# Inverted Double Planar Pendulum on a Cart: Feedback Stabilization Using the Method of Controlled Lagrangians

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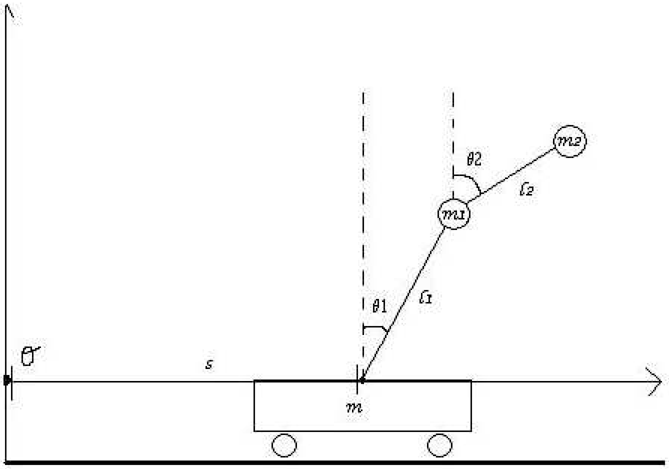
This paper is inspired by the work of Anthony Bloch, Naomi Leonard and Jerrold Marsden [2000] in Controlled Lagrangians and the Stabilization of Mechanical Systems I: The First Matching Theorem [2000], which develops a theoretical basis for a method of stabilization that uses kinetic shaping (altering the kinetic energy to incorporate feedback controls while preserving symmetry) to stabilize modulo a symmetric group.

Controlled Lagrangians and the Stabilization of Mechanical Systems I introduces two mechanical systems as examples. In the first system, an upside-down pendulum is attached by a hinge to a cart free to move back and forth on an axis. In the second, an inverted pendulum is attached by a hinge to a cart able to move on a plane. In both cases the authors used the method of controlled Lagrangians to stabilize the pendulums (at the upright equilibrium) modulo the cart position. This paper will examine the inverted double planar pendulum on a cart. (Two upside-down pendulums, supported by stiff rods and connected by a hinge, are attached, also by a hinge, to a cart free to move on an axis).

Inverted Double Planar Pendulum on Cart.



This system is intuitively much different from the systems discussed in Controlled Lagrangians and the Stabilization of Mechanical Systems I. With the one-pendulum systems, the direction in which one must push the carts to force the pendulums upright is clear (modulo rotation for the second system, where the cart moves on a plane). In the two-pendulum system it is not always clear where the cart should be pushed. If one angle is positive and the other negative (with respect to the vertical), then who is to say whether I should push it to the right or to the left? If this is difficult to determine, then how can I begin to know how to make the pendulums stand upright?

The purpose of this paper is to show that the method of controlled Lagrangians indeed stabilizes the two-pendulum system (modulo the cart position).

## 1

Suppose $m$ and $s$ are the mass and position of the cart, respectively,

$m_1$ and $y_1$ are the mass and position of the first pendulum on the plane,

$m_2$ and $y_2$ are the mass and position of the second pendulum,

$l_1$ and $l_1$ are the lengths of the rods supporting the pendulums, and

$\theta_1$ and $\theta_2$ are the angles formed by the rods with the vertical. Then

$$y_1 = \begin{bmatrix} s + \ell_1 \sin\theta_1 \\ \ell_1 \cos\theta_1 \end{bmatrix}, y_2 = \begin{bmatrix} s + \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2 \\ \ell_1 \cos\theta_1 + \ell_2 \cos\theta_2 \end{bmatrix},$$

and

$$KE = 1/2 \left( m\dot{s}^{\mathbf{2}} + m_1 \dot{y}_1^{\mathbf{2}} + m_2 \dot{y}_2^{\mathbf{2}} \right)$$

$$= 1/2 \left( \begin{array}{l} m\dot{s}^{\mathbf{2}} + m_1 \left( \dot{s} + \dot{\theta}_1 \ell_1 \cos\theta_1 \right)^{\mathbf{2}} + m_1 \left( -\dot{\theta}_1 \ell_1 \sin\theta_1 \right)^{\mathbf{2}} \\ +m_2 \left( \dot{s} + \dot{\theta}_1 \ell_1 \cos\theta_1 + \dot{\theta}_2 \ell_2 \cos\theta_2 \right)^{\mathbf{2}} + m_2 \left( -\dot{\theta}_1 \ell_1 \sin\theta_1 - \dot{\theta}_2 \ell_2 \sin\theta_2 \right)^{\mathbf{2}} \end{array} \right),$$

$$PE = -\left( g m_1 \ell_1 + g m_2 \ell_1 \right) \cos\theta_1 - g m_2 \ell_2.$$

The potential energy taken from the kinetic energy, called the Lagrangian ($L$), is:

$$\begin{array}{l} L = 1/2 \gamma \dot{s}^{\mathbf{2}} + 1/2 \left( \alpha_1 + \alpha_3 \right) \dot{\theta}_1^{\mathbf{2}} + 1/2 \alpha_2 \dot{\theta}_2^{\mathbf{2}} + \left( \beta_1 + \beta_3 \right) \dot{s} \dot{\theta}_1 \cos\theta_1 + \beta_2 \dot{s} \dot{\theta}_2 \cos\theta_2 \\ +\xi \left( \dot{\theta}_1 \dot{\theta}_2 \cos\theta_1 \cos\theta_2 + \dot{\theta}_1 \dot{\theta}_2 \sin\theta_1 \sin\theta_2 \right) + \left( D_1 + D_2 \right) \cos\theta_1 + D_3 \cos\theta_2 \end{array} \tag{1.1}$$

where

$$\begin{array}{l} \gamma = m + m_1 + m_2, \alpha_1 = m_1 \ell_1^{\mathbf{2}}, \alpha_2 = m_2 \ell_2^{\mathbf{2}}, \alpha_3 = m_2 \ell_1^{\mathbf{2}}, \\ \beta_1 = m_1 \ell_1, \beta_2 = m_2 \ell_2, \beta_3 = m_2 \ell_1, \xi = m_2 \ell_1 \ell_2, \\ D_1 = -g m_1 \ell_1, D_2 = -g m_2 \ell_1, D_3 = -g m_2 \ell_2 \end{array}$$

I will use the Euler-Lagrange equations $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$ for generalized coordinates $\{q_i\}$ with the Lagrangian $L$ and coordinates $\theta_1, \theta_2$ and $s$ and to describe the motion of the cart and pendulums.

