Ternary Basic Digit Sets

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1 Introduction

A radix representation of a real number, for an integral base $B \geq 2$ and digit set $\mathcal{D}$, is a string of digits $(a_n...a_1a_0...)_B = \sum_{i=-n}^{\infty} a_{-i}B^{-i}$, with $a_k \in \mathcal{D}$. It is commonly known that using a digit set of the form $\{0, 1, ..., B-1\}$, any positive real number has a B-ary radix representation. This is simply the standard B-ary representation of a number.

Alternative digit representations have been of practical use in some computer applications. Such representations have been used to optimize computer arithmetic (Reitwiesner [7]) and also the specific computations needed in some cryptographic algorithms (Gordon [3]). In addition, radix representations have been studied in connection with self-affine tilings (Gilbert [2]), (Curry [1]) and fractal sets (Heuberger [4]).

We are interested in the digit sets of the form $\mathcal{D}_{a,b} = \{0, a, -b\}$ such that any real number r can be represented by a 3-ary representation $(r_n...r_1r_0...)_3 = \sum_{i=-n}^{\infty} r_{-i}3^{-i}$ using digits in $\mathcal{D}_{a,b}$ (that is, without the use of a "sign" to designate a number as positive or negative). For base 3, the most common example of these sets is the balanced ternary digit set $\{0, 1, -1\}$.

One of the first to study digit sets that can be used to produce representations of all real numbers in depth was D. W. Matula [5]. We define the following:

Definition 1.1. Let $\mathcal{D}$ be a finite set of integers with $0 \in \mathcal{D}$. Then $\mathcal{D}$ is *basic* for base B if and only if every integer i has a unique finite B-ary representation

$$i = \sum_{j=0}^{N} d_j B^j \textit{ with } d_i \in \mathcal{D}.$$

Matula [5, Theorem 10] showed that using a basic digit set $\mathcal{D}$, any real number has a B-ary representation using digits from $\mathcal{D}$.

Work has been done in extending Matula's results on certain classes of digit sets. Muir [6] has done work to this extent by studying digit sets of the form $\{0, 1, -x\}$ for non-adjacent binary representations. We seek to extend the work of Matula and others by seeking to find some underlying structure and characteristics of the sets $\mathcal{X}_a := \{b \in \mathbb{Z} \mid \{0, a, -b\} \textit{ is basic}\}$. Most of the work done here is regarding $\mathcal{X}_1$.

We first start with a review of the necessary and sufficient conditions found by Matula that characterize basic digit sets. Then, in Section 3 we discuss some empirical observations made on a sample of digits in $\mathcal{X}_1$ (produced by the program in Section 8). The necessary conditions for $\mathcal{D}_{1,b}$ to be basic used in this computation are discussed in Section 4, where an optimization to Matula's algorithm is also discussed. In Section 5, a theorem is proven that explains an important observation about $\mathcal{X}_1$, and a consequence of this theorem on the upper asymptotic density of $\mathcal{X}_1$ is discussed in Section 6. Finally, in Section 7 we discuss some infinite families of basic $\mathcal{D}_{1,b}$.

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2 Basic Digit Sets

It is convenient to use polynomials as a way to represent the B-ary expansion of a number. Let $\mathcal{P}_{\mathcal{D}}[x]$ represent the set of all polynomials with coefficients in $\mathcal{D}$. Taking the base B, for each term $a_i x^i$ in the polynomial $P(x) \in \mathcal{P}_{\mathcal{D}}[x]$, i represents the position of the digit a_i in the base B expansion of the number $P(B)$.

This gives us an equivalent statement of Definition 1.1: $\mathcal{D}$ is basic for base B if and only if for every integer i, there exists a unique $P(x) \in \mathcal{P}_{\mathcal{D}}[x]$ such that

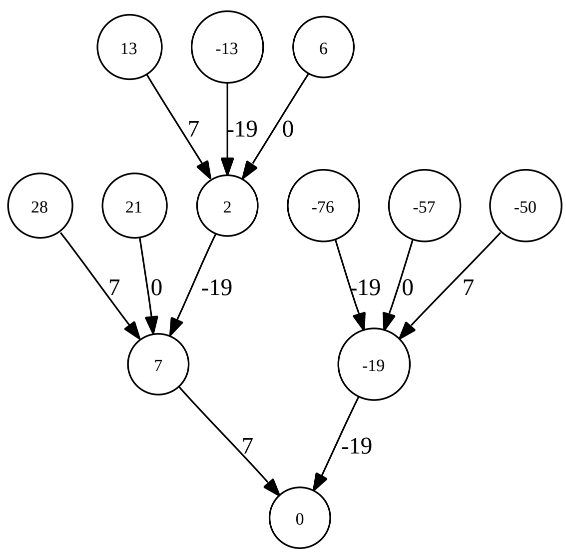

Figure 2.1: Graph of $\Phi(i)$ for the basic digit set $\mathcal{D}_{7,19} = \{0, 7, -19\}$.

