

# Stabilization of Inverted Pendulum Systems

David G. Aristoff

Department of Mathematics, University of Michigan

## Abstract

This paper presents a condition that stabilizes a general inverted pendulum system (IPS) and outlines a control law for the coordinated stabilization of several IPS's. It is shown that a system of any number of IPS's is an SMC system, and relates the IPS concept to systems like the inverted planar pendulum on a cart and the spherical pendulum on a puck.

## 1 Introduction

Most of this paper draws from the Method of Controlled Lagrangians in Bloch, Leonard and Marsden [2000] and Bloch, Chang, Leonard and Marsden [2000], hereafter referred to as [BLM] and [BCLM]. The graphs showing asymptotic stability were obtained by following energy methods outlined in [BCLM]. Kapitza [1951] analyzes a different method of stabilization.

## 2 Inverted Pendulum Systems

An  $n$ -inverted pendulum system ( $n$ -IPS) in  $\mathfrak{R}^3$  consists of point masses  $m_0, m_1, \dots, m_n$  with displacement vectors  $\mathbf{s}, \mathbf{x}_1, \dots, \mathbf{x}_n$ , respectively, in a given fixed inertial frame  $Oxyz$ . The system is subject to the constraints  $\|\mathbf{s} - \mathbf{x}_1\| = d_1$  and  $\|\mathbf{x}_i - \mathbf{x}_{i+1}\| = d_{i+1}$  for  $i = 1, \dots, n - 1$ , where the  $d_i$  are constants. The point masses experience a gravitational potential. The system's energy is

$$\frac{1}{2}m_0\dot{\mathbf{s}} \cdot \dot{\mathbf{s}} + \frac{1}{2} \sum_{i=1}^n m_i \dot{\mathbf{x}}_i \cdot \dot{\mathbf{x}}_i + V,$$

where  $V$  is the potential energy.

### 3 Generalized Coordinates and Lagrangian

Let  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  be an orthonormal basis for the space frame  $Oxyz$ . Write

$$\mathbf{s} = \sum_{i=1}^3 s_i \mathbf{e}_i, \mathbf{x}_1 = \sum_{i=1}^3 x_{1i} \mathbf{e}_i, \dots, \mathbf{x}_n = \sum_{i=1}^3 x_{ni} \mathbf{e}_i$$

so that one has Cartesian coordinates  $(s_1, s_2, s_3, x_{11}, x_{12}, x_{13}, \dots, x_{n1}, x_{n2}, x_{n3})$  describing the position of the IPS. Obtain generalized coordinates  $(s_1, s_2, s_3, \varphi_1, \psi_1, \dots, \varphi_n, \psi_n)$  by putting the  $(i+1)$ th point mass in spherical coordinates relative to the position of the  $i$ th mass:

$$\begin{aligned} \mathbf{x}_i - \mathbf{x}_{i-1} &= d_i \cos \varphi_i \sin \psi_i \mathbf{e}_1 + d_i \sin \varphi_i \sin \psi_i \mathbf{e}_2 + d_i \cos \psi_i \mathbf{e}_3 \\ \mathbf{x}_1 - \mathbf{s} &= d_1 \cos \varphi_1 \sin \psi_1 \mathbf{e}_1 + d_1 \sin \varphi_1 \sin \psi_1 \mathbf{e}_2 + d_1 \cos \psi_1 \mathbf{e}_3 \end{aligned}$$

The angles  $\varphi_i$  and  $\psi_i$  are measured in the inertial frame with respect to the basis  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ . It is clear that the configuration space  $Q$  is  $\mathfrak{R}^3 \times S^2 \times \dots \times S^2$  (there are  $n+1$  factors). Let  $\mathbf{a}_i$  be a unit vector in the direction of  $\mathbf{x}_i - \mathbf{x}_{i-1}$  and suppose the potential energy is given by gravity times the sum of the products of the point masses  $m_0, m_1, \dots, m_n$  and the components in the direction of  $\mathbf{e}_3$  of the corresponding displacement vectors  $\mathbf{s}, \mathbf{x}_1, \dots, \mathbf{x}_n$ . That is, let  $V = gm_0 \mathbf{s} \cdot \mathbf{e}_3 + gm_1 \mathbf{x}_1 \cdot \mathbf{e}_3 + \dots + gm_n \mathbf{x}_n \cdot \mathbf{e}_3$ . The Lagrangian, in this case the kinetic minus potential energy, is then

$$L = \sum_{i,j=1}^n \left( \frac{1}{2} \alpha_{ij} d_i d_j \dot{\mathbf{a}}_i \cdot \dot{\mathbf{a}}_j + \alpha_i d_i \dot{\mathbf{a}}_i \cdot \dot{\mathbf{s}} + \frac{1}{2} \alpha_0 \dot{\mathbf{s}} \cdot \dot{\mathbf{s}} - g \alpha_i d_i \mathbf{a}_i \cdot \mathbf{e}_3 - g \alpha_0 \mathbf{s} \cdot \mathbf{e}_3 \right)$$

where  $\alpha_i = \sum_{k=i}^n m_k$  and  $\alpha_{ij} = \sum_{k=\max(i,j)}^n m_k$ . Let  $f^\alpha(x) = \begin{cases} \cos x, & \alpha = 0 \\ \sin x, & \alpha = 1 \end{cases}$  and let  $P_{ij}^{\alpha\beta\gamma\delta} = f^\alpha(\varphi_i) f^\beta(\psi_i) f^\gamma(\varphi_j) f^\delta(\psi_j)$ ,  $Q_i^{\alpha\beta} = f^\alpha(\varphi_i) f^\beta(\psi_i)$ ,  $R_{ij}^{\alpha\beta} = f^\alpha(\psi_i) f^\beta(\psi_j)$ ,  $T_i^\alpha = f^\alpha(\psi_i)$ . Now

