Optimal bidding strategies for digital advertising with social interactions

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Third Van Eenam Lecture
University of Michigan
April 14, 2022
Digital advertising and real-time bidding

- **Targeted advertising** vs traditional advertising (newspapers, TV, billboards, etc)
- Companies/advertisers can minimise ad costs by targeting directly individuals/users potentially interested by their product/service

**Auctions** for Ad display (in milliseconds):

- Ad exchange sends data to advertisers about page content and user’s profile
- Advertisers place bids for ad display (impression) by publisher to a given user
- The highest bidder wins the ad space
Introduction

Literature on advertising models

- Classical approach: modelling of macroscopic variables, e.g. sales process, affected by (traditional) advertising expenditure
  - Vidale, Wolfe (57), Nerlove, Arrow (62)
  - Sethi advertising models: Sethi and collaborators (1973-2020)

- Auction for digital advertising: Levin, Milgrom (10), Goke et al (21)

- Optimisation in digital advertising:
  - Supply side (publisher) perspective: Balseiro et al. (14,15), Yuan (14,15)
  - Demand side (bidders) perspective:
    - discrete time and MDP models: Amin et al. (12), Tillberg et al. (20)
    - stochastic control in continuous-time and HJB equation: Fernandez-Tapia, Guéant, Lasry (17): maximisation of number of banners displayed
Our purpose and basic framework

We address the following problem from the demand-side perspective:

- **Agent A** (company/association) willing to spread Information \( I \) to Users, e.g.
  - the existence of a new product, a new service: **commercial advertising**
  - the danger of some behaviour (drug, virus, etc): **social marketing**

- **Impression \( \rightarrow \) Click/Conversion**: Once they get the information \( I \), Users can decide to make an action, e.g.
  - purchase of the new product, subscribe to the new service
  - stop behaving unsafely

- **Attribution problem**: how to efficiently diffuse \( I \) by means of “modern” online channels (digital ad, social networks, etc) in order to generate conversion?
  - We propose a **continuous-time model for optimal digital advertising strategies**:
    - online behaviours of users, social interactions: microscopic modelling of the population
    - advertising auctions
    - targeted vs non-targeted advertising
Outline

1. The commercial advertising model
2. Social marketing model
Online behaviour of User

The User can connect at any (random) time to:

- Website providing the Information (e.g. company own website):
  \[ N^I \text{ Poisson process with intensity } \eta^I : \text{ number of connections} \]

- Publisher T (social networks, search engine) not containing a priori but displaying Targeted ad:
  \[ N^T \text{ Poisson process with intensity } \eta^T : \text{ number of connections}. \]

→ Data collected by Ad exchange and sent to Advertisers
Targeted ad auctions and bidding strategies

- **Targeted ad auction**: each time the User connects to a Publisher displaying targeted ads, advertisers compete to win the right to display their ad to him.

- **Model the maximal bid made by other bidders (other than the agent A)**:
  
  $B_k^T$: maximal bid of other bidders during the $k$-th ad auction

  We assume that $B_k^T, k \in \mathbb{N}$ are i.i.d. nonnegative r.v.

- **Bidding strategies for Agent A**. Non anticipative $\mathbb{R}^+$-valued process $\beta = (\beta_t)_{t \geq 0}$:
  
  $\beta_t$: bid that A makes if user is connecting to a Publisher at time $t$

  predictable w.r.t. to data information: $\sigma\{N_s^I, N_s^T, B_{N_s^T}^T, s \leq t\}$.