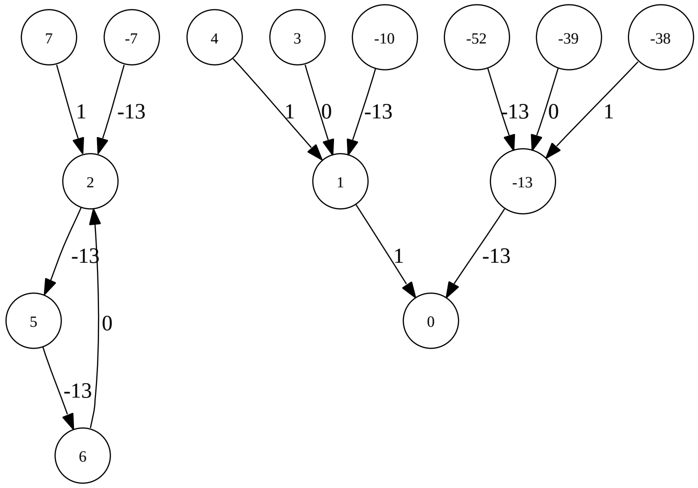

Figure 2.2: Graph of $\Phi(i)$ for the digit set $\mathcal{D}_{1,13} = \{0, 1, -13\}$
Since there is a loop, $\mathcal{D}_{1,13}$ cannot be basic.

$P(B) = i$.

The following characterization of basic digit sets is quite useful.

Theorem 2.1 (Matula). *Let $\mathcal{D}$ be a finite set of integers with $0 \in \mathcal{D}$. Then $\mathcal{D}$ is basic if and only if the following conditions hold:*

1. *$\mathcal{D}$ is a complete residue system modulo B and*
2. *The set $\mathcal{D}^n = \{P(B) \mid P(x) \in \mathcal{P}_\mathcal{D}[x]$ and $deg(P(x)) \leq n-1\}$ contains no nonzero multiple of $B^n - 1$ for any $n \geq 1$.*

The second condition of Theorem 2.1 cannot be verified in finitely many computations. Matula found an algorithm to verify that a given digit set is basic, as summarized in Theorem 2.2.

It is helpful to use a function to build the expansion of a given number. Let $||i||_\mathcal{D}$ denote the digit $d \in \mathcal{D}$ such that $d \equiv i \ (mod\ 3)$. Define the function

$$\Phi(i) = \frac{i - ||i||_\mathcal{D}}{3} \tag{2.1}$$

where $\mathcal{D}$ is the digit set in the current context. We also adopt the convetion $\Phi^n(i) = \Phi(\Phi^{n-1}(i))$. By Theorem 2.3, $\mathcal{D}_{a,b}$ is basic if and only if for all i, if $j \neq k$ and $\Phi^j(i) \neq 0$ then $\Phi^j(i) \neq \Phi^k(i)$. This means that the graph of the $\Phi(i)$ function must be a tree rooted at 0. Figure 2.1 and Figure 2.2 illustrate this, as well as that $\Phi(i)$ is the number represented by $(...||\Phi^{n+1}(i)||_\mathcal{D}||\Phi(i)||_\mathcal{D}||i||_\mathcal{D})_3$ when $\mathcal{D}$ is basic, ignoring leading zeroes.

Theorem 2.3 puts the $\Phi(i)$ function to use in the characterization of basic digit sets.

Theorem 2.2 (Matula). *Let $B > 0$ be an integer. Let $\mathcal{D}$ be a complete residue system modulo B. Let $d_{min} = \min \mathcal{D}$, $d_{max} = \max \mathcal{D}$. Then $\mathcal{D}$ is basic for base B if and only if for all i such that*

$$\frac{-d_{max}}{B-1} \le i \le \frac{-d_{min}}{B-1} \tag{2.2}$$

there exists a $P(x) \in \mathcal{P}_{\mathcal{D}}[x]$ such that $P(B) = i$.

Theorem 2.3 (Matula). *There is a radix representation of an integer i given by digits in $\mathcal{D}$ if and only if the sequence $(i, \Phi(i), \Phi^2(i), ...)$ does not have any duplicate elements, except for* 0.