$$L = \sum_{i,j=1}^n \left( \frac{1}{2} \alpha_{ij} d_i d_j \left( P_{ij}^{1111} + P_{ij}^{0101} \right) \dot{\varphi}_i \dot{\varphi}_j + \frac{1}{2} \alpha_{ij} d_i d_j \left( R_{ij}^{11} + P_{ij}^{0000} + P_{ij}^{1010} \right) \dot{\psi}_i \dot{\psi}_j \right. \\ \left. + \alpha_{ij} d_i d_j \left( P_{ij}^{0110} - P_{ij}^{1100} \right) \dot{\varphi}_i \dot{\psi}_j - \alpha_i d_i Q_i^{11} \dot{\varphi}_i \dot{s}_1 + \alpha_i d_i Q_i^{00} \dot{\psi}_i \dot{s}_1 \right. \\ \left. + \alpha_i d_i Q_i^{01} \dot{\varphi}_i \dot{s}_2 + \alpha_i d_i Q_i^{10} \dot{\psi}_i \dot{s}_2 - \alpha_i d_i T_i^1 \dot{\psi}_i \dot{s}_3 \right. \\ \left. + \frac{1}{2} \alpha_0 \dot{s}_1^2 + \frac{1}{2} \alpha_0 \dot{s}_2^2 + \frac{1}{2} \alpha_0 \dot{s}_3^2 - g \alpha_i d_i T_i^0 - g \alpha_0 s_3 \right)$$

Note that the dynamics are invariant to translations of  $s_1$  and  $s_2$ , since the Lagrangian does not depend on these variables. Let the additive group  $\mathbb{R}^2$  act on the configuration space, where the action of  $(a, b) \in \mathbb{R}^2$  on  $\mathbb{R}^3 \times S^2 \times \dots \times S^2$  is given by

$$(a, b) \cdot (s_1, s_2, s_3, \varphi_1, \psi_1, \dots, \varphi_n, \psi_n) = (s_1 + a, s_2 + b, s_3, \varphi_1, \psi_1, \dots, \varphi_n, \psi_n)$$

so that  $G$  is a symmetry group. Note that the shape space is  $\mathbb{R} \times S^2 \times \dots \times S^2$ . The Euler-Lagrange equations of motion for the  $n$ -IPS are

$$\varepsilon_{\varphi_i}(L) = \sum_{j=1}^n \left( \begin{array}{l} \alpha_{ij} d_i d_j \left( P_{ij}^{1111} + P_{ij}^{0101} \right) \ddot{\varphi}_j + \alpha_{ij} d_i d_j \left( P_{ij}^{0110} - P_{ij}^{1100} \right) \ddot{\psi}_j \\ + \alpha_{ij} d_i d_j \left( P_{ij}^{1101} - P_{ij}^{0111} \right) \dot{\varphi}_j^2 + \alpha_{ij} d_i d_j \left( P_{ij}^{1101} - P_{ij}^{0111} \right) \dot{\psi}_j^2 \\ + 2\alpha_{ij} d_i d_j \left( P_{ij}^{0100} + P_{ij}^{1110} \right) \dot{\varphi}_j \dot{\psi}_j - \alpha_i d_i Q_i^{11} \ddot{s}_1 + \alpha_i d_i Q_i^{01} \ddot{s}_2 \end{array} \right) = 0$$

$$\varepsilon_{\psi_i}(L) = \sum_{j=1}^n \left( \begin{array}{l} \alpha_{ij} d_i d_j \left( R_i^{11} + P_{ij}^{0000} + P_{ij}^{1010} \right) \ddot{\psi}_j + \alpha_{ij} d_i d_j \left( P_{ij}^{1001} - P_{ij}^{0011} \right) \ddot{\varphi}_j \\ + \alpha_{ij} d_i d_j \left( R_i^{10} - P_{ij}^{0001} - P_{ij}^{1011} \right) \dot{\psi}_j^2 + \alpha_{ij} d_i d_j \left( -P_{ij}^{0001} - P_{ij}^{1011} \right) \dot{\varphi}_j^2 \\ + 2\alpha_{ij} d_i d_j \left( P_{ij}^{1000} - P_{ij}^{0010} \right) \dot{\varphi}_j \dot{\psi}_j + \alpha_i d_i k Q_i^{00} \ddot{s}_1 + \alpha_i d_i Q_i^{10} \ddot{s}_2 \\ - \alpha_i d_i T_i^1 \ddot{s}_3 - g \alpha_i d_i T_i^1 \end{array} \right) = 0$$

$$\varepsilon_{s_1}(L) = \sum_{j=1}^n \left( \begin{array}{l} \alpha_0 \ddot{s}_1 - \alpha_j d_j Q_j^{11} \ddot{\varphi}_j + \alpha_j d_j Q_j^{00} \ddot{\psi}_j \\ - \alpha_j d_j Q_j^{01} \dot{\varphi}_j^2 - \alpha_j d_j Q_j^{01} \dot{\psi}_j^2 - 2\alpha_j d_j Q_j^{10} \dot{\varphi}_j \dot{\psi}_j \end{array} \right) = 0$$

$$\varepsilon_{s_2}(L) = \sum_{j=1}^n \left( \begin{array}{l} \alpha_0 \ddot{s}_2 + \alpha_j d_j Q_j^{01} \ddot{\varphi}_j + \alpha_j d_j Q_j^{10} \ddot{\psi}_j \\ - \alpha_j d_j Q_j^{11} \dot{\varphi}_j^2 - \alpha_j d_j Q_j^{11} \dot{\psi}_j^2 + 2\alpha_j d_j Q_j^{00} \dot{\varphi}_j \dot{\psi}_j \end{array} \right) = 0$$

$$\varepsilon_{s_3}(L) = \sum_{j=1}^n \left( \alpha_0 \ddot{s}_3 - \alpha_j d_j T_j^1 \ddot{\psi}_j - \alpha_j d_j T_j^0 \dot{\psi}_j^2 + g \alpha_0 \right) = 0$$

where  $\epsilon_x$  is the Euler-Lagrange operator defined by  $\epsilon_x(L) = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x}$ .