- **Agent A wins bid at time $t$ ↔ $N_t^T$-ad auction, if**:
  
  $\beta_t \geq B_{N_t^T}^T.$
Conversion dynamic of the User

- Conversion state in \( \{0, 1\} \):
  - \( x = 0 \): user not aware of I
  - \( x = 1 \): user aware of I and clicks for conversion

His conversion state \( X = X^\beta \) is affected by the bidding strategy of Agent:

\[
X^\beta_{0-} = 0, \\
dX^\beta_t = (1 - X^\beta_t)(dN^I_t + 1_{\beta_t \geq B^T_N} dN^T_t), \quad t \geq 0.
\]
Conversion dynamic of the User

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  - \( x = 0 \): user not aware of \( I \)
  - \( x = 1 \): user aware of \( I \) and clicks for conversion

His conversion state \( X = X^\beta \) is affected by the bidding strategy of Agent:

\[
X_0^\beta = 0,
\]
\[
dX_t^\beta = (1 - X_t^\beta)(dN_t^I + 1_{\beta_t \geq B_{N_t}} dN_t^T), \quad t \geq 0.
\]

- Assumption of spontaneous **click/conversion**: once the user gets information, he purchases the product
  - In reality, multi-stage process before a purchase decision: **conversion funnel**, see Abhisek et al (12), Jordan et al. (12), Berman (18)
  - A simplified modeling of conversion funnel can be considered here by replacing \( \eta^T \) by \( \eta^T \times q^T \), where \( q^T \) is the probability of conversion when seeing the ad (idem for \( \eta^I \leftrightarrow \eta^I \times q^I \))
Optimal bidding problem for pay-per-conversion Agent

Maximise over bidding strategies $\beta$ the purchase-based gain function:

$$V(\beta) = \mathbb{E} \left[ \int_0^\infty e^{-\rho t} K dX_t^\beta \right] - C(\beta),$$

where $\rho \geq 0$ is a discount rate, $K$ is the punctual payment from the User to the Agent when he gets informed and clicks/converts, and $C(\beta)$ is the ad cost:

$$C(\beta) = \mathbb{E} \left[ \int_0^\infty e^{-\rho t} 1_{\beta_t \geq B_{NT}}^T c(\beta_t, B_{NT}^T) dX_t^\beta dN_t^T \right].$$

where $c$ is the paying rule of the auction:

- **First-price auction**: $c(b, B) = b$
- **Second-price (Vickrey) auction**: $c(b, B) = B$. 
Explicit solution

\[ V^* := \sup_{\beta} V(\beta) = \sup_{b \in \mathbb{R}_+} V(\beta^b), \]

where \( \beta^b \) is the constant bidding strategy: \( \beta^b_t = (1 - X_{t-}^b) b \), with gain function:

\[ V(\beta^b) = \frac{\eta^T K + \eta^T \mathbb{E}[(K - c(b, B_1^T))1_{b \geq B_1^T}]}{\eta^T + \rho + \eta^T \mathbb{P}[b \geq B_1^T]}, \quad b \in \mathbb{R}_+. \]

Furthermore, any \( b_* \in \arg\max_{b \in \mathbb{R}_+} V(\beta^b) \) yields an optimal constant bidding strategy \( \beta^{b_*} \).
Properties of the solution

**Monotonicity w.r.t. parameters**

- $V^*$ is increasing w.r.t. $\eta^I, \eta^T$, and decreasing in $\rho$
- The smallest optimal bid policy $b^* = \min_{b \in \mathbb{R}_+} \arg\max V(\beta^b)$ is decreasing w.r.t. $\eta^I, \eta^T$, and increasing in $\rho$

**Upper bound on optimal bids**

$$b^*_* \leq K - V^* \leq \frac{\rho K}{\eta^I + \rho}.$$
Outline

1. The commercial advertising model

2. Social marketing model
Welfare purpose

• Population of $M$ users behaving unsafely ($D$ danger!) over time

$N^{m,D}$ Poisson process with intensity 1: counting the times of $D$ of $m \in [1, M]$

→ This incurs a cost $K$ to Agent $A$ (association) as long as population is in $D$

• $A$ willing to alert population of users about $D$ (and how to protect against) so that once they get the information $I$ and are converted:
  • stop behaving unsafely (no more in $D$)
→ Cancels the cost for $A$. 
Online behaviour of the population → information channels