In Theorem 4.1, we present an improvement on Matula's algorithm (applying Theorem 2.3 to each i that satisfies (2.2)) that reduces the number of computations needed to check that a given digit set is basic. An implementation of the improved Matula algorithm based on Theorem 4.1 is presented as a computer program in Section 8.

3 Computational Results

The program in Section 8 was run to find the following basic digit sets.

Table 3.1: All $b < 10,000$ such that $\mathcal{D}_{1,b}$ is basic

1	7	25	31	37	73	79	85	97	103
193	241	253	271	313	319	337	343	361	517
553	661	703	721	727	733	745	751	781	799
805	865	925	943	967	1015	1039	1081	1087	1633
1687	1705	1837	1981	2125	2137	2143	2185	2191	2233
2257	2263	2341	2581	2593	2605	2641	2719	2761	2797
2815	2833	2857	2887	2893	2911	3127	3145	3193	3199
3247	3265	3277	4417	4717	4783	4903	5065	5095	5143
5497	5503	5533	5713	5743	5791	6001	6121	6127	6193
6337	6367	6391	6421	6463	6481	6493	6505	6517	6541
6553	6559	6577	6583	6625	6631	6637	6661	6703	6727
6751	6793	6853	7087	7093	7105	7177	7201	7207	7213
7225	7261	7273	7279	7303	7351	7357	7375	7441	7453
7519	7687	7801	7873	7885	7897	7945	7981	8113	8425
8461	8521	8533	8605	8665	8677	8695	8737	8743	8785
8821	8857	8893	9013	9097	9367	9391	9397	9439	9463
9601	9637	9679	9721	9745	9751				

Table 3.2: All $b < 50,000$ such that $\mathcal{D}_{7,b}$ is basic

1	19	241	529	625	1531	1759	2059	2185	2449
2467	3571	3841	3985	4153	4399	4609	4759	4915	5035
5059	5119	5155	5305	5371	5527	5647	5665	5851	5875
5881	5911	6067	6289	6553	6793	6889	6931	6985	6991
7351	7435	7459	11155	11593	11611	13231	13339	13345	13495
13513	13969	14257	14647	14761	15019	15121	15151	15307	15331
15391	15475	15697	15745	15817	15865	15895	16735	16987	17011
17065	17239	17635	17785	18937	19051	19657	19675	20281	20641
20695	20887	21139	21295	21979	22153	22177	22651	22681	22921
32059	33841	35887	38305	38611	39745	39937	40975	41359	41425
41875	42871	43087	43375	44305	45361	45697	45817	45985	46099
46201	46339	46351	46435	46471	46531	46555	46705	46825	46939
47155	47431	47659	48211	48751	49495	49531			

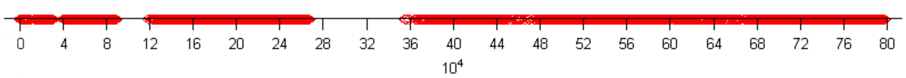

Figure 3.1: All $b \leq 900,000$ such that $\mathcal{D}_{1,b}$ is basic plotted on a number line

On a first glance of the digits of $\mathcal{X}_1$, it appears that there is no particular pattern to the set of digits that makes $\mathcal{D}_{1,b}$ basic. There are, however, some interesting patterns that the set exhibits on a larger scale. Below, we summarize some empirical observations based on the available data.

Observation 3.1. $f(3x) \approx 2f(x)$.

Since we are interested in the density of $\mathcal{X}_1$ in the integers, we define the counting function $f(x) := \#\{b \mid b \leq x \text{ and } \mathcal{D}_{1,b} \text{ is basic}\}$. As can be seen in Figure 3.2, the shape of the $f(x)$ function when plotted against x is approximately self similar. A plot of the points $(\frac{x}{3}, \frac{f(x)}{2})$ produces a curve that is almost identical in shape and position as the curve produced by plotting $(x, f(x))$.

Observation 3.2. $f(x) < x^c$ for $c \approx 0.588$ for $x \leq 10^6$.

This is shown by Figure 3.2.

Observation 3.3. For increasingly large integers b, there are increasingly large intervals on which there are no b such that $D_{1,b}$ is basic (*dead spots*).

This pattern can be seen in Figure 3.1. We will rigorously justify this observation in Theorem 5.1, where a sufficient condition on b is given that excludes $D_{1,b}$ from being basic.