## 4 Stabilization

The goal of this paper is to stabilize the IPS about the upright equilibria  $\psi_1 = \dot{\psi}_1 = \dots = \psi_n = \dot{\psi}_n = 0$ ,  $\varphi_1 = \dots = \varphi_n = 0$ ,  $\dot{s}_1 = \xi_1$ ,  $\dot{s}_2 = \xi_2$ ,  $\dot{s}_3 = 0$ , where the  $\xi_i$  are constants, by letting  $\mathfrak{R}^2$  act on the configuration space as discussed in the last section. That is, the point mass  $m_0$  will be directed to move in a plane parallel to the one spanned by  $\mathbf{e}_1$  and  $\mathbf{e}_2$  so that the other masses will be balanced near the upright position. Interestingly, it has been shown that a 1-IPS may be stabilized by moving the mass  $m_0$  vertically (i.e., in the subspace spanned by  $\mathbf{e}_3$ ). See, for example, Kapitza [1951].

This paper will apply the Method of Controlled Lagrangians from Bloch, Leonard and Marsden [2000] (hereafter called [BLM]) to stabilize the system. Note that the upright equilibria described above are relative equilibria, since they are solutions of the Euler-Lagrange equations that lie inside  $\mathfrak{R}^2$ -orbits. See [BLM].

In the Method of Controlled Lagrangians, each vector  $v$  in  $TQ$  is decomposed into two components:  $Ver(v)$ , a vector tangent to the orbits of the  $G$ -action, and  $Hor(v)$ , a vector in the metric orthogonal space to the space of vertical vectors. The components are called the vertical and horizontal parts, respectively, of  $v$ . One may think of the vertical part of a tangent vector as a piece in the  $\mathfrak{R}^2$  (group) direction. The horizontal part can be visualized as a piece in the  $S^2 \times \dots \times S^2$  (shape) direction.

This problem will require kinetic shaping only (as opposed to kinetic and potential shaping; for a reference, see Bloch, Chang, Leonard and Marsden [2001]) since the group  $\mathfrak{R}^2$  acts only on symmetry variables. The kinetic energy will be reshaped with a new choice of horizontal space,  $Hor_\tau$ , and an adjustment by a scalar factor,  $\sigma$ , of the metric acting on horizontal vectors.

The Controlled Lagrangian  $L_{\tau,\sigma}$  incorporates the new kinetic energy. The goal is to match the Euler-Lagrange equations corresponding to the Controlled Lagrangian (CL) with the controlled  $L$ -equations, called the controlled cart (CC) equations, which are the Euler-Lagrange equations for  $L$  with controls affecting the symmetry variables. The Simplified Matching Conditions (SMC) of [BLM] are sufficient to guarantee the existence of parameters  $\tau$  and  $\sigma$  such that the CC and CL equations match. Hence if the SMC hold, the Energy-Momentum Method can be used to check stability, since the CL system is Lagrangian (i.e., it satisfies the Euler-Lagrange equations). For a reference on the Energy-Momentum Method, see Bloch, Marsden and Zenkov [1998].

Note that since  $m_0$  will be moved only in the plane spanned by  $\mathbf{e}_1$  and  $\mathbf{e}_2$ ,

the (nonsymmetry) position variable  $s_3$  will remain constant. Hence without loss of generality, suppose  $s_3 \equiv 0$ . (If  $s_3 \equiv c$ ,  $c \neq 0$ , then the potential energy is changed by the constant factor  $g\alpha_0 c$  and the kinetic energy is unchanged, so the Lagrangian differs by a constant, which vanishes in the Euler-Lagrange equations.) In light of this, the SMC for kinetic shaping hold, since  $\alpha_0$  is a constant and

$$\begin{aligned} \frac{\partial(-Q_i^{11})}{\partial\varphi_i} &= -Q_i^{01} = \frac{\partial(Q_i^{00})}{\partial\psi_i}, & \frac{\partial(-Q_i^{11})}{\partial\psi_i} &= -Q_i^{10} = \frac{\partial(Q_i^{00})}{\partial\varphi_i}, \\ \frac{\partial(Q_i^{01})}{\partial\varphi_i} &= -Q_i^{11} = \frac{\partial(Q_i^{10})}{\partial\psi_i}, & \frac{\partial(Q_i^{01})}{\partial\psi_i} &= Q_i^{00} = \frac{\partial(Q_i^{10})}{\partial\varphi_i}, \\ \frac{\partial(-Q_i^{11})}{\partial\varphi_j} &= \frac{\partial(-Q_i^{11})}{\partial\psi_j} = \frac{\partial(Q_i^{00})}{\partial\varphi_j} = \frac{\partial(Q_i^{00})}{\partial\psi_j} = \frac{\partial(Q_i^{01})}{\partial\varphi_j} = \\ \frac{\partial(Q_i^{01})}{\partial\psi_j} &= \frac{\partial(Q_i^{10})}{\partial\varphi_j} = \frac{\partial(Q_i^{01})}{\partial\psi_j} = 0 \end{aligned}$$

for  $j \neq i$ . See [BLM]. The conditions define  $\tau$  and  $\sigma$ , producing control laws  $u_j^{cons}$  and a CL  $L_{\tau,\sigma}$  such that the systems

$$\begin{aligned} \varepsilon_{\varphi_i}(L) &= \varepsilon_{\psi_i}(L) = 0, \quad i = 1, \dots, n, \\ \varepsilon_{s_1}(L) &= u_1, \quad \varepsilon_{s_2}(L) = u_2 \end{aligned}$$

and

$$\varepsilon_{\varphi_i}(L_{\tau,\sigma}) = \varepsilon_{\psi_i}(L_{\tau,\sigma}) = \varepsilon_{s_1}(L_{\tau,\sigma}) = \varepsilon_{s_2}(L_{\tau,\sigma}) = 0, \quad i = 1, \dots, n$$