# 2

The Euler-Lagrange equations for the system are:

$$\begin{array}{l}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = \\
\ddot{s}\left(\beta_1 + \beta_3\right)\left(\cos\theta_1\right) + \ddot{\theta}_1\left(\alpha_1 + \alpha_3\right) \\
+\ddot{\theta}_2\left(\xi\cos\theta_1\cos\theta_2\right) \\
+\dot{\theta}_2^2\left(\xi\sin\theta_1\cos\theta_2 - \xi\cos\theta_1\sin\theta_2\right) \\
+\left(D_1 + D_2\right)\sin\theta_1 = 0
\end{array} \tag{2.1}$$

$$\begin{array}{l}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = \\
\ddot{s}\left(\beta_2\cos\theta_2\right) + \ddot{\theta}_1\left(\xi\cos\theta_1\cos\theta_2\right) \\
+\dot{\theta}_1^2\left(\xi\cos\theta_1\sin\theta_2 - \xi\sin\theta_1\cos\theta_2\right) \\
+\ddot{\theta}_2\left(\alpha_2\right) \\
+D_3\sin\theta_2 = 0
\end{array} \tag{2.2}$$

$$\begin{array}{l}
\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = \\
\gamma\ddot{s} + \left(\beta_1 + \beta_3\right)\left(\ddot{\theta}_1\cos\theta_1 - \dot{\theta}_1^2\sin\theta_1\right) \\
+\beta_2\left(\ddot{\theta}_2\cos\theta_2 - \dot{\theta}_2^2\sin\theta_2\right) = 0
\end{array} \tag{2.3}$$

Using (2.3) with (2.1) and (2.2) to eliminate the $s$ variable, I can rewrite these equations with $\theta_1.\theta_2, \dot{\theta}_1, \dot{\theta}_2, \ddot{\theta}_1, \ddot{\theta}_2$ in terms of each other. This is convenient because I am not concerned with the motion of the cart.

$$\begin{array}{l}
\ddot{\theta}_1\left(\alpha_1 + \alpha_3 - \frac{\left(\beta_1+\beta_3\right)^2}{\gamma}\cos^2\theta_1\right) \\
+\dot{\theta}_1^2\left(-\frac{\left(\beta_1+\beta_3\right)^2}{\gamma}\sin\theta_1\cos\theta_1\right) \\
+\ddot{\theta}_2\left(\xi\cos\theta_1\cos\theta_2 - \frac{\beta_2\left(\beta_1+\beta_3\right)}{\gamma}\cos\theta_1\cos\theta_2\right) \\
+\dot{\theta}_2^2\left(\xi\sin\theta_1\cos\theta_2 - \xi\cos\theta_1\sin\theta_2 - \frac{\beta_2\left(\beta_1+\beta_3\right)}{\gamma}\cos\theta_1\sin\theta_2\right) \\
+\left(D_1 + D_2\right)\sin\theta_1 = 0
\end{array} \tag{2.4}$$

$$\begin{array}{l}
\ddot{\theta}_1\left(\xi\cos\theta_1\cos\theta_2 - \frac{\beta_2\left(\beta_1+\beta_3\right)}{\gamma}\cos\theta_1\cos\theta_2\right) \\
+\dot{\theta}_1^2\left(\xi\cos\theta_1\sin\theta_2 - \xi\sin\theta_1\cos\theta_2 - \frac{\beta_2\left(\beta_1+\beta_3\right)}{\gamma}\sin\theta_1\cos\theta_2\right) \\
+\ddot{\theta}_2\left(\alpha_2 - \frac{\beta_2^2}{\gamma}\cos^2\theta_2\right) \\
+\dot{\theta}_2^2\left(-\frac{\beta_2^2}{\gamma}\cos\theta_2\sin\theta_2\right) \\
+D_3\sin\theta_2 = 0
\end{array}$$

(2.5)

What are the dynamics like when a control force is applied to the cart? Suppose I manipulate the momentum conjugate to $s$, say $\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = u = -\frac{d}{dt}\left(k_1\dot{\theta}_1 + k_2\dot{\theta}_2\right)$, where $k_1 = k_1(\theta_1)$ and $k_2 = k_2(\theta_2)$ are feedback controls applied in the s-direction. According to work by A. Bloch, N. Leonard, and J. Marsden [2000], the controlled system is rewriteable, under certain conditions, as a closed-loop system:

$$
\begin{aligned}
&L_{\xi,\sigma} = 1/2\gamma\left(\dot{s} + k_1\dot{\theta}_1 + k_2\dot{\theta}_2\right)^{\mathbf{2}} + 1/2\left(\alpha_1 + \alpha_3\right)\dot{\theta}_1^{\mathbf{2}} + 1/2\alpha_2\dot{\theta}_2^{\mathbf{2}}\\
&+\left(\beta_1\dot{\theta}_1\cos\theta_1 + \beta_2\dot{\theta}_2\cos\theta_2 + \beta_3\dot{\theta}_1\cos\theta_1\right)\left(\dot{s} + k_1\dot{\theta}_1 + k_2\dot{\theta}_2\right)\\
&+\xi\dot{\theta}_1\dot{\theta}_2\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right) + \sigma\gamma/2\left(k_1\dot{\theta}_1 + k_2\dot{\theta}_2\right)^{\mathbf{2}}\\
&\left(D_1 + D_2\right)\cos\theta_1 + D_3\cos\theta_2
\end{aligned}
\tag{2.6}
$$

where $\sigma$ is some dimensionless constant. If there exist formulas for the controls so that the equations of motion corresponding to the controlled Lagrangian (2.6) match with the equations corresponding to the free Lagrangian (1.1) with the control law outlined above, then this revision of the Lagrangian makes sense.

