earning potential $S_T - S_t$ from the stock, of course still in a trade off against the liquidity costs $\Lambda$.
Remark 1:

The monetary value of being able to peek ahead by $\Delta$ is best described by the term

$$
\frac{1}{2} \int_0^T \frac{(s \wedge \Delta) \rho}{1 + (s \wedge \Delta) \sqrt{\rho} \tanh \left( \sqrt{\rho}(T - s) \right)} \, ds
$$

determines the extra utility afforded to our investor by her ability to look $\Delta$ time units ahead. It can be computed explicitly, but the resulting formulae turn out to be not more informative than the above integral and are therefore omitted. Interestingly, it does not depend on the stock’s risk premium $\mu$, but is determined by the risk/liquidity ratio $\rho = \alpha \sigma^2 / \Lambda$, the investor’s time horizon $T$ and the time units $\Delta$ she can look ahead.
Remark 2:

Let $\tau : [0, T] \rightarrow [0, T]$ be a nondecreasing and continuous function such that $\tau(t) > t$ for all $t < T$ (in particular $\tau(T) = T$). Suppose that the investor observes the filtration $(\mathcal{F}^S_{\tau(t)})_{t \geq 0}$. Observe that as soon as the investor knows the all path $S_{[0, T]}$ there is no need in additional information, i.e. by applying the map $\tau \rightarrow \tau \land T$ there is no loss of generality to assume that $\tau(T) = T$. Define the inverse function $\tau^{-1} : [0, T] \rightarrow [0, T]$ by $\tau^{-1}(s) := \inf\{ t : \tau(t) \geq s \}$. We can prove that the monetary value of being able to peek ahead is given by

$$
\frac{1}{2} \int_0^T \frac{(s - \tau^{-1}(s))\rho}{1 + (s - \tau^{-1}(s))\sqrt{\rho} \tanh(\sqrt{\rho}(T - s))} \, ds.
$$

Our case is: $\tau(t) := T \land (t + \Delta)$, $t \in [0, T]$ is the most trackable.
Duality:

Theorem:

\[
\max_{\phi \in \mathcal{A}} \left\{ -\frac{1}{\alpha} \log \mathbb{E} \left[ \exp \left( -\alpha V^\phi_{T} \right) \right] \right\} = \inf_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[ \Phi_0(S_T - S_0) + \frac{1}{\alpha} \log \left( \frac{dQ}{dP} \right) \right.
\]
\[
\left. + \frac{1}{2\Lambda} \int_0^T \left| \mathbb{E}_Q (S_T | \mathcal{G}_t^\Lambda) - S_t \right|^2 dt \right].
\]

Optimal Strategy:

\[
\hat{\phi}_t = \frac{\mathbb{E}_Q [S_T | \mathcal{G}_t^\Lambda] - S_t}{\Lambda}, \quad t \in [0, T].
\]
Continuation:

- Without loss of generality: \( S = W \) (rescale parameters).

\[
\frac{dQ}{dP} = \exp \left( - \int_0^T \theta_t dW_t - \frac{1}{2} \int_0^T \theta_t^2 dt \right).
\]

Martingale representation theorem:

\[
\theta_t = a_t + \int_0^t l_{t,s} dW_s^Q, \quad t \in [0, T].
\]
The dual problem is reduced to the deterministic control problem in $a$ and $l$. 

\[ E_Q \left[ -\phi_0(S_T - S_0) + \frac{1}{\alpha} \log \left( \frac{dQ}{dP} \right) + \frac{1}{2\lambda} \int_0^T \left( E_Q \left[ S_T \bigg| G_t^\Delta \right] - S_t \right)^2 dt \right] \]

\[ = -\phi_0 \int_0^T a_t dt + \frac{1}{2\alpha} \int_0^T a_t^2 dt + \frac{1}{2\lambda} \int_0^T \left( \int_t^T a_u du \right)^2 dt \]

\[ + \int_0^T E_Q \left[ \frac{1}{2\alpha} \int_s^T l_{t,s}^2 dt + \frac{1}{2\lambda} \int_s^T \left( \int_t^T l_{u,s} du \right)^2 dt \right] \]

\[ + \frac{s \wedge \Delta}{2\lambda} \left( 1 - \int_s^T l_{u,s} du \right)^2 \] ds.

The dual problem is reduced to the deterministic control problem in $a$ and $l$. 