are equivalent. See Theorem 3.2, [BLM]. Following [BLM], the controls are

$$\begin{aligned} u_1^{cons} &= - \sum_{i=1}^n \frac{d}{dt} \left( \frac{1}{\sigma} \alpha_i d_i Q_i^{11} \dot{\varphi}_i - \frac{1}{\sigma} \alpha_i d_i Q_i^{00} \dot{\psi}_i \right) \\ u_2^{cons} &= - \sum_{i=1}^n \frac{d}{dt} \left( -\frac{1}{\sigma} \alpha_i d_i Q_i^{01} \dot{\varphi}_i - \frac{1}{\sigma} \alpha_i d_i Q_i^{10} \dot{\psi}_i \right) \end{aligned}$$

and the CL is

$$L_{\tau,\sigma} = \sum_{i,j=1}^n \left( \begin{array}{l} \frac{1}{2}\alpha_{ij}d_id_j \left( P_{ij}^{1111} + P_{ij}^{0101} \right) \dot{\varphi}_i \dot{\varphi}_j \\ + \frac{1}{2}\alpha_{ij}d_id_j \left( R_{ij}^{11} + P_{ij}^{0000} + P_{ij}^{1010} \right) \dot{\psi}_i \dot{\psi}_j \\ + \alpha_{ij}d_id_j \left( P_{ij}^{0110} - P_{ij}^{1100} \right) \dot{\varphi}_i \dot{\psi}_j \\ + \alpha_i d_i \left( Q_i^{00} \dot{\psi}_i - Q_i^{11} \dot{\varphi}_i \right) \left( \dot{s}_1 + \frac{1}{\sigma} \frac{\alpha_j}{\alpha_0} d_j Q_j^{11} \dot{\varphi}_j - \frac{1}{\sigma} \frac{\alpha_j}{\alpha_0} d_j Q_j^{00} \dot{\psi}_j \right) \\ + \alpha_i d_i \left( Q_i^{01} \dot{\varphi}_i + Q_i^{10} \dot{\psi}_i \right) \left( \dot{s}_2 - \frac{1}{\sigma} \frac{\alpha_j}{\alpha_0} d_j Q_j^{01} \dot{\varphi}_j - \frac{1}{\sigma} \frac{\alpha_j}{\alpha_0} d_j Q_j^{10} \dot{\psi}_j \right) \\ - \alpha_i d_i T_i^1 \dot{\psi}_i \dot{s}_3 \\ + \frac{1}{2} \alpha_0 \left( \dot{s}_1 + \frac{1}{\sigma} \frac{\alpha_i}{\alpha_0} d_i Q_i^{11} \dot{\varphi}_i - \frac{1}{\sigma} \frac{\alpha_i}{\alpha_0} d_i Q_i^{00} \dot{\psi}_i \right)^2 \\ + \frac{1}{2} \alpha_0 \left( \dot{s}_2 - \frac{1}{\sigma} \frac{\alpha_i}{\alpha_0} d_i Q_i^{01} \dot{\varphi}_i - \frac{1}{\sigma} \frac{\alpha_i}{\alpha_0} d_i Q_i^{10} \dot{\psi}_i \right)^2 \\ + \frac{1}{2} \alpha_0 \dot{s}_3 \\ + \frac{1}{2} \left( \frac{1}{\sigma} \frac{\alpha_i}{\alpha_0} d_i Q_i^{11} \dot{\varphi}_i - \frac{1}{\sigma} \frac{\alpha_i}{\alpha_0} d_i Q_i^{00} \dot{\psi}_i \right) \left( -\alpha_j d_j Q_j^{11} \dot{\varphi}_j + \alpha_j d_j Q_j^{00} \dot{\psi}_j \right) \\ + \frac{1}{2} \left( -\frac{1}{\sigma} \frac{\alpha_i}{\alpha_0} d_i Q_i^{01} \dot{\varphi}_i - \frac{1}{\sigma} \frac{\alpha_i}{\alpha_0} d_i Q_i^{10} \dot{\psi}_i \right) \left( \alpha_j d_j Q_j^{01} \dot{\varphi}_j + \alpha_j d_j Q_j^{10} \dot{\psi}_j \right) \\ - g \alpha_i d_i T_i^0 - g \alpha_0 s_3 \end{array} \right)$$

## 5 Stability

In this section it will become apparent that it is difficult to stabilize a general  $n$ -IPS for  $n \geq 2$  using the Method of Controlled Lagrangians. If the point masses  $m_1, \dots, m_n$  are constrained to move in a plane (by, for example, forcing  $\varphi_1 = \dots = \varphi_n = c \in [0, 2\pi]$  at any time  $t$ ), however, there is a nice condition for stabilization.

One may check that the Euler-Lagrange equations for the CL match the CC equations, and in fact, the systems corresponding to the Lagrangians  $L$  and  $L_{\tau,\sigma}$  have the same relative equilibria. See Prop. 3.3, [BLM]. Hence the CC system is Lyapunov stable at the relative equilibria  $\psi_1 = \dot{\psi}_1 = \dots = \psi_n = \dot{\psi}_n = 0$ ,  $\dot{\varphi}_1 = \dots = \dot{\varphi}_n = 0$ ,  $\dot{s}_1 = \xi_1$ ,  $\dot{s}_2 = \xi_2$ ,  $\dot{s}_3 = 0$  if the second variation of  $E = KE_\tau + \tilde{V}$  is definite there, where  $KE_\tau$  is the  $hor_\tau$  kinetic energy and  $\tilde{V}$  is the amended potential as defined in [BLM].