- Any user \( m \in [1, M] \) can browse through
  - Website providing the information \( I \):
    \[
    N^{m,I} \text{ Poisson process with intensity } \eta^I : \text{ number of connections of user } m
    \]
  - Publisher \( T \) (search engine) displaying targeted ad:
    \[
    N^{m,T} \text{ Poisson process with intensity } \eta^T : \text{ number of connections of user } m
    \]
  - Platform \( NT \) displaying non-targeted ad:
    \[
    N^{m,NT} \text{ Poisson process with intensity } \eta^{NT} : \text{ number of connections of user } m
    \]
    \[
    N^{NT} := \sum_{m=1}^{M} N^{m,NT} : \text{ total number of connections to } NT \text{ of the population}
    \]
- Social interactions. \( N^{m,i,S} \) Poisson process with intensity \( \eta^S \): counting the social interactions between users \( m \) and \( i \).
- \((N^{m,D}, N^{m,I}, N^{m,T}, N^{m,NT}, N^{m,i,S})\), \( m, i = 1, \ldots, M \), are independent
Targeted and non-targeted ad auctions

- **Targeted ad auction**: each time User $m$ connects to a Publisher displaying targeted ads, advertisers compete to win the right to display their ad.

  - Model the maximal bid made by other bidders (other than the agent $A$):

    $B_{m,T}^k : \text{maximal bid of other bidders during the } k\text{-th ad auction for user } m$

  We assume that $B_{m,T}^k, k \in \mathbb{N}, m \in [1, M]$, are i.i.d. nonnegative r.v.

- **Non-Targeted ad auction**: Bids are indifferent w.r.t. users of the population.

  - Model the maximal bid made by other bidders (other than the agent $A$):

    $B_{NT}^k : \text{maximal bid of other bidders during the } k\text{-th ad auction for any user}$

  We assume that $B_{NT}^k, k \in \mathbb{N}$ are i.i.d. nonnegative r.v., and independent of $(B_{m,T}^k)_{k,m}$.
Advertising bidding map strategies

Non-anticipative process $\beta = \{(\beta_t^m)_{m=0,...,M}, t \geq 0\}$ valued in $\mathbb{R}_{+}^{M+1}$:

- $\beta_t^0$ is the bid that $A$ makes when any user is connecting to the Platform $NT$.
- $\beta_t^m$, $m = 1, \ldots, M$, is the bid that $A$ makes if user $m$ is connecting to a Publisher $T$ at time $t$. 
Conversion dynamic of the population of users

Conversion state $X_{m}^{m, \beta}$ in $\{0, 1\}$ of user $m \in [1, M]$ influenced by the bidding map strategy of Agent, and the other users (social interaction):

$$
\begin{aligned}
X_{0-}^{m, \beta} & = 0, \\
\frac{dX_{t}^{m, \beta}}{dt} & = (1 - X_{t-}^{m, \beta}) \left[ dN_{t}^{m,1} + 1_{\beta_{t}^{m} \geq B_{m,T}^{m}} dN_{t}^{m,T} ight. \\
& \quad \left. + 1_{\beta_{t}^{0} \geq B_{NT}^{m}} dN_{t}^{m,NT} + \sum_{i \neq m} X_{t-}^{i, \beta} dN_{t}^{m,i,S} \right], \quad t \geq 0.
\end{aligned}
$$
Optimal bidding problem for Agent

Minimize over bidding map strategies $\beta = (\beta^m)_{m \in [0, M]}$ the cost function:

$$V(\beta) = \sum_{m=1}^{M} \mathbb{E} \left[ \int_{0}^{\infty} K(1 - X_{t-}^{m, \beta}) dN_{t}^{m, D} + \int_{0}^{\infty} 1_{\beta_t^m \geq B_{N_t^m, T}^m} c(\beta_t^m, B_{N_t^m, T}^m) dN_{t}^{m, T} \right.$$  

$$+ \int_{0}^{\infty} 1_{\beta_t^0 \geq B_{N_t^0, T}^{NT}} c(\beta_t^0, B_{N_t^0, T}^{NT}) dN_{t}^{m, NT} \right].$$

where $c$ is the paying rule of the auctions:

- **First-price auction**: $c(b, B) = b$
- **Second-price (Vickrey) auction**: $c(b, B) = B$.