# 3

The Euler-Lagrange equations for the controlled cart are:

$$
\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = \ddot{s}\left(\gamma k_1 + \left(\beta_1 + \beta_3\right)\cos\theta_1\right)\\
&+\ddot{\theta}_1\left(\alpha_1 + \alpha_3 + \gamma k_1^{\mathbf{2}} + \sigma\gamma k_1^{\mathbf{2}} + 2\left(\beta_1 + \beta_3\right)k_1\cos\theta_1\right)\\
&+\dot{\theta}_1^{\mathbf{2}}\left(\gamma k_1\underline{k_1'} + \sigma\gamma k_1\underline{k_1'}\right)\\
&+\ddot{\theta}_2\left(\gamma k_1 k_2 + \sigma\gamma k_1 k_2 + \left(\beta_1 + \beta_3\right)k_2\cos\theta_1 + \beta_2 k_1\cos\theta_2 + \xi\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right)\right)\\
&+\dot{\theta}_2^{\mathbf{2}}\left(\gamma k_1\underline{k_2'} + \sigma\gamma k_1\underline{k_2'} + \left(\beta_1 + \beta_3\right)\underline{k_2'}\cos\theta_1 - \beta_2 k_1\sin\theta_2 + \xi\left(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2\right)\right)\\
&+\left(D_1 + D_2\right)\sin\theta_1 = 0
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = \ddot{s}\left(\gamma k_2 + \beta_2\cos\theta_2\right)\\
&+\ddot{\theta}_1\left(\gamma k_1 k_2 + \sigma\gamma k_1 k_2 + \beta_2 k_1\cos\theta_2 + \left(\beta_1 + \beta_3\right)k_2\cos\theta_1 + \xi\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right)\right)\\
&+\dot{\theta}_1^{\mathbf{2}}\left(\gamma\underline{k_1'}k_2 + \sigma\gamma\underline{k_1'}k_2 - \left(\beta_1 + \beta_3\right)k_2\sin\theta_1 + \beta_2\underline{k_1'}\cos\theta_2 + \xi\left(\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2\right)\right)\\
&\ddot{\theta}_2\left(\alpha_2 + \gamma k_2^{\mathbf{2}} + \sigma\gamma k_2^{\mathbf{2}} + 2\beta_2 k_2\cos\theta_2\right)\\
&+\dot{\theta}_2^{\mathbf{2}}\left(\gamma k_2\underline{k_2'} + \sigma\gamma k_2\underline{k_2'}\right)\\
&+D_3\sin\theta_2 = 0
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
&\tfrac{d}{dt}\tfrac{\partial L}{\partial \dot{s}} = \ddot{s}\left(\gamma\right) + \dot{\theta}_1^{\mathbf{2}}\left(\gamma \underline{k_1'} - \left(\beta_1 + \beta_3\right)\sin\theta_1\right)\\
&+\dot{\theta}_2^{\mathbf{2}}\left(\gamma \underline{k_2'} - \beta_2 \sin\theta_2\right) + \ddot{\theta}_1\left(\gamma k_1 + \left(\beta_1 + \beta_3\right)\cos\theta_1\right) + \ddot{\theta}_2\left(\gamma k_2 + \beta_2 \cos\theta_2\right) = 0
\end{aligned}
\tag{3.3}
$$

The underlined derivatives are taken with respect to $\theta_i$, and the controls are defined by:

$$
\begin{aligned}
k_1 &= -\tfrac{1}{\sigma\gamma}\left(\left(\beta_1 + \beta_3\right)\cos\theta_1\right),\\
k_2 &= -\tfrac{1}{\sigma\gamma}\left(\beta_2 \cos\theta_2\right)
\end{aligned}
\tag{3.4}
$$

These expressions can be derived from the general matching procedure developed by A. Bloch, N. Leonard, and J. Marsden [2000], or simply by determining under what conditions the equations corresponding to the free Lagrangian with the control law match the equations corresponding to the closed-loop system.

Combining (3.3) with (3.1) and (3.2) and using (3.4) to simplify notation, the controlled equations reduce to:

$$
\begin{aligned}
&\ddot{\theta}_1\left(\alpha_1 + \alpha_3 + \sigma\gamma k_1^{\mathbf{2}} - \sigma^{\mathbf{2}}\gamma k_1^{\mathbf{2}}\right)\\
&+\dot{\theta}_1^{\mathbf{2}}\left(\sigma\gamma k_1 \underline{k_1'} + \sigma^{\mathbf{2}}\gamma k_1 \underline{k_1'}\right)\\
&+\ddot{\theta}_2\left(\sigma\gamma k_1 k_2 - \sigma^{\mathbf{2}}\gamma k_1 k_2 + \xi\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right)\right)\\
&+\dot{\theta}_2^{\mathbf{2}}\left(\sigma\gamma k_1 \underline{k_2'} + \sigma^{\mathbf{2}}\gamma k_1 \underline{k_2'} + \xi\left(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2\right)\right)\\
&+\left(D_1 + D_2\right)\sin\theta_1 = 0
\end{aligned}
\tag{3.5a}
$$

$$
\begin{aligned}
&\ddot{\theta}_1\left(\sigma\gamma k_1 k_2 - \sigma^{\mathbf{2}}\gamma k_1 k_2 + \xi\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right)\right)\\
&+\dot{\theta}_1^{\mathbf{2}}\left(\sigma\gamma \underline{k_1'} k_2 + \sigma^{\mathbf{2}}\gamma \underline{k_1'} k_2 + \xi\left(\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2\right)\right)\\
&\ddot{\theta}_2\left(\alpha_2 + \sigma\gamma k_2^{\mathbf{2}} - \sigma^{\mathbf{2}}\gamma k_2^{\mathbf{2}}\right)\\
&+\dot{\theta}_2^{\mathbf{2}}\left(\sigma\gamma k_2 \underline{k_2'} + \sigma^{\mathbf{2}}\gamma k_2 \underline{k_2'}\right)\\
&+D_3 \sin\theta_2 = 0
\end{aligned}
\tag{3.5b}
$$

# 4

According to theory developed by A. Bloch, N. Leonard, and J. Marsden [2000], the system is stable if the second variation of

$$(D_1+D_2)\cos\theta + D_3\cos\theta_2 + 1/2\begin{bmatrix}\dot\theta_1 & \dot\theta_2\end{bmatrix} *$$

$$\begin{bmatrix} \left(\frac{1}{\sigma\gamma}-\frac{1}{\gamma}\right)(\beta_1\cos\theta_1+\beta_3\cos\theta_1)^{\mathbf{2}}+\alpha_1+\alpha_3 & \begin{array}{c}\left(\frac{1}{\sigma\gamma}-\frac{1}{\gamma}\right)(\beta_1\cos\theta_1+\beta_3\cos\theta_1)(\beta_2\cos\theta_2)\\ +\xi\left(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\right)\end{array} \\ \begin{array}{c}\left(\frac{1}{\sigma\gamma}-\frac{1}{\gamma}\right)(\beta_1\cos\theta_1+\beta_3\cos\theta_1)(\beta_2\cos\theta_2)\\ +\xi\left(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\right)\end{array} & \left(\frac{1}{\sigma\gamma}-\frac{1}{\gamma}\right)(\beta_2\cos\theta_2)^{\mathbf{2}}+\alpha_2 \end{bmatrix}\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix} \tag{4.1}$$