Let  $EQ$  be the set of vectors in  $TQ$  satisfying  $\psi_1 = \dot{\psi}_1 = \dots = \psi_n = \dot{\psi}_n = 0$ ,  $\dot{\varphi}_1 = \dots = \dot{\varphi}_n = 0$ ,  $\varphi_1 = \dots = \varphi_n$ ,  $\dot{s}_1 = \xi_1$ ,  $\dot{s}_2 = \xi_2$ , and  $\dot{s}_3 = 0$ , where the  $\xi_i$  are constants. Think of  $EQ$  as the set of upright relative equilibria for a  $n$ -IPS with point masses constrained to move in a plane. Let  $\Gamma_i = -g\alpha_i d_i$ , and for  $i \leq j$ , let  $\beta_{ij} = \left( \alpha_j + \left( \frac{1}{\sigma} - 1 \right) \frac{\alpha_i \alpha_j}{\alpha_0} \right) d_i d_j$ . Let

$$B_k = \begin{pmatrix} 2\beta_{11} & \beta_{12} & \cdots & \beta_{1k} \\ \beta_{12} & 2\beta_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ \beta_{1k} & \cdots & \cdots & 2\beta_{kk} \end{pmatrix}, \quad C = \begin{pmatrix} \Gamma_1 & 0 & \cdots & 0 \\ 0 & \Gamma_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \Gamma_n \end{pmatrix},$$

$$D = \begin{pmatrix} \delta\dot{\varphi}_1 \\ \vdots \\ \delta\dot{\varphi}_n \end{pmatrix}, \quad E = \begin{pmatrix} \delta\varphi_1 \\ \vdots \\ \delta\varphi_n \end{pmatrix}, \quad X = \begin{pmatrix} \frac{B_n}{2} & 0 \\ 0 & C \end{pmatrix}$$

*Lemma 1.1.* The system  $\varepsilon_{\varphi_i}(L) = \varepsilon_{\psi_i}(L) = 0$ ,  $\varepsilon_{s_1}(L) = u_1^{cons}$  and  $\varepsilon_{s_2}(L) = u_2^{cons}$  is stable at the relative equilibrium  $\mathbf{v} \in EQ$  if the matrix  $X$  is definite.

*Proof.* From [BLM], the system is stable at a relative equilibrium if the second variation of  $E_\tau$ , the sum of the  $\tau$ -horizontal energy and the amended potential energy, is definite there. Following formulas (3.13) and (3.20) from [BLM], the  $\tau$ -horizontal energy is

$$\begin{aligned} & \frac{1}{2} \sum_{i,j=1}^n \left( \begin{aligned} & \alpha_{ij} d_i d_j \left( P_{ij}^{1111} + P_{ij}^{0101} \right) \\ & + (-\alpha_i d_i Q_i^{11}) \left( -\frac{1}{\sigma\alpha_0} + \frac{1}{\alpha_0} \right) (-\alpha_j d_j Q_j^{11}) \\ & + (\alpha_i d_i Q_i^{01}) \left( -\frac{1}{\sigma\alpha_0} + \frac{1}{\alpha_0} \right) (\alpha_j d_j Q_j^{01}) \end{aligned} \right) \dot{\varphi}_i \dot{\varphi}_j \\ & + \frac{1}{2} \sum_{i,j=1}^n \left( \begin{aligned} & \alpha_{ij} d_i d_j \left( R_{ij}^{11} + P_{ij}^{0000} + P_{ij}^{1010} \right) \\ & + (\alpha_i d_i Q_i^{00}) \left( -\frac{1}{\sigma\alpha_0} + \frac{1}{\alpha_0} \right) (\alpha_j d_j Q_j^{00}) \\ & + (\alpha_i d_i Q_i^{10}) \left( -\frac{1}{\sigma\alpha_0} + \frac{1}{\alpha_0} \right) (\alpha_j d_j Q_j^{10}) \end{aligned} \right) \dot{\psi}_i \dot{\psi}_j \\ & + \frac{1}{2} \sum_{i,j=1}^n \left( \begin{aligned} & \alpha_{ij} d_i d_j \left( P_{ij}^{0110} - P_{ij}^{1100} \right) \\ & + (-\alpha_i d_i Q_i^{11}) \left( -\frac{1}{\sigma\alpha_0} + \frac{1}{\alpha_0} \right) (\alpha_j d_j Q_j^{00}) \\ & + (\alpha_i d_i Q_i^{01}) \left( -\frac{1}{\sigma\alpha_0} + \frac{1}{\alpha_0} \right) (\alpha_j d_j Q_j^{01}) \end{aligned} \right) \dot{\varphi}_i \dot{\psi}_j \end{aligned}$$

Noting that  $Q_i^{\alpha\beta} Q_j^{\gamma\delta} = P_{ij}^{\alpha\beta\gamma\delta}$  and  $T_i^\alpha T_j^\beta = R_{ij}^{\alpha\beta}$ , this becomes

$$\frac{1}{2} \sum_{i,j=1}^n \left( \begin{aligned} & \frac{1}{2} \left( P_{ij}^{1111} + P_{ij}^{0101} \right) \left( \alpha_{ij} d_i d_j + \left( \frac{1}{\sigma} - 1 \right) \frac{\alpha_i \alpha_j}{\alpha_0} \right) \dot{\varphi}_i \dot{\varphi}_j \\ & + \frac{1}{2} \left( R_{ij}^{11} + P_{ij}^{0000} + P_{ij}^{1010} \right) (\alpha_{ij} d_i d_j) \dot{\psi}_i \dot{\psi}_j \\ & + \frac{1}{2} \left( P_{ij}^{0000} + P_{ij}^{1010} \right) \left( \frac{1}{\sigma} - 1 \right) \left( \frac{\alpha_i \alpha_j}{\alpha_0} \right) \dot{\psi}_i \dot{\psi}_j \\ & + \frac{1}{2} \left( P_{ij}^{0110} - P_{ij}^{1100} \right) \left( \alpha_{ij} d_i d_j + \left( \frac{1}{\sigma} - 1 \right) \frac{\alpha_i \alpha_j}{\alpha_0} \right) \dot{\varphi}_i \dot{\psi}_j \end{aligned} \right)$$