*(For simplicity of notations, we assume here the same auction rule $c$ on $T$ and $NT$ but they can differ)*
Explicit solution

- Minimal cost

\[ V^* := \inf_{\beta} V(\beta) = \sum_{p \in \left[ \frac{0,M}{M} \right]} v(p) \]

(here \( \left[ \frac{0,M}{M} \right] = \{ \frac{k}{M} : k = 0, \ldots, M - 1 \} \)), where \( v(p) = \inf_{b^T,b^{NT} \in \mathbb{R}_+} v^{b^T,b^{NT}}(p) \), with

\[ v^{b^T,b^{NT}}(p) = \frac{K + \eta^T \mathbb{E} \left[ \mathbf{c}(b^T, B_1^T) \mathbf{1}_{b^T \geq B_1^T} \right] + \eta^{NT} \mathbb{E} \left[ \frac{\mathbf{c}(b^{NT}, B_1^{NT})}{1-p} \mathbf{1}_{b^{NT} \geq B_1^{NT}} \right]}{\eta^I + \eta^T \mathbb{P} \left[ b^T \geq B_1^T \right] + \eta^{NT} \mathbb{P} \left[ b^{NT} \geq B_1^{NT} \right] + p \eta^S} \].
Explicit solution

- Minimal cost

\[ V^* := \inf_{\beta} V(\beta) = \sum_{p \in \mathbb{I}_{0,M}} v(p) \]

(here \( \mathbb{I}_{0,M} = \{k/M : k = 0, \ldots, M-1 \} \)), where \( v(p) = \inf_{b^T, b^{NT} \in \mathbb{R}_+} b^T, b^{NT} (p) \), with

\[ v^{b^T, b^{NT}} (p) = \frac{K + \eta^T \mathbb{E}[c(b^T, B^T_1)\mathbb{1}_{b^T \geq B^T_1}] + \eta^{NT} \mathbb{E}[c(b^{NT}, B^{NT}_1)\mathbb{1}_{b^{NT} \geq B^{NT}_1}]}{\eta^T \mathbb{P}[b^T \geq B^T_1] + \eta^{NT} \mathbb{P}[b^{NT} \geq B^{NT}_1] + p \eta^S}. \]

- Optimal bidding map policies based on proportion of informed users:

\[ (b^T_*(p), b^{NT}_*(p)) \in \arg\min_{b^T, b^{NT} \in \mathbb{R}_+} v^{b^T, b^{NT}} (p), \quad p \in \mathbb{I}_{0,M}, \]

→ optimal bidding map strategy \( \beta^* = (\beta^*, m)_{m \in \mathbb{I}_{0,M}} \) with \( p^\beta_t := \frac{1}{M} \sum_{i=1}^M X^i_{t^{\beta}}, \)

\[ \begin{cases} 
\beta^*_t = b^T_*(p^\beta_{t-})(1 - X^m_{t^{\beta}}), & m = 1, \ldots, M, \\
\beta^*_0 = b^{NT}_*(p^\beta_{t-})1_{p^\beta_{t-} < 1}, & t \geq 0.
\end{cases} \]
Remarks on proof

- Direct arguments: do not rely on dynamic programming or maximum principle methods

- Change of variable: reformulate the problem $V(\beta)$ defined as a sum over the Poisson processes to an integral over proportion of converted users $p^\beta$
  - martingale tools using intensity process of Point process