is definite when evaluated at the equilibrium $\dot\theta_1=\dot\theta_2=\theta_1=\theta_2=\dot s=0$, which occurs when the matrix

$$B=\begin{bmatrix} D_1+D_2 & 0 & 0 & 0\\ 0 & D_3 & 0 & 0\\ 0 & 0 & \left(\frac{1}{\sigma\gamma}-\frac{1}{\gamma}\right)(\beta_1+\beta_3)^{\mathbf{2}}+\alpha_1+\alpha_3 & \left(\frac{1}{\sigma\gamma}-\frac{1}{\gamma}\right)(\beta_2\beta_1+\beta_2\beta_3)+\xi\\ 0 & 0 & \left(\frac{1}{\sigma\gamma}-\frac{1}{\gamma}\right)(\beta_2\beta_1+\beta_2\beta_3)+\xi & \left(\frac{1}{\sigma\gamma}-\frac{1}{\gamma}\right)\beta_2^{\mathbf{2}}+\alpha_2 \end{bmatrix} \tag{4.2}$$

(where $\begin{bmatrix}\delta\theta_1 & \delta\theta_2 & \delta\dot\theta_1 & \delta\dot\theta_2\end{bmatrix} B \begin{bmatrix}\delta\theta_1\\ \delta\theta_2\\ \delta\dot\theta_1\\ \delta\dot\theta_2\end{bmatrix}$ is the expression which results from taking variations of (4.1) twice and then evaluating at the equilibrium) is negative definite, meaning

$$\frac{1}{\sigma}<1-\frac{\gamma(\alpha_1+\alpha_3)}{(\beta_1+\beta_3)^{\mathbf{2}}}$$

and

$$\frac{1}{\gamma}-\frac{1}{\sigma\gamma}>\frac{\xi^{\mathbf{2}}-\alpha_2(\alpha_1+\alpha_3)}{2\xi(\beta_2\beta_1+\beta_2\beta_3)-\alpha_2(\beta_1+\beta_3)^{\mathbf{2}}-(\alpha_1+\alpha_3)\beta_2^{\mathbf{2}}}$$

or

$$\kappa>\frac{m}{m_1+m_2}$$

and

$$\kappa>\frac{-(m+m_1+m_2)}{m_1+m_2}-1 \tag{4.3}$$

I can also analyze the linear system to find the stabilizing values of $\kappa$ . Taking small variations of (3.5a) and (3.5b) about the upright equilibrium, I find the system

$$\begin{bmatrix}\delta\dot\phi_1\\ \delta\dot\phi_2\\ \delta\dot\theta_1\\ \delta\dot\theta_2\end{bmatrix}=A^{-1}\begin{bmatrix}\delta\phi_1\\ \delta\phi_2\\ \delta\theta_1\\ \delta\theta_2\end{bmatrix}$$

where

$$A=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ b & c & 0 & 0 \\ d & e & 0 & 0 \end{bmatrix}$$

and

$$\begin{aligned} b &= \frac{\alpha_1+\alpha_3+\sigma\gamma k_1^{\mathbf{2}}(e)-\sigma^{\mathbf{2}}\gamma k_1^{\mathbf{2}}(e)}{-(D_1+D_2)}, \\ c &= \frac{\sigma\gamma k_1(e)\,k_2(e)-\sigma^{\mathbf{2}}\gamma k_1(e)\,k_2(e)+\xi}{-(D_1+D_2)}, \\ d &= \frac{\sigma\gamma k_1(e)\,k_2(e)-\sigma^{\mathbf{2}}\gamma k_1(e)\,k_2(e)+\xi}{-D_3}, \\ e &= \frac{\alpha_2+\sigma\gamma k_2^{\mathbf{2}}(e)-\sigma^{\mathbf{2}}\gamma k_2^{\mathbf{2}}(e)}{-D_3}, \end{aligned} \tag{4.4}$$

and $k_i(e)$ are the values of the controls at the equilibrium position. For spectral stability the eigenvalues of $A^{-1}$ must be on or to the left of the imaginary axis. In this case the system is spectrally stable if over time the pendulums angles remain within a neighborhood of zero. The pendulum angles will not necessarily approach zero, and in general the system is not asymptotically stable without dissipation (friction).

Using (4.4), spectral stability occurs when both $b<0$ and $\det\left(\begin{bmatrix} b & c \\ d & e \end{bmatrix}\right)>0$, or when (4.3) is true.

# 5

The control law, or momentum conjugate to $s$, is defined by $u=-\frac{d}{dt}\left(k_1\dot{\theta}_1+k_2\dot{\theta}_2\right)$. The acceleration terms can be eliminated from the uncontrolled equations (2.4, 2.5):

$$\ddot{\theta}_1=f\left(\theta_1,\theta_2,\dot{\theta}_1,\dot{\theta}_2\right)=$$

$$-\frac{\left(\begin{array}{l} \dot{\theta}_1^{\mathbf{2}}\left(-\frac{(\beta_1+\beta_3)^{\mathbf{2}}}{\gamma}\sin\theta_1\cos\theta_1\right) \\ -\left(\xi\cos\theta_1\cos\theta_2-\frac{\beta_2(\beta_1+\beta_3)}{\gamma}\cos\theta_1\cos\theta_2\right)\dfrac{\left(\begin{array}{l}\dot{\theta}_1^{\mathbf{2}}\left(\xi\cos\theta_1\sin\theta_2-\xi\sin\theta_1\cos\theta_2-\frac{\beta_2(\beta_1+\beta_3)}{\gamma}\sin\theta_1\cos\theta_2\right)\\ +\dot{\theta}_2^{\mathbf{2}}\left(-\frac{\beta_2^{\mathbf{2}}}{\gamma}\cos\theta_2\sin\theta_2\right)+D_3\sin\theta_2\end{array}\right)}{\left(\alpha_2-\frac{\beta_2^{\mathbf{2}}}{\gamma}\cos^{\mathbf{2}}\theta_2\right)} \\ +\dot{\theta}_2^{\mathbf{2}}\left(\xi\sin\theta_1\cos\theta_2-\xi\cos\theta_1\sin\theta_2-\frac{\beta_2(\beta_1+\beta_3)}{\gamma}\cos\theta_1\sin\theta_2\right) \\ +(D_1+D_2)\sin\theta_1 \end{array}\right)}{\left(\alpha_1+\alpha_3-\frac{(\beta_1+\beta_3)^{\mathbf{2}}}{\gamma}\cos^{\mathbf{2}}\theta_1\right)\left(1-\dfrac{\left(\xi\cos\theta_1\cos\theta_2-\frac{\beta_2(\beta_1+\beta_3)}{\gamma}\cos\theta_1\cos\theta_2\right)\left(\xi\cos\theta_1\cos\theta_2-\frac{\beta_2(\beta_1+\beta_3)}{\gamma}\cos\theta_1\cos\theta_2\right)}{\left(\alpha_1+\alpha_3-\frac{(\beta_1+\beta_3)^{\mathbf{2}}}{\gamma}\cos^{\mathbf{2}}\theta_1\right)\left(\alpha_2-\frac{\beta_2^{\mathbf{2}}}{\gamma}\cos^{\mathbf{2}}\theta_2\right)}\right)}$$