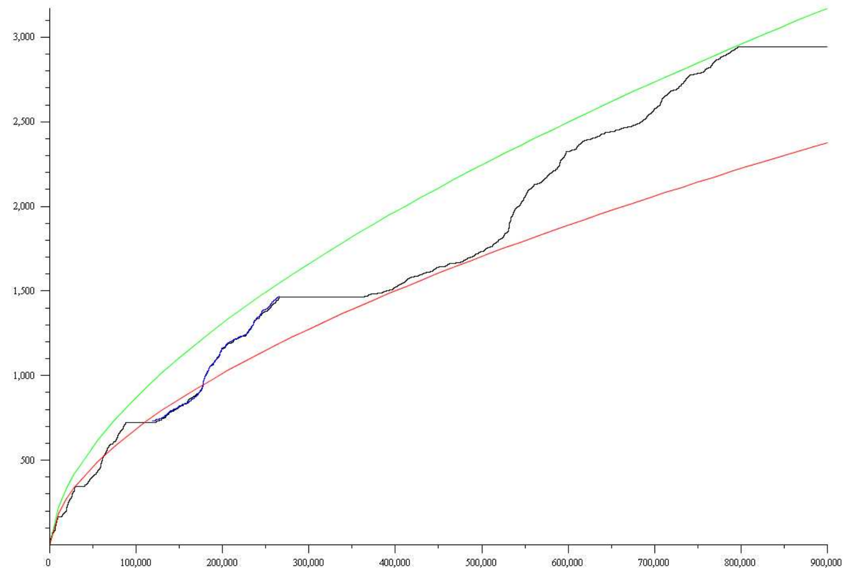

Figure 3.2: Black: The counting function $f(x)$ vs. x. Blue: $(\frac{x}{3}, \frac{f(x)}{2})$ for $354289 \leq x \leq 797155$. Green: $x^{0.588}$. Red: $x^{0.567}$.

4 Necessary Conditions for $\mathcal{D}_{1,b}$ Basic

In searching for the digits in $\mathcal{X}_1$, we first start with some necessary conditions to narrow the search. If some integer $i \equiv 2$ $(mod\ 3)$, then the last digit in the expansion of i must be equivalent to 2 $(mod\ 3)$ (since if $i = ... + a_1 3 + a_0$, then $i \equiv a_0$ $(mod\ 3)$). Thus, if all integers are to be represented using digits from $\mathcal{D}_{1,b}$, then $b \equiv 2$ $(mod\ 3)$. Futhermore, b must be odd, for if it weren't and $\mathcal{D}_{1,b}$ were basic, then $b = -2k$ for some integer k and we would have

$$k = 3^n a_n + ... + 3a_1 + a_0$$

and

$$3k - 2k = 3^{n+1} a_n + ... + 3^2 a_1 + 3a_0 - 2k$$

which would mean that the integer k has two different representations, $(a_n ... a_1 a_0)_3$ and $(a_n ... a_1 a_0 b)_3$, which contradicts the uniqueness of representation of integers with digits from basic digit sets. Finally, b must be negative since all integers must be represented using only these digits. Thus, we restrict b to integers of the form $6k + 5$, where the integer $k < 0$. Theorem 2.1 generalizes these conditions for any digit set $\mathcal{D}$.

To establish that $\mathcal{D}_{a,b}$ is basic, it is necessary and sufficient by Theorem 2.2 to check that there exists a ternary expansion for all i such that

$$-\frac{a}{2} \leq i \leq \frac{b}{2} \tag{4.1}$$

For large a and/or large b, this becomes computationally difficult. We present in Theorem 4.1 a method of reducing the number of computations needed to check $D_{a,b}$ basic.

Theorem 4.1. *Suppose that $\mathcal{D}_{a,b}$ is not basic. Then there exists some integer i that cannot be expressed uniquely using digits from $\mathcal{D}_{a,b}$. Furthermore, there exists an integer i' that cannot be expressed uniquely using digits from $\mathcal{D}_{a,b}$ such that $|i'| \leq |i|$. If $i < 0$, then the i' such that $|i'|$ is minimal satisfies $i' \equiv 1 \ (mod\ 3)$. If $i > 0$, then the i' such that $|i'|$ is minimal satisfies $i' \equiv 2 \ (mod\ 3)$.*

Proof. Suppose $\mathcal{D}_{a,b}$ is not basic and $i > 0$ cannot be expressed using digits from $\mathcal{D}_{a,b}$. Then we have $\Phi^j(i) = \Phi^k(i) \neq 0$ for some $j > k \geq 0$ by Theorem 2.3. Suppose $i \not\equiv 2 \ (mod\ 3)$. We have

$$\Phi(i) = \begin{cases} \frac{i-a}{3} & \text{if } i \equiv 1 \ (mod\ 3) \\ \frac{i}{3} & \text{if } i \equiv 0 \ (mod\ 3) \end{cases}$$

and so $i' = \Phi(i) < i$, with $\Phi^{j-1}(i) = \Phi^{k-1}(i)$. Thus i' is a smaller integer that cannot be expressed with digits from $\mathcal{D}_{a,b}$. The case $i < 0$ is similar. □

Theorem 4.1 allows us to reduce the number of values of i for which the sequence $(i, \Phi(i), \Phi^2(i), ...)$ needs to be computed by checking only the i satisfying (4.1) such that $i \equiv 1 \ (mod\ 3)$ when $i < 0$ and $i \equiv 2 \ (mod\ 3)$ when $i > 0$. This reduces the number of sequences that need to be checked by an approximate factor of 3. The number of such sequences that need to be checked is reduced further by checking the i satisfying (4.1) that are closest 0, and working outwards.