At any point  $\mathbf{v}$  in  $EQ$ , the second variation of this is

$$\frac{1}{2} \sum_{i,j=1}^n \left( \alpha_{ij} d_i d_j + \left( \frac{1}{\sigma} - 1 \right) \frac{\alpha_i \alpha_j}{\alpha_0} \right) \delta \dot{\psi}_i \delta \dot{\psi}_j$$

Note that the second variation at a relative equilibrium point not satisfying  $\varphi_1 = \dots = \varphi_n$  depends on these angles; this proof requires that the angles be the same. See Remark 1. Following formula (3.19) from [BLM], the amended potential energy is

$$\sum_{i=1}^n \left( g \alpha_i d_i T_i^0 + \frac{1}{2} \alpha_0 (\xi_1^2 + \xi_2^2) \right)$$

where the  $\xi_i$  are the components of  $\dot{\mathbf{s}}$  at the equilibrium. The second term in the parentheses is a constant, so it vanishes in the variations. It follows that the second variation of the amended potential at any point  $\mathbf{v}$  in  $EQ$  is

$$- \sum_{i=1}^n g \alpha_i d_i (\delta \psi_i)^2$$

Hence  $\delta^2 E|_{EQ}$  is

$$\begin{pmatrix} D \\ E \end{pmatrix}^T X \begin{pmatrix} D \\ E \end{pmatrix}$$

which is definite if and only if  $X$  is a definite matrix. This completes the proof.

*Theorem 1.1.* The system  $\varepsilon_{\varphi_i}(L) = \varepsilon_{\psi_i}(L) = 0$ ,  $\varepsilon_{s_1}(L) = u_1^{cons}$  and  $\varepsilon_{s_2}(L) = u_2^{cons}$  is stable at the relative equilibrium  $\mathbf{v} \in EQ$  if  $-\frac{1}{\sigma} > \frac{\alpha_0 - \alpha_1}{\alpha_1}$ .

*Proof.* Note that since  $\Gamma_i = -g \alpha_i d_i < 0$  for  $i = 1, \dots, n$ ,  $X$  is definite if and only if  $B_n$  is negative definite.

Let

$$Y_k = \underbrace{\begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 1 & \dots & \dots & 2 \end{pmatrix}}_{k \times k}, \quad \tilde{\beta}_{ij} = \begin{cases} 2\beta_{ii}, & i = j \\ \beta_{ij}, & i \neq j \end{cases}$$

$$f(i) = \left( 1 + \left( \frac{1}{\sigma} - 1 \right) \frac{\alpha_i}{\alpha_0} \right) d_i, \quad g(j) = \alpha_j d_j$$

Denote the group of permutations of  $\{1, \dots, k\}$  by  $S_k$ . For any  $\gamma \in S_k$ ,  $k = 1, \dots, n$ , let  $O_\gamma^k = \{i \in \{1, \dots, k\} : \gamma(i) = i\}$  and define  $\eta_k : S_k \rightarrow \{1, \dots, k\}$  by  $\eta_k(\gamma) = |O_\gamma^k|$ . Note that

$$\det(Y_k) = \det \left( \underbrace{I + \begin{pmatrix} 0 & \cdots & 0 & -1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -1 \\ 1 & \cdots & 1 & 1 \end{pmatrix}}_{k \times k} \right) = k + 1$$

Now

$$\begin{aligned} \det(B_k) &= \sum_{\gamma \in S_k} \operatorname{sgn}(\gamma) \tilde{\beta}_{1\gamma(1)} \cdots \tilde{\beta}_{k\gamma(k)} \\ &= \sum_{\gamma \in S_k} 2^{\eta_k(\gamma)} \operatorname{sgn}(\gamma) \beta_{1\gamma(1)} \cdots \beta_{k\gamma(k)} \\ &= \sum_{\gamma \in S_k} 2^{\eta_k(\gamma)} \operatorname{sgn}(\gamma) f(1) g(\gamma(1)) \cdots f(k) g(\gamma(k)) \\ &= \sum_{\gamma \in S_k} 2^{\eta_k(\gamma)} \operatorname{sgn}(\gamma) f(1) g(1) \cdots f(k) g(k) \\ &= \sum_{\gamma \in S_k} 2^{\eta_k(\gamma)} \operatorname{sgn}(\gamma) \beta_{11} \cdots \beta_{kk} \\ &= \beta_{11} \cdots \beta_{kk} \left( \sum_{\gamma \in S_k} 2^{\eta_k(\gamma)} \operatorname{sgn}(\gamma) \right) \\ &= \beta_{11} \cdots \beta_{kk} (\det(Y_k)) \\ &= (k + 1) \beta_{11} \cdots \beta_{kk} \end{aligned}$$

Hence  $B_n$  is negative definite if and only if  $\beta_{ii} < 0$  for  $i = 1, \dots, n$ . Note that the latter is true

$$\begin{aligned} &\Leftrightarrow \left( \alpha_i - \left(1 - \frac{1}{\sigma}\right) \frac{\alpha_i^2}{\alpha_0} \right) d_i^2 < 0, \quad i = 1, \dots, n \\ &\Leftrightarrow -\frac{1}{\sigma} > \frac{\alpha_0}{\alpha_i} - 1, \quad i = 1, \dots, n \\ &\Leftrightarrow -\frac{1}{\sigma} > \frac{\alpha_0}{\alpha_1} - 1 = \frac{\alpha_0 - \alpha_1}{\alpha_1} \end{aligned}$$

So  $X$  is negative definite if and only if  $-\frac{1}{\sigma} > \frac{\alpha_0 - \alpha_1}{\alpha_1}$ . By the lemma, the system is stable if  $\sigma$  satisfies this condition. This completes the proof. See figs. 1a, 1b.

## 6 Stability of Planar IPS

The set of relative equilibria  $EQ$  is rather artificial, since one cannot expect  $\varphi_1 = \dots = \varphi_n$  near an upright equilibrium point. Some new definitions are therefore in order.