$$\tag{5.1a}$$

$$\ddot{\theta}_2 = g\left(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2\right) =$$

$$-\frac{\left(\begin{array}{l} -\left(\xi\cos\theta_1\cos\theta_2 - \frac{\beta_2(\beta_1+\beta_3)}{\gamma}\cos\theta_1\cos\theta_2\right)\dfrac{\left(\begin{array}{l}\dot{\theta}_1^{\mathbf{2}}\left(-\frac{(\beta_1+\beta_3)^{\mathbf{2}}}{\gamma}\sin\theta_1\cos\theta_1\right)\\ +\dot{\theta}_2^{\mathbf{2}}\left(\xi\sin\theta_1\cos\theta_2 - \xi\cos\theta_1\sin\theta_2 - \frac{\beta_2(\beta_1+\beta_3)}{\gamma}\cos\theta_1\sin\theta_2\right)\\ +(D_1+D_2)\sin\theta_1\end{array}\right)}{\left(\alpha_1+\alpha_3-\frac{(\beta_1+\beta_3)^{\mathbf{2}}}{\gamma}\cos^{\mathbf{2}}\theta_1\right)}\\ +\dot{\theta}_1^{\mathbf{2}}\left(\xi\cos\theta_1\sin\theta_2 - \xi\sin\theta_1\cos\theta_2 - \frac{\beta_2(\beta_1+\beta_3)}{\gamma}\sin\theta_1\cos\theta_2\right)\\ +\dot{\theta}_2^{\mathbf{2}}\left(-\frac{\beta_2^{\mathbf{2}}}{\gamma}\cos\theta_2\sin\theta_2\right)\\ +D_3\sin\theta_2\end{array}\right)}{\left(\alpha_2 - \frac{\beta_2^{\mathbf{2}}}{\gamma}\cos^{\mathbf{2}}\theta_2\right)\left(1 - \frac{\left(\xi\cos\theta_1\cos\theta_2 - \frac{\beta_2(\beta_1+\beta_3)}{\gamma}\cos\theta_1\cos\theta_2\right)\left(\xi\cos\theta_1\cos\theta_2 - \frac{\beta_2(\beta_1+\beta_3)}{\gamma}\cos\theta_1\cos\theta_2\right)}{\left(\alpha_1+\alpha_3-\frac{(\beta_1+\beta_3)^{\mathbf{2}}}{\gamma}\cos^{\mathbf{2}}\theta_1\right)\left(\alpha_2-\frac{\beta_2^{\mathbf{2}}}{\gamma}\cos^{\mathbf{2}}\theta_2\right)}\right)} \tag{5.1b}$$

The control law can now be expressed in terms of the angles and their velocities.

$$\begin{array}{l} u = \dot{\theta}_1^{\mathbf{2}}\frac{((\beta_1+\beta_3)\sin\theta_1)}{\sigma\gamma} + \dot{\theta}_2^{\mathbf{2}}\frac{(\beta_2\sin\theta_2)}{\sigma\gamma}\\ -\frac{((\beta_1+\beta_3)\cos\theta_1)}{\sigma\gamma}f\left(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2\right) - \frac{(\beta_2\cos\theta_2)}{\sigma\gamma}g\left(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2\right)\end{array} \tag{5.2}$$

Matlab simulations (included below) show that this indeed stabilizes the system.

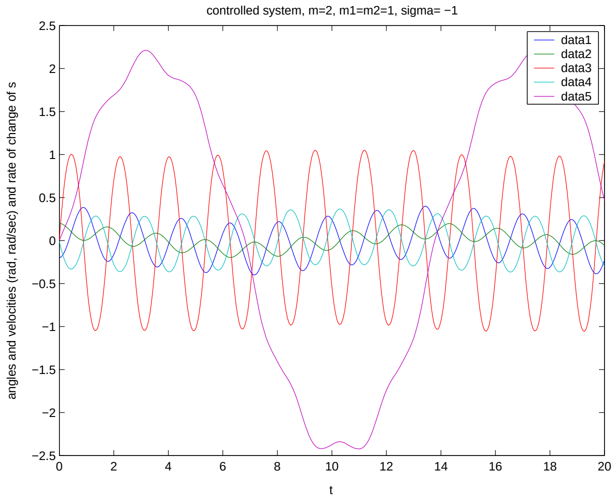
The stability condition $\kappa > \frac{-(m+m_1+m_2)}{m_1+m_2} - 1$ partially explains the conundrum presented in the introduction (in which direction do I push the cart?). As it turns out, both positive and negative feedback will stabilize, though it is still not intuitively clear- at least to me- why they do.

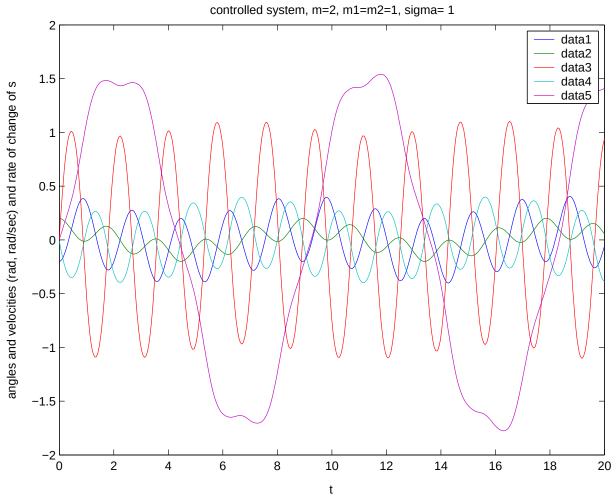
Both of the one-pendulum systems I discussed in the introduction were stabilized by positive feedback only.