An implementation of the improved Matula algorithm based on Theorem 4.1 in the C programming language is included in Section 8.

5 Sufficient Conditions Excluding $\mathcal{D}_{1,b}$ Basic

In looking for digits that make $\mathcal{D}_{1,b}$ basic, there are certain digits which satisfy the necessary conditions in Section 4 but cannot be used to produce a basic digit set. Here we present some sufficient conditions for exclusion of a digit from a basic digit set, which we believe play an important role in the structure of the set $\mathcal{X}_1$.

Theorem 2.1 is useful for computing families of non-basic digits, as shown in the following example.

Example 5.1. Consider $\mathcal{D}_{1,b}$. If $i \in \mathcal{D}_{1,b}^2$ with $i = 8k$ for some nonzero integer k, then $\mathcal{D}_{1,b}$ is not basic by Theorem 2.1. We consider the representations of all numbers in $\mathcal{D}_{1,b}^2$: $(0)_3$, $(01)_3$, $(0b)_3$, $(10)_3$, $(11)_3$, $(1b)_3$, $(b0)_3$, $(b1)_3$, $(bb)_3$. We have that $(b1)_3 = 3b + 1 = 8k$ for some integer $k \neq 0$ if $b \equiv 5 \ (mod\ 8)$. This gives us an infinite family of non-basic digits: $b = 19, 43, 67, 91, ...$

Definition 5.1. An integer k is called a *bad digit factor* if for any basic $\mathcal{D}_{1,b}$ we have that $k \nmid b$.

We start by showing the characteristics of one class of bad digit factors in the following lemma.

Lemma 5.1. *If $\mathcal{D}_{1,b} = \{0, 1, -b\}$ is not basic and some nonzero multiple of $3^n - 1$ has a radix representation of length less than n using only the digits 0 and $-b$, then b is a bad digit factor.*

Proof. Suppose $\mathcal{D}_{1,b}$ is not basic, and $\bar{d} := (d_{n-1}...d_1d_0)_3 = k(3^n - 1)$ for some $k \in \mathbb{Z}$, and $d_i \in \{0, -b\}$. Then for any nonzero integer m, $m \cdot d \in \mathcal{D}^n_{1,mb}$. Since $3^n - 1 | m \cdot d$, we have that $\mathcal{D}_{1,mb}$ is not basic, and we have that b is a bad digit factor. □

We term some bad digit factors as "trivial." All nonzero multiples of $3^n - 1$ for $n \geq 1$ (i.e., all even integers) are trivial bad digit factors (that they are bad digit factors follows directly from Theorem 2.1). All multiples of 3 are also trivial bad digit factors, since any multiple of 3 is never congruent to 1 (mod 6). Thus far the classes of nontrivial bad digit factors that have been found are quite sparse. We start by showing a family of such digits. Here we show one family of bad digit factors of the type referred to in Lemma 5.1.

Lemma 5.2. *For $n \geq 1$, $k = 1 + 3 + ... + 3^n$ is a bad digit factor.*

Proof. Take the integer $j = 3^n k + k$. Then $j \in \mathcal{D}^{n+1}_{1,k}$. Since $k = \frac{3^{n+1}-1}{2}$ and $3^n + 1$ is even, we must have that $3^{n+1} - 1 | (3^n + 1)k = j$, so $\mathcal{D}_{1,k}$ is not basic. Thus $\mathcal{D}_{1,k}$ satisfies the conditions of Lemma 5.1, and so k is a bad digit factor. □

Upon plotting the b such that $\mathcal{D}_{1,b}$ is basic on a number line, as in Figure 3.1, one notices the dead spots as described in Observation 3.3. The following theorem accounts for these dead spots. This theorem relies on an elementary property of radix representations, as shown in Lemma 5.3. The original idea for Lemma 5.3 comes from (Muir [6]).