Let  $Q_\theta$  be the set of vectors in  $Q$  satisfying  $\varphi_1 = \dots = \varphi_n = \theta \in [0, 2\pi]$ , and let  $TQ_\theta$  be the corresponding tangent space. Let  $L_\theta$  be the restriction of the Lagrangian to  $TQ_\theta$ . That is, given  $\mathbf{v} \in TQ_\theta$ , let  $L_\theta(\mathbf{v}) = L(\mathbf{v})$ . Then  $L_\theta$  is an SMC system (since  $L$  is); let  $u_1^\theta, u_2^\theta$  and  $L_{\theta,\tau,\sigma}$  be the corresponding controls and CL, respectively, obtained by applying the Method of Controlled Lagrangians. Let  $EQ_\theta$  be the set of vectors in  $TQ_\theta$  satisfying  $\psi_1 = \dot{\psi}_1 = \dots = \psi_n = \dot{\psi}_n = 0$  and  $\dot{s}_1 = \xi_1, \dot{s}_2 = \xi_2$ , where the  $\xi_i$  are constants.

*Corollary 1.1.* The system  $\varepsilon_{\psi_1}(L_\theta) = \dots = \varepsilon_{\psi_n}(L_\theta) = 0, \varepsilon_{s_1}(L_\theta) = u_1^\theta$  and  $\varepsilon_{s_2}(L_\theta) = u_2^\theta$  is stable at the relative equilibrium  $\mathbf{v} \in EQ_\theta$  if  $-\frac{1}{\sigma} > \frac{\alpha_0 - \alpha_1}{\alpha_1}$ .

*Proof.* At any point  $\mathbf{v} \in EQ_\theta$ , the second variation of  $E_\theta$  (the sum of the  $hor_\tau^\theta$  kinetic energy and amended potential energy), is the same as the second variation of  $E$  as defined in the lemma. Hence the proof of Theorem 1.1 holds, confirming the condition given above.

## 7 Remarks

*Rmk. 1.* Note that the second variation of the  $hor_\tau$  kinetic energy at a relative equilibrium point not satisfying  $\varphi_1 = \dots = \varphi_n$  is

$$\sum_{i,j=1}^n (\cos \varphi_i \cos \varphi_j + \sin \varphi_i \sin \varphi_j) \left( \alpha_{ij} d_i d_j + \left( \frac{1}{\sigma} - 1 \right) \frac{\alpha_i \alpha_j}{\alpha_0} \right) \delta \psi_i \delta \dot{\psi}_j$$

*Rmk. 2.* Note that the proof of Theorem 1.1 holds for the special case of the 1-IPS with mass  $m_0$  moving in the plane and  $m_1$  moving in space. See the section on the spherical pendulum on a puck from [BLM]; the stability condition of Theorem 1.1 matches the condition given in that paper.

*Rmk. 3.* The 1-IPS constrained to move in a plane is the inverted pendulum on a cart, also from [BLM]. The stability condition given in Theorem 1.1 is consistent with the conditions presented in that paper.

*Rmk.* 3. Notice that the stability condition is equivalent to

$$-\frac{1}{\sigma} > \frac{m_0}{\sum_{i=1}^n m_i}$$

## 8 Asymptotic Stabilization and Full Phase Space Stabilization

Given stability in the sense of Lyapunov (as achieved in section 6), asymptotic stabilization can be relatively easily achieved by using energy methods outlined, for example, in [BLM]. Complete phase space stabilization requires a bit more work, but a general method can be found in Bloch, Chang, Leonard and Marsden [2001]. See figs 2a, 2b, 3a, and 3b.

## 9 Figures

All figures are of IPS's moving in the plane, as previously discussed, with configuration spaces  $Q_0 \subset Q$ . Figs. 1a, 2a and 3a are 1-IPS systems, where  $(data1, data2, data3, data4) = \psi_1, \dot{\psi}_1, s_1, \dot{s}_1$ , and figs. 1b, 2b and 3b are 2-IPS systems, where  $(data1, data2, data3, data4, data5, data6) = \psi_1, \dot{\psi}_1, \psi_2, \dot{\psi}_2, s_1, \dot{s}_1$ .