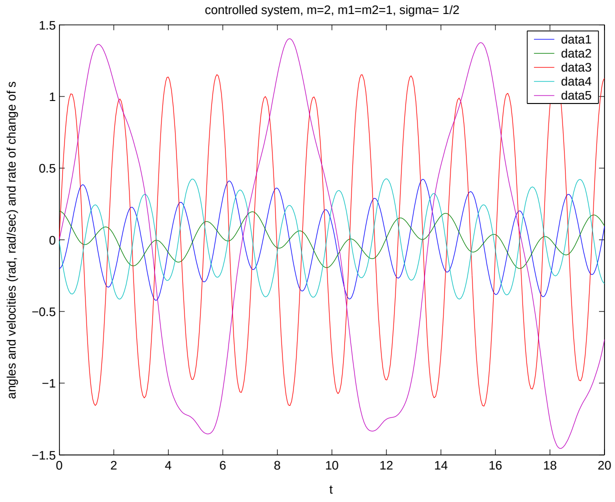
If I were to examine a more complicated system- for example, n inverted pendulums on a cart- I would find the similarities between the 1-pendulum equations (outlined in Controlled Lagrangians and the Stabilization of Mechanical Systems) and the 2-pendulum equations quite promising. However, I don't think a direct calculation of the n-pendulum equations would be very economic. Since calculating the two-pendulum equations seemed exponentially harder than calculating the one-pendulum equations, I would expect the three-pendulum equations to be even harder, and so on.

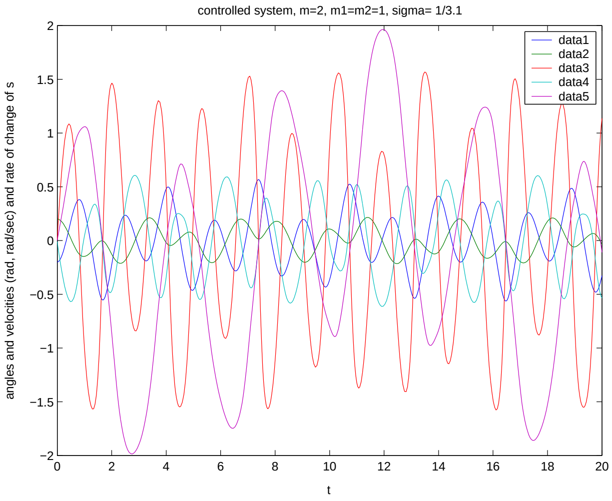
On the other hand, the symmetric nature of the equations suggests that there may be a nice inductive formula for the n-pendulum equations. I will leave this problem to another paper.

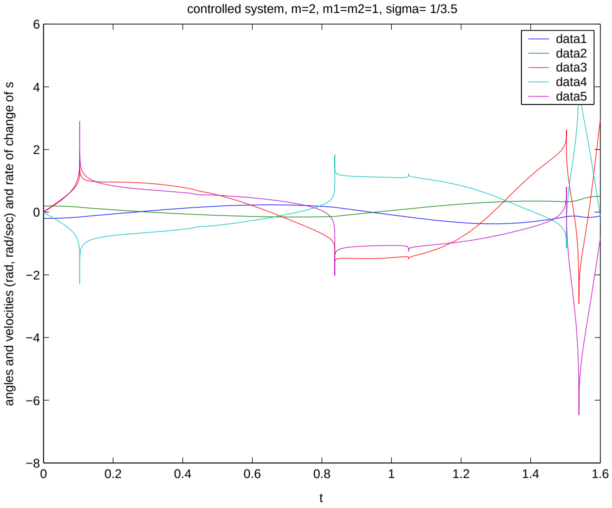
In the following simulations, I used the initial condition [x1, x2, x3, x4, x5]= [-.2 .2 .01 -.01 0] where x1= $\theta_1$ , x2= $\theta_2$ , x3= $\dot{\theta}_1$ , x4= $\dot{\theta}_5$, x5= $\dot{s}$. In fact, any initial condition close to the equilibrium position will work (I used this because it made nice graphs). In the following graphs, $\theta_1 = data1$, $\theta_2 = data2$, $\dot{\theta}_1 = data3$, $\dot{\theta}_2 = data4$, $\dot{s} = data5$.


controlled system, m=2, m1=m2=1, sigma= −1
angles and velocities (rad, rad/sec) and rate of change of s
t
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
0
2
4
6
8
10
12
14
16
18
20
data1
data2
data3
data4
data5


controlled system, m=2, m1=m2=1, sigma= 1
angles and velocities (rad, rad/sec) and rate of change of s
t
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
0
2
4
6
8
10
12
14
16
18
20
data1
data2
data3
data4
data5


controlled system, m=2, m1=m2=1, sigma= 1/2
angles and velocities (rad, rad/sec) and rate of change of s
t
1.5
1
0.5
0
−0.5
−1
−1.5
0
2
4
6
8
10
12
14
16
18
20
data1
data2
data3
data4
data5





# 6 Appendix

The Matlab code used for the simulations is given here.

function xdot=fivevar(t,x)
$m0 = 2; m1 = 1; m2 = 1; l1 = 1; l2 = 2; g = 9.8;$
$y = m0 + m1 + m2; a1 = m1 * l1^2; a2 = m2 * l2^2; a3 = m2 * l1^2; b1 = m1 * l1;$
$b2 = m2 * l2; b3 = m2 * l1; r = m2 * l1 * l2; d1 = -g * m1 * l1; d2 = -g * m2 * l1;$
$d3 = -g * m2 * l2;\ o = 1/100;$
$xdot = [x(3); x(4); (-(((-((b1+b3)*cos(x(1))+((b1+b3)*cos(x(1))/(-o*y)))*(((-x(3)^2*$
$(r*cos(x(1))*sin(x(2))-r*sin(x(1))*cos(x(2)))-d3*sin(x(2)))-x(4)^2*(r*sin(x(1))*$
$cos(x(2))-r*cos(x(1))*sin(x(2)))-(d1+d2)*sin(x(1)))/((a1+a3)*(1-(r^2*cos(x(1))^2*$
$cos(x(2))^2)/(a1+a3))))-(b2*cos(x(2))+(b2*cos(x(2))/(-o*y)))*((-((-(-x(3)^2*(r*$
$cos(x(1))*sin(x(2))-r*sin(x(1))*cos(x(2)))-d3*sin(x(2)))-x(4)^2*(r*sin(x(1))*$