- Bound from below minimal cost

- Achieve the lower bound with a suitable bidding policy.
Properties of the solution

**Monotonicity w.r.t. intensity parameters** $\eta = (\eta^I, \eta^T, \eta^{NT}, \eta^S)$

- $V^*$ is decreasing w.r.t. $\eta$
- The smallest optimal bid policies $b^T_*(p), b^{NT}_*(p)$ are decreasing w.r.t. $\eta$

**Monotonicity w.r.t. proportion of converted users** $p$

- The smallest optimal bid policy for non-targeted ad $b^{NT}_*(p)$ is decreasing in $p$
- The smallest optimal bid policy for targeted ad $b^T_*(p)$ is
  - decreasing in $p$ when there is no non-targeted ad ($\eta^{NT} = 0$)
  - increasing in $p$ when there is no social interactions ($\eta^S = 0$)

**Upper bound on optimal bids**

$$b^T_*(p), b^{NT}_*(p) \leq v(p) \leq \frac{K}{\eta^I + p\eta^S}.$$
Computational cost of optimal bids

- Algo implementation of optimal bids require to compute:

\[ b_T(p), \ b_{NT}(p), \quad \text{for all } p = \frac{k}{M}, \ k = 0, \ldots, M - 1. \]

→ This is a priori quite expensive when \( M \) is large!
Computational cost of optimal bids

• Algo implementation of optimal bids require to compute:

\[ b^T_*(p), \ b^{NT}_*(p), \ \text{for all } p = \frac{k}{M}, \ k = 0, \ldots, M - 1. \]

\[ \rightarrow \] This is a priori quite expensive when \( M \) is large!

• But, taking advantage of the monotonicity in \( p \) of \( b^T_*(p), \ b^{NT}_*(p) \), one can proceed by dichotomy

\[ \rightarrow \] Computational complexity is of order \( O(\ln_2(M)) \)

e.g. for \( M = 7 \times 10^9 \), we have \( \ln_2(M) = 30 \).
Mean-field problem: $M \to \infty$

The average of the minimal cost $V^* = V^*_M$ converges to:

$$\frac{1}{M} V^*_M \to \int_0^1 \nu(p) dp.$$ 

This corresponds formally to the optimal control problem on the proportion of converted users:

$$\frac{dp^\beta_t}{dt} = (1 - p^\beta_t) \left( \eta^I + \eta^T \mathbb{P}[\beta^T_t \geq B^T_1] + \eta^{NT} \mathbb{P}[\beta^{NT}_t \geq B^{NT}_1] + \eta^S p^\beta_t \right), \quad t \geq 0,$$

with deterministic control $\beta = (\beta^T, \beta^{NT})$ and cost functional

$$V^*_\infty(\beta) = \int_0^\infty \left\{ (1 - p^\beta_t) \left( K + \eta^T \mathbb{E} [c^T(\beta^T_t, B^T_1) 1_{\beta^T_t \geq B^T_1}] \right) 
+ \eta^{NT} \mathbb{E} [c^{NT}(\beta^{NT}_t, B^{NT}_1) 1_{\beta^{NT}_t \geq B^{NT}_1}] \right\} dt.$$ 

$$\to \inf_{\beta} V^*_\infty(\beta) = \int_0^1 \nu(p) dp.$$
Conclusion

- Formulation and (explicit) resolution of some advertising problems
  - Microscopic modelling of users: online behaviour
  - Digital feature of advertising, auctions for ad display
  - Quantitative comparison between targeted vs non-targeted advertising
  - Role of social interactions between users

- Enrich models for more realism while keeping tractable
  - Conversion funnel for user to be receptive or not to the information:
    - purchase or not a product
    - stop or continue to behave unsafely
  - Some heterogeneity in the population
  - Auctions:
    - maximal bid of others bidders by Markov process
    - several bidding agents in fictitious play to learn the law of the maximal bid

Thank you for your attention