Fig. 1a

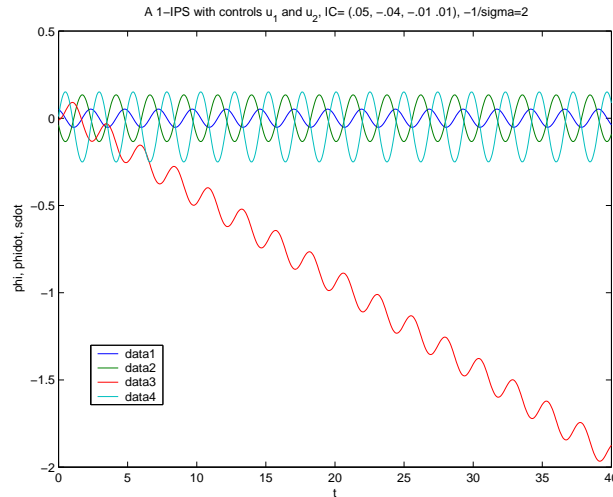


Fig. 1b

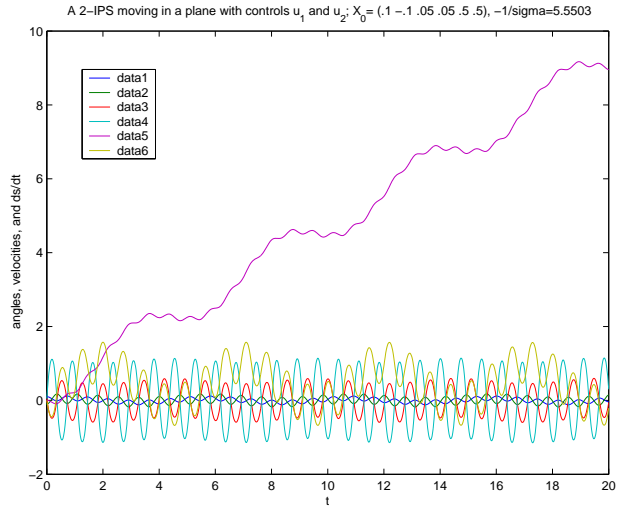


Fig. 2a

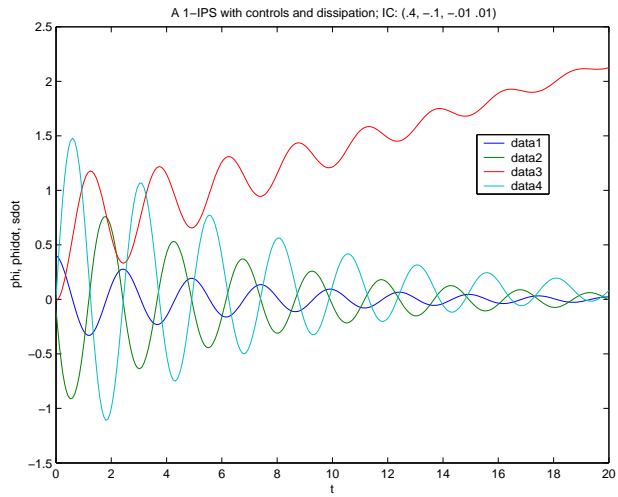


Fig. 2b

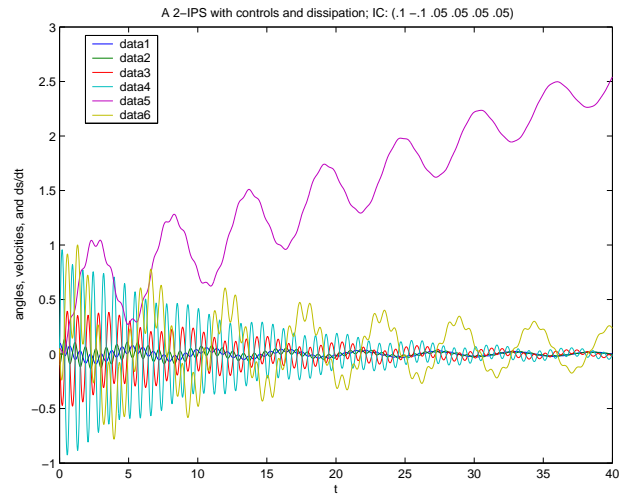


Fig. 3a

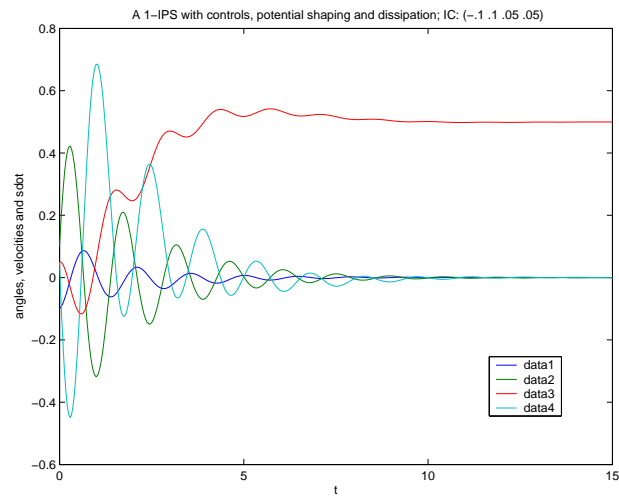
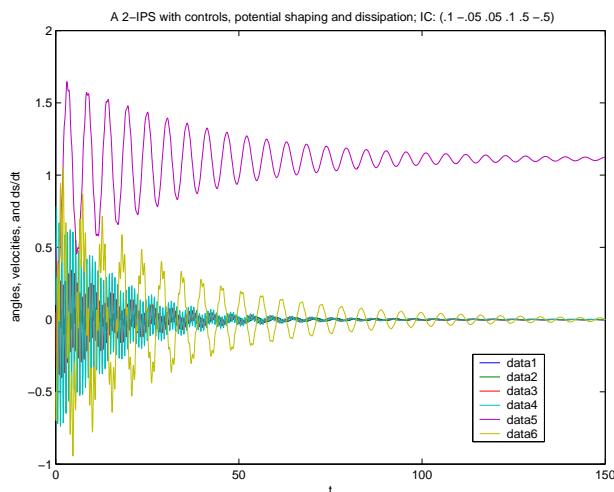


Fig. 3b



### References

Bloch, A.M., D.E. Chang, N.E. Leonard and J.E. Marsden [2001], Controlled Lagrangians and the Stabilization of Mechanical Systems II: Potential Shaping, IEEE

Bloch, A.M., N.E. Leonard and J.E. Marsden [2000], Controlled Lagrangians and the Stabilization of Mechanical Systems I: The First Matching Theorem, Proc. 45th IEEE Conf. Decision and Control, Phoenix, AZ, 2253-2270.

Bloch, A.M., J.E. Marsden and D.V. Zenkov [1998], The Energy-Momentum Method for the Stability of Nonholonomic Systems, Dyn. Stab. Of Systems., 13, 123-166. Trans. Automat. Control, 45, 1483-1491

P. L. Kapitza [1951], Dynamic Stability of the Pendulum With Vibrating Suspension Point, Sov. Phys. JETP 21 (5), 588-597, see also D. Ter Haar [1965], Collected Papers of P. L. Kapitza, Vol.2, 714-726, Pergamon, London, 1965

Leonard, N.E., L. Moreau and S. Nair [2003], Coordinated Control of Networked Mechanical Systems with Unstable Dynamics, Proc. 42nd IEEE Conf. Decision and Control, 2003