Lemma 5.3. *Let $\mathcal{D}$ be digit set for the base $B > 0$, containing at least one positive integer. Let $i = \max \mathcal{D}$, $j = \min\{x \in \mathcal{D} \mid x > 0\}$. Then for $n \geq 1$, any integer x such that*

$$i \cdot \frac{B^n - 1}{B - 1} < x < j \cdot B^n \tag{5.1}$$

cannot be expressed using the digits in $\mathcal{D}$.

Proof. Let $n \geq 1$. The largest number with a representation of length n that can be expressed with the digits in $\mathcal{D}$ is

$$(i...i)_B = i \cdot \sum_{k=0}^{n-1} B^k = i \cdot \frac{B^n - 1}{B - 1}.$$

The smallest number with a representation of length $n+1$ that is not less than $i \cdot \frac{B^n-1}{B-1}$ is $(j0...0)_B = j \cdot B^n$. □

Example 5.2. Take the base $B = 3$ and digits $\{0, 1\}$ Then $i = j = 1$. If we take $n = 2$, we consider $(11)_3 = 4$ and $(100)_3 = 9$. Obviously, the numbers $5, 6, 7, 8$ cannot be expressed using only the digits 0 and 1.

Here we adopt the following convention: k represents an integer, $\bar{k}$ is the string of digits that represents the expansion of k using the digit set used in the current context, and $|\bar{k}|$ is the non-negative integer representing the length of $\bar{k}$.

Theorem 5.1 (Dead Spots for $\{0, 1, -b\}$). *If $\frac{3^{n+1}-3}{2} < x < 2 \cdot 3^n$ for $n \geq 2$, then $\mathcal{D}_{1,x} = \{0, 1, -x\}$ is not basic.*

Proof. Suppose $\mathcal{D}_x$ is basic. Choose any $2 < k < 3^n$ with $k \equiv 2 \ (mod\ 3)$. Since $\Phi(i) = \frac{i - ||i||_{\mathcal{D}}}{3}$, if $3^n > i$ then $\frac{3^n+x}{3} > \frac{i+x}{3} \geq \Phi(i)$. Since $2 \cdot 3^n > x$, we have $3^n > \Phi(i)$. Also, if $i > 2$, then $\Phi(i) \geq \frac{i-1}{2} \geq 1$. So we have

$$0 \leq \Phi^m(i) < 3^n \ for\ m \geq 0\ whenever\ 2 < i < 3^n \tag{5.2}$$

Since k is positive, the first digit in $\bar{k}$ cannot be x, since $\bar{k} = (a_{|\bar{k}|-1} ... a_1, a_0)_3$ and if the first digit $a_{|\bar{k}|-1} = -x$, then

$$k = \sum_{t=0}^{|\bar{k}|} a_t 3^t \leq \frac{3^{|\bar{k}|} - 1}{2} - x \cdot 3^{|\bar{k}|-1}$$

which would imply that $k \leq 0$. Thus any x in $\bar{k}$ must have a 1 to its left. Let $\bar{k}'$ be the string created by removing all digits to the right of the leftmost occurrence of x in $\bar{k}$. Then $\Phi^{|\bar{k}|-|\bar{k}'|}(k) = k'$, the number represented by $\bar{k}'$.

Since $k' \equiv 2 \ (mod\ 3)$, we have that $0 < k' < 3^n$ by (5.2). So let $(b_{|\bar{k}'|-1} ... b_1 b_0)_3$ represent $\bar{k}'$. Then $b_{|\bar{k}'|}, ... b_1 \in \{0, 1\}$ and $b_0 = -x$, by construction.

$$\frac{3^{n+1} - 3}{2} < x < k' + x < 3^n + x < 3^{n+1}$$

$$\frac{3^n - 1}{2} < \frac{k' + x}{3} < 3^n$$

but the expansion of $\frac{k'+x}{3}$ is $(b_{|\bar{k}'|-1}, ..., b_1)_3$, a string of numbers in $\{0, 1\}$, which contradicts Lemma 5.3.

□

Theorem 5.1 justifies Observation 3.3 by exhibiting the intervals on which there are no b such that $D_{1,b}$ is basic. The bounds on the interval given in Theorem 5.1 seems to be relatively sharp in the available data.

6 Density of Basic Digit Sets {0, 1, -b}

We define upper asymptotic density of the set $\mathcal{A}$ of non-negative integers as

$$\bar{d}(\mathcal{A}) := \limsup_{t \to \infty} \frac{\#\{x \ \in \ \mathcal{A} \mid x \ \leq \ t\}}{t}$$

Theorem 5.1 has implications on the density of good digits, which we discuss here.

From the plots of b such that $\mathcal{D}_{1,b}$ is basic, it appears that the set $\mathcal{X}_1$ is quite sparse.