$cos(x(2)) - r * cos(x(1)) * sin(x(2))) - (d1 + d2) * sin(x(1)))/((a1 + a3) * (1 - (r^2 * cos(x(1))^2 *$
$cos(x(2))^2)/(a1 + a3)))) * (r * cos(x(1)) * cos(x(2))) - x(3)^2 * (r * cos(x(1)) * sin(x(2)) -$
$r * sin(x(1)) * cos(x(2))) - d3 * sin(x(2)))/(a2)) - ((b1 + b3) * sin(x(1)) - ((b1 + b3) *$
$sin(x(1))/(o*y))) * x(3)^2 - (b2 * sin(x(2)) - (b2 * sin(x(2))/(o*y))) * x(4)^2)/y)/(1 - (1/((a1+$
$a3) * y)) * (-(b1 + b3) * cos(x(1)) - (b1 + b3) * cos(x(1))/(o * y)) * (-(b1 + b3) * cos(x(1)) + r *$
$cos(x(1)) * cos(x(2))^2 * b2/a2) - (1/(a2 * y)) * (b2 * cos(x(2))/(o * y)) * (b2 * cos(x(2)) + (b1+$
$b3) * cos(x(1)) + r * cos(x(1)) * cos(x(2))^2 * b2/a2))) * (b1 + b3) * cos(x(1)) - (r * cos(x(1)) *$
$cos(x(2))/a2) * (-(((-((b1 + b3) * cos(x(1)) + ((b1 + b3) * cos(x(1))/(-o * y))) * (((-x(3)^2 *$
$(r * cos(x(1)) * sin(x(2)) - r * sin(x(1)) * cos(x(2))) - d3 * sin(x(2))) - x(4)^2 * (r * sin(x(1)) *$
$cos(x(2)) - r * cos(x(1)) * sin(x(2))) - (d1 + d2) * sin(x(1)))/((a1 + a3) * (1 - (r^2 * cos(x(1))^2 *$
$cos(x(2))^2)/(a1 + a3)))) - (b2 * cos(x(2)) + (b2 * cos(x(2))/(-o * y))) * ((-((-(-x(3)^2 * (r *$
$cos(x(1)) * sin(x(2)) - r * sin(x(1)) * cos(x(2))) - d3 * sin(x(2))) - x(4)^2 * (r * sin(x(1)) *$
$cos(x(2)) - r * cos(x(1)) * sin(x(2))) - (d1 + d2) * sin(x(1)))/((a1 + a3) * (1 - (r^2 * cos(x(1))^2 *$
$cos(x(2))^2)/(a1 + a3)))) * (r * cos(x(1)) * cos(x(2))) - x(3)^2 * (r * cos(x(1)) * sin(x(2)) - r *$
$sin(x(1)) * cos(x(2))) - d3 * sin(x(2)))/(a2)) - ((b1 + b3) * sin(x(1)) - ((b1 + b3) * sin(x(1))/(o*$
$y))) * x(3)^2 - (b2 * sin(x(2)) - (b2 * sin(x(2))/(o * y))) * x(4)^2)/y)/(1 - (1/((a1 + a3) * y)) * (-(b1+$
$b3) * cos(x(1)) - (b1 + b3) * cos(x(1))/(o * y)) * (-(b1 + b3) * cos(x(1)) + r * cos(x(1)) * cos(x(2))^2 *$
$b2/a2) - (1/(a2 * y)) * (b2 * cos(x(2))/(o * y)) * (b2 * cos(x(2)) + (b1 + b3) * cos(x(1)) + r *$
$cos(x(1)) * cos(x(2))^2 * b2/a2))) * b2 * cos(x(2)) - x(3)^2 * (r * cos(x(1)) * sin(x(2)) - r * sin(x(1)) *$
$cos(x(2))) - d3 * sin(x(2))) - x(4)^2 * (r * sin(x(1)) * cos(x(2)) - r * cos(x(1)) * sin(x(2))) -$
$(d1 + d2) * sin(x(1)))/((a1 + a3) * (1 - (r^2 * cos(x(1))^2 * cos(x(2))^2)/(a1 + a3))); (-(((-((b1+$
$b3) * cos(x(1)) + ((b1 + b3) * cos(x(1))/(-o * y))) * (((-x(3)^2 * (r * cos(x(1)) * sin(x(2)) - r *$
$sin(x(1)) * cos(x(2))) - d3 * sin(x(2))) - x(4)^2 * (r * sin(x(1)) * cos(x(2)) - r * cos(x(1)) *$
$sin(x(2))) - (d1 + d2) * sin(x(1)))/((a1 + a3) * (1 - (r^2 * cos(x(1))^2 * cos(x(2))^2)/(a1 + a3)))) -$
$(b2 * cos(x(2)) + (b2 * cos(x(2))/(-o * y))) * ((-((-(-x(3)^2 * (r * cos(x(1)) * sin(x(2)) - r *$
$sin(x(1)) * cos(x(2))) - d3 * sin(x(2))) - x(4)^2 * (r * sin(x(1)) * cos(x(2)) - r * cos(x(1)) *$
$sin(x(2))) - (d1 + d2) * sin(x(1)))/((a1 + a3) * (1 - (r^2 * cos(x(1))^2 * cos(x(2))^2)/(a1 + a3)))) *$
$(r * cos(x(1)) * cos(x(2))) - x(3)^2 * (r * cos(x(1)) * sin(x(2)) - r * sin(x(1)) * cos(x(2))) - d3 *$
$sin(x(2)))/(a2)) - ((b1 + b3) * sin(x(1)) - ((b1 + b3) * sin(x(1))/(o*y))) * x(3)^2 - (b2 * sin(x(2)) -$
$(b2 * sin(x(2))/(o*y))) * x(4)^2)/y)/(1 - (1/((a1 + a3) * y)) * (-(b1 + b3) * cos(x(1)) - (b1 + b3) *$
$cos(x(1))/(o*y)) * (-(b1 + b3) * cos(x(1)) + r * cos(x(1)) * cos(x(2))^2 * b2/a2) - (1/(a2*y)) * (b2*$
$cos(x(2))/(o*y)) * (b2 * cos(x(2)) + (b1 + b3) * cos(x(1)) + r * cos(x(1)) * cos(x(2))^2 * b2/a2))) *$
$b2 * cos(x(2)) - ((-(((-((b1 + b3) * cos(x(1)) + ((b1 + b3) * cos(x(1))/(-o * y))) * (((-x(3)^2 *$
$(r * cos(x(1)) * sin(x(2)) - r * sin(x(1)) * cos(x(2))) - d3 * sin(x(2))) - x(4)^2 * (r * sin(x(1)) *$
$cos(x(2)) - r * cos(x(1)) * sin(x(2))) - (d1 + d2) * sin(x(1)))/((a1 + a3) * (1 - (r^2 * cos(x(1))^2 *$
$cos(x(2))^2)/(a1 + a3)))) - (b2 * cos(x(2)) + (b2 * cos(x(2))/(-o * y))) * ((-((-(-x(3)^2 * (r *$
$cos(x(1)) * sin(x(2)) - r * sin(x(1)) * cos(x(2))) - d3 * sin(x(2))) - x(4)^2 * (r * sin(x(1)) *$
$cos(x(2)) - r * cos(x(1)) * sin(x(2))) - (d1 + d2) * sin(x(1)))/((a1 + a3) * (1 - (r^2 * cos(x(1))^2 *$
$cos(x(2))^2)/(a1 + a3)))) * (r * cos(x(1)) * cos(x(2))) - x(3)^2 * (r * cos(x(1)) * sin(x(2)) - r *$
$sin(x(1)) * cos(x(2))) - d3 * sin(x(2)))/(a2)) - ((b1 + b3) * sin(x(1)) - ((b1 + b3) * sin(x(1))/(o*$
$y))) * x(3)^2 - (b2 * sin(x(2)) - (b2 * sin(x(2))/(o * y))) * x(4)^2)/y)/(1 - (1/((a1 + a3) * y)) *$
$(-(b1 + b3) * cos(x(1)) - (b1 + b3) * cos(x(1))/(o * y)) * (-(b1 + b3) * cos(x(1)) + r * cos(x(1)) *$
$cos(x(2))^2 * b2/a2) - (1/(a2*y)) * (b2*cos(x(2))/(o*y)) * (b2*cos(x(2)) + (b1+b3) * cos(x(1))+$
$r * cos(x(1)) * cos(x(2))^2 * b2/a2))) * (b1 + b3) * cos(x(1)) - (r * cos(x(1)) * cos(x(2))/a2) *$