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Euler–Lagrange equation

\[ \int_0^T L(q(t), \dot{q}(t), t) dt \rightarrow \min. \]

\[ \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}. \]
Lemma: The minimum of the functional

\[- \Phi_0 \int_0^T a_t dt + \frac{1}{2\alpha} \int_0^T a_t^2 dt + \frac{1}{2\Lambda} \int_0^T \left( \int_t^T a_u du \right)^2 dt\]

over \(a \in L^2([0, T], dt)\) is attained for \(\hat{a} \Phi_0\) where

\[\hat{a}_t = \frac{\alpha \cosh (\sqrt{\rho}(T - t))}{\cosh (\sqrt{\rho} T)}, \quad t \in [0, T].\]

The resulting minimum value is \(-\hat{A}_T \Phi_0^2\) where

\[\hat{A}_T = \Lambda \sqrt{\rho} \tanh(\sqrt{\rho} T)/2.\]
**Lemma:** For any $s \in [0, T]$, the minimum of the functional

$$\frac{1}{2\alpha} \int_s^T l_t^2 dt + \frac{1}{2\Lambda} \int_s^T \left( \int_t^T l_u du \right)^2 dt + \frac{s \wedge \Delta}{2\Lambda} \left( 1 - \int_s^T l_u du \right)^2$$

is attained at

$$\hat{l}_{t,s} = \frac{\rho(s \wedge \Delta) \cosh(\sqrt{\rho}(T - t))}{\cosh(\sqrt{\rho}(T - s)) + \sqrt{\rho}(s \wedge \Delta) \sinh(\sqrt{\rho}(T - s))}, \quad t \in [s, T].$$

The corresponding minimum value is

$$\hat{L}_s = \frac{1}{2\Lambda} \frac{s \wedge \Delta}{1 + (s \wedge \Delta) \sqrt{\rho} \tanh(\sqrt{\rho}(T - s))}.$$
Deal with Gaussian stochastic Volterra equation

- Candidate for dual solution: $\hat{Q} \sim P$ with density represented by
  \[ \hat{\theta}_t = \hat{a}_t + \int_0^t \hat{l}_{t,s} dW_s^{\hat{Q}}, \quad t \in [0, T]. \]

- For the associated Brownian motion $\hat{W} := W^{\hat{Q}} = W + \int_0^t \hat{\theta}_t dt$ this implies the Gaussian Volterra-type integral equation
  \[ W_t + \int_0^t \hat{a}_s \Phi_0 ds = \hat{W}_t - \int_0^t \int_0^s \hat{l}_{s,r} d\hat{W}_r ds, \quad t \in [0, T]. \]
Theorem: (Hitsuda ’68, Hida, Hitsuda ’93) The Gaussian Volterra-type integral equation has the unique solution

$$\hat{W}_t = W_t - \int_0^t \int_0^s \hat{k}_{s,r} dW_r ds + \Phi_0 \left( \int_0^t \hat{a}_s ds - \int_0^t \int_0^s \hat{k}_{s,r} a_r dr ds \right)$$

where $\hat{k}$ is the associated resolvent kernel characterized by

$$\hat{k}_{t,s} + \hat{l}_{t,s} = \int_s^t \hat{l}_{t,u} \hat{k}_{u,s} du, \quad 0 \leq s \leq t \leq T.$$

$$\hat{k}_{t,s} = -\exp \left( \int_s^t \hat{l}_{u,u} du \right) \hat{l}_{t,s}, \quad 0 \leq s \leq t \leq T.$$
Finding the optimal strategy

\[ \hat{\phi}_t = \frac{E_{\hat{Q}}[W_T|G_t^\Delta] - W_t}{\Lambda}, \quad t \in [0, T]. \]

\[ E_{\hat{Q}}[W_T|G_t^\Delta] = \int_0^{(t+\Delta)\wedge T} \left( 1 - \int_s^T \hat{l}_{u,s} du \right) d\hat{W}_s - \int_0^T \hat{a}_u du \Phi_0 \]

which gives an “open loop” description of the optimal policy.

A rather tedious computation finally leads to the initially described optimal feedback policy:

\[ \hat{\phi}_t = \frac{1}{\Lambda} \left( \bar{S}_t^\Delta - S_t \right) + \frac{\gamma^\Delta(T - t)}{\Delta} \left( \frac{\mu}{\alpha \sigma^2} - \hat{\Phi}_t \right). \]
Thank you very much