Conjecture 6.1. *The upper asymptotic density of $\mathcal{X}_1$ is 0.*

In fact, if the upper asymptotic density of $\mathcal{X}_1$ is 0 then $f(x)$ will be bound for all x by x^c for some $c < 1$. As in Observation 3.2, x^c upper bounds $f(x)$ with $c \approx 0.59$ for $0 \leq x < 10^6$. This perhaps suggests that Conjecture 6.1 can be proven.

We prove a nontrivial upper bound on the upper asymptotic density of $\mathcal{X}_1$.

Theorem 6.1. *The upper asymptotic density of b such that $\mathcal{D}_{1,b} = \{0, 1, -b\}$ is basic is at most $\frac{5}{36}$.*

Proof. Let $I_n := [2 \cdot 3^n, \frac{3^{n+2}-3}{2}]$ for $n \geq 2$, that is, the interval between the n-th and $(n+1)$-th dead spot.

Let $\mathcal{B} := [0, 2 \cdot 3^2] \cup (\cup_{n=2}^{\infty} I_n)$. We find the upper asymptotic density of $\mathcal{B}$. $l_n = \frac{5 \cdot 3^n - 3}{2}$ for $n \geq 2$ represents the length of I_n. We have that $L_n := \sum_{i=2}^{n} l_i = \frac{5 \cdot 3^{n+1} + 6(1-n) - 40}{4}$ for $n \geq 2$.

Let $a_t := \frac{\#\{x \in \mathcal{B} \mid x \leq t\}}{t}$. For each $n \geq 2$, for $\frac{3^{n+2}-3}{2} \leq t \leq \frac{3^{n+3}-3}{2}$ we have that a_t is largest at the right endpoint (as seen in Figure 3.1, the maximum of a_t for bounded t occurs at the left endpoint of a dead spot). Thus we take the sequence $t_n = \frac{3^{n+2}-3}{2}$. Then

$$\bar{d}(\mathcal{B}) = \limsup_{n \to \infty} \frac{L_n}{t_n} = \limsup_{n \to \infty} \frac{5 \cdot 3^{n+1} + 6(1-n) - 40}{2(3^{n+2} - 3)} = \frac{5}{6}$$

Since if $b \in \mathcal{X}_1$, then $b \equiv 5 \ (mod\ 6)$ as discussed in Section 4, the frequency at which the $b \in \mathcal{X}_1$ occur in $\mathcal{B}$ is at most 1/6. Thus we have the relationship $\bar{d}(\mathcal{X}_1) \leq \frac{1}{6} \cdot \bar{d}(\mathcal{B})$, and we have that $\bar{d}(\mathcal{X}_1) \leq \frac{5}{36}$.

□

A theorem of Szemeredi states that given a set of integers with positive upper asymptotic density, the set contains arithmetic progressions of arbitrary length. This suggests one possible route to prove that the density of $\mathcal{X}_1$ is zero: show an upper bound on the length of arithmetic progressions in $\mathcal{X}_1$. Here is one elementary result relating arithmetic progressions in $\mathcal{X}_1$ to the underlying digit sets.

Theorem 6.2. *Suppose $\mathcal{X}_1$ has an arithmetic progression $a_n = a_1 + (n-1)d$ with $1 \leq n \leq l$. If $b < l$ and b is a prime bad digit factor, then $b \mid d$.*

Proof. Suppose not. Then $gcd(b, d) = 1$. Then $(n-1)d \equiv -a_1 \ (mod\ b)$ has a solution for $0 \leq n - 1 \leq b$. The solution t in this range gives $(t-1)d + a_1 \equiv 0 \ (mod\ b)$. We have that $b \mid a_t$, thus a_t is not in $\mathcal{X}_1$. Since $t \leq l$, this is a contradiction. □

7 Infinite Families of Basic Digit Sets

Theorem 7.1. *If $b_n = 3^n - 2$ for $n \geq 1$, then $\mathcal{D}_{1,b_n}$ is basic.*

Proof. Every positive integer has a unique ternary expansion using the digit set $\{0, 1, 2\}$. Every positive integer less than 3^n has an expansion using $\{0, 1, 2\}$ which has length no greater than n.

Let $0 \leq k < 3^n$ be an integer, and $\bar{k} = (k_m...k_0)_3$ represent its expansion using the digit set $\{0, 1, 2\}$. For every $k_i = 2$, re-define $k_i := b_n$ and $k_{i+(n+1)} := 1$, and do nothing otherwise.