$$(-(((-((b1+b3)*cos(x(1))+((b1+b3)*cos(x(1))/(-o*y)))*(((-x(3)^2*(r*cos(x(1))* sin(x(2))-r*sin(x(1))*cos(x(2)))-d3*sin(x(2)))-x(4)^2*(r*sin(x(1))*cos(x(2))-r* cos(x(1))*sin(x(2)))-(d1+d2)*sin(x(1)))/((a1+a3)*(1-(r^2*cos(x(1))^2*cos(x(2))^2)/(a1+ a3))))-(b2*cos(x(2))+(b2*cos(x(2))/(-o*y)))*((-((-(-x(3)^2*(r*cos(x(1))*sin(x(2))- r*sin(x(1))*cos(x(2)))-d3*sin(x(2)))-x(4)^2*(r*sin(x(1))*cos(x(2))-r*cos(x(1))* sin(x(2)))-(d1+d2)*sin(x(1)))/((a1+a3)*(1-(r^2*cos(x(1))^2*cos(x(2))^2)/(a1+a3))))* (r*cos(x(1))*cos(x(2)))-x(3)^2*(r*cos(x(1))*sin(x(2))-r*sin(x(1))*cos(x(2)))-d3* sin(x(2)))/(a2))-((b1+b3)*sin(x(1))-((b1+b3)*sin(x(1))/(o*y)))*x(3)^2-(b2*sin(x(2))- (b2*sin(x(2))/(o*y)))*x(4)^2)/y)/(1-(1/((a1+a3)*y))*(-(b1+b3)*cos(x(1))-(b1+b3)* cos(x(1))/(o*y))*(-(b1+b3)*cos(x(1))+r*cos(x(1))*cos(x(2))^2*b2/a2)-(1/(a2*y))*(b2* cos(x(2))/(o*y))*(b2*cos(x(2))+(b1+b3)*cos(x(1))+r*cos(x(1))*cos(x(2))^2*b2/a2)))* b2*cos(x(2))-x(3)^2*(r*cos(x(1))*sin(x(2))-r*sin(x(1))*cos(x(2)))-d3*sin(x(2)))- x(4)^2*(r*sin(x(1))*cos(x(2))-r*cos(x(1))*sin(x(2)))-(d1+d2)*sin(x(1)))/((a1+a3)* (1-(r^2*cos(x(1))^2*cos(x(2))^2)/(a1+a3))))*(r*cos(x(1))*cos(x(2)))-x(3)^2*(r*cos(x(1))* sin(x(2))-r*sin(x(1))*cos(x(2)))-d3*sin(x(2)))/(a2);(((-((b1+b3)*cos(x(1))+((b1+ b3)*cos(x(1))/(-o*y)))*(((-x(3)^2*(r*cos(x(1))*sin(x(2))-r*sin(x(1))*cos(x(2)))- d3*sin(x(2)))-x(4)^2*(r*sin(x(1))*cos(x(2))-r*cos(x(1))*sin(x(2)))-(d1+d2)* sin(x(1)))/((a1+a3)*(1-(r^2*cos(x(1))^2*cos(x(2))^2)/(a1+a3))))-(b2*cos(x(2))+(b2* cos(x(2))/(-o*y)))*((-((-(-x(3)^2*(r*cos(x(1))*sin(x(2))-r*sin(x(1))*cos(x(2)))- d3*sin(x(2)))-x(4)^2*(r*sin(x(1))*cos(x(2))-r*cos(x(1))*sin(x(2)))-(d1+d2)* sin(x(1)))/((a1+a3)*(1-(r^2*cos(x(1))^2*cos(x(2))^2)/(a1+a3))))*(r*cos(x(1))* cos(x(2)))-x(3)^2*(r*cos(x(1))*sin(x(2))-r*sin(x(1))*cos(x(2)))-d3*sin(x(2)))/(a2))- ((b1+b3)*sin(x(1))-((b1+b3)*sin(x(1))/(o*y)))*x(3)^2-(b2*sin(x(2))-(b2* sin(x(2))/(o*y)))*x(4)^2)/y)/(1-(1/((a1+a3)*y))*(-(b1+b3)*cos(x(1))-(b1+b3)* cos(x(1))/(o*y))*(-(b1+b3)*cos(x(1))+r*cos(x(1))*cos(x(2))^2*b2/a2)-(1/(a2*y))*(b2* cos(x(2))/(o*y))*(b2*cos(x(2))+(b1+b3)*cos(x(1))+r*cos(x(1))*cos(x(2))^2*b2/a2))]$$

I used this command to run the program:
$t0 = 0tmax = 30x0 = [-.2.2.01 - .010]$
$[t, x] = ode45('fivevar', [t0tmax], x0)$
plot(t,x),xlabel('t'),ylabel('angles and velocities (rad, rad/sec) and rate of change of s'),title('controlled system')