This procedure replaces every $2 \cdot 3^i$ with $(3^n - b_n) \cdot 3^i$, thus the value of k is unchanged. Since $k_i = 0$ for $i > n$ in the expansion using digits from $\{0, 1, 2\}$, this procedure doesn't re-define any k_i more than once. Therefore, $(k_{m+n+1}...k_0)_3$ is a valid ternary expansion for k using digits from $\mathcal{D}_{1,b_n}$.

Since all i satisfying (4.1), specifically

$$0 \leq i \leq \frac{x_n}{2} = \frac{3^n}{2} - 1$$

have an expansion using digits from $\mathcal{D}_{1,b_n}$, it follows that $\mathcal{D}_{1,b_n}$ is basic. □

Conjecture 7.1. *For $n \geq 1$, $\mathcal{D}_{1,b_n}$ is basic for each of the following:*

1. $b_n = 7 \cdot 3^{2n+1} + 4$
2. $b_n = 3^{2n} - 8$

It has been verified that each of the $\mathcal{D}_{1,b_n}$ listed above, for $b_n \leq 10^6$, is basic.

8 Code

Originally, Maple was used for the checking of digit sets. It was found that a standalone implementation would find results much faster, since Maple is not designed to handle large computations. The following is a standalone implementation of the improved Matula algorithm (using Theorem 4.1).

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>

typedef struct phiSeq {
  int phiVal;
  struct phiSeq *next;
} seq;

int findResDigit(int findVal, int digitSet[]) {
 int i;
 for (i = 0; i < 3; i++) {
  int wDigit = digitSet[i] > 0 ? digitSet[i] : (abs(digitSet[i] * 3) + digitSet[i]);
  if ((wDigit % 3) == (findVal % 3)) {return digitSet[i];}
 }
 return 0;
}

int checkDuplicate (seq *startElt) {
 seq *temp1, *temp2;
 int i, j;
 i = 0;
 j = 0;
```

```
 temp1 = startElt;
 temp2 = startElt;
 while (temp1 != NULL) {
  while (temp2 != NULL) {
   if (i != j) {
    if (temp1->phiVal == temp2->phiVal) {
     return 0;
    }
   }
   temp2 = temp2->next;
   j++;
  }
  temp1 = temp1->next;
  j = 0;
  temp2 = startElt;
  i++;
 }
 return 1;
}

int cleanSequence(seq *firstElt) {
 seq *curElt, *nextElt;
 curElt = firstElt;
 while (curElt != NULL) {
  nextElt = curElt->next;
  free(curElt);
  curElt = nextElt;
 }
 return 1;
}

int phiSequenceCheck(int startNum, int digitSet[]) {
 int i, j, lastVal;
 seq *first = malloc(sizeof(seq));
 seq *temp;
 first->next = NULL;
 first->phiVal = startNum;
 temp = first;
 lastVal = startNum;
 while (1) {
  int curVal = (lastVal - findResDigit(lastVal, digitSet)) / 3;
  if (curVal == 0) {
   cleanSequence(first);
   return 1;
  }
  seq *nextElt = malloc(sizeof(seq));
  nextElt->phiVal = curVal;
  temp->next = nextElt;
  temp = nextElt;
  temp->next = NULL;
  if (checkDuplicate(first) == 0) {
   cleanSequence(first);
   return 0;
  }
  lastVal = curVal;
 }
 return 1;
}

int checkDigit(int digit) {
 int digitSet[3] = {0, 1, -digit};
 int dmin = -digit;
 int dmax = 1;
 int lowerLimit = (int)ceil((-dmax / 2));
 int upperLimit = (int)floor((-dmin / 2));
 int i = 0;
 while (1) {
  int posDigit = 2*i;
  int negDigit = 2*(-i) - 1;
  int wChk = 0;
  if (posDigit <= upperLimit) {
   if (phiSequenceCheck(2*i, digitSet) == 0) {return 0;}
   wChk = 1;
  }
  if (negDigit >= lowerLimit) {
   if (phiSequenceCheck(2*(-i) - 1, digitSet) == 0) {return 0;}
   wChk = 1;
  }
  i++;
  if (wChk == 0) {break;}
 }
 return 1;
}

int main(int argc, char **argv) {
 if (argc < 3) {
  printf("Usage: %s <start> <end>\n", argv[0]);
  return 1;
 }
 int firstDigit = atoi(argv[1]);
 int lastDigit = atoi(argv[2]);
```

```
 int i;
 for (i = firstDigit; i <= lastDigit; i++) {
  printf("%d is %s\n", (6*i - 5), checkDigit(6*i - 5) ? "good" : "bad");
 }
 return 1;
}
```

Listing 1: Basic Digit Finder (Improved Matula Algorithm)

References

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