THE PROBLEM OF TYPE

CHARLES CRISSMAN

ABSTRACT. We discuss the background of the type problem as well as several open sub-problems, namely:

- The type of surfaces formed by gluing a half-strip with the gluing function $x \mapsto x + x^\alpha$, $0 \le \alpha \le 1$, $\alpha \in \Re$.
- The type of surfaces formed by placing countable numbers of z^2 inserts vertically originating from the real line with particular spacings.

1. BACKGROUND

Definition 1.1. A *conformal map* is a map that is both analytic and bijective. Note that if the map $z = (x+iy) \mapsto \phi(x+iy) = u(x+iy) + iv(x+iy)$ is a conformal map, we have $u_x = v_y$ and $v_x = -u_y$, hence:

$$J_\phi(z) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x = u_x^2 + v_x^2 = |\phi'(z)|^2. \tag{1.2}$$

Definition 1.3. The Uniformization Theorem states that if F is an open, simply connected Riemann surface, then there exists a conformal map of F onto either $\mathbb{C}$, the complex plane, or $\mathbb{D}$, the unit disk. In the case of $\mathbb{C}$, the surface F is said to be of *parabolic type.* In the case of $\mathbb{D}$, the surface F is said to be of *hyperbolic type.* Note that there exists no conformal map between $\mathbb{C}$ and $\mathbb{D}$.

Definition 1.4. Let Γ be a curve family on the smooth surface F. We call a piecewise continuous function $\rho\colon F \to (0, \infty)$ admissible for Γ if for all curves $\gamma \in \Gamma$ we have $\int_\gamma \rho \, ds \ge 1$. Then the *modulus* of Γ (written $\operatorname{mod}(\Gamma)$) is defined as $\inf(\{\int_F \rho^2 \colon \rho \text{ admissible for } \Gamma\})$.

In general, a curve family will have a large modulus if it is composed of many short curves, and a small modulus if it is composed of few long curves.

Theorem 1.5. *Modulus is a conformal-invariant, that is, if Γ is a curve family on the smooth surface F and is contained in some domain Ω, and ϕ is a conformal mapping of some subset of F containing Ω, then* $\operatorname{mod}(\Gamma) = \operatorname{mod}(\{\phi(\gamma)\colon \gamma \in \Gamma\}) =: \operatorname{mod}(\phi \circ \Gamma)$.

Date: August 22, 2004.

Proof. "$\leftarrow$": $\operatorname{mod}(\Gamma) \leq \operatorname{mod}(\phi \circ \Gamma)$:
Call $\phi(\Omega) =: \Omega'$. Let λ be an admissible function for $\phi \circ \Gamma$. Define $\rho(z) := \lambda(\phi(z)) \cdot |\phi'(z)|$ for $z \in \Omega$. Then for $\gamma \in \Gamma$, $\gamma\colon (a,b) \to \Omega$ a smooth path in Ω and $\tilde{\gamma} := \phi(\gamma)$, we have:

$$\begin{aligned}\int_{\gamma} \rho(z)|dz| &= \int_a^b \rho(\gamma(t)) \cdot |\gamma'(t)| dt \\ &= \int_a^b \lambda(\phi(\gamma(t))) \cdot |\phi'(\gamma(t))| \cdot |\gamma'(t)| dt \\ &= \int_{\tilde{\gamma}} \lambda(w)|dw| \geq 1.\end{aligned}$$

Hence ρ is admissible for Γ. Furthermore,

$$\int_{\Omega} \rho^2(z) \overset{1.2}{=} \int_{\Omega} \lambda^2(\phi(z)) J_{\phi}(z) = \int_{\Omega'} \lambda^2.$$

Hence $\operatorname{mod}(\Gamma) \leq \operatorname{mod}(\phi \circ \Gamma)$.

"$\rightarrow$": $\operatorname{mod}(\Gamma) \geq \operatorname{mod}(\phi \circ \Gamma)$:
Since the inverse map ϕ^{-1} is also a conformal map, we apply the first portion of the proof to this function. It follows that $\operatorname{mod}(\Gamma) \geq \operatorname{mod}(\phi \circ \Gamma)$.
Thus $\operatorname{mod}(\Gamma) = \operatorname{mod}(\phi \circ \Gamma)$. □

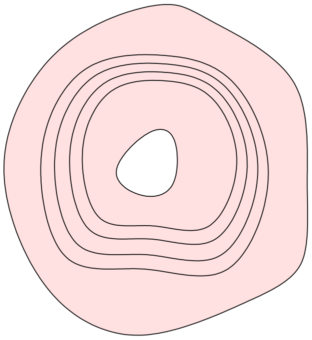
Definition 1.6. Let $\gamma\colon [0,1) \to F$ be an open curve on some open surface F. We say $\gamma(t)$ *goes to infinity* as t goes to 1 if γ eventually leaves every compact set on F, that is, if for $K \subset F$ compact, there exists a $t_0 \in [0,1)$ such that for all $t > t_0$, we have $\gamma(t) \notin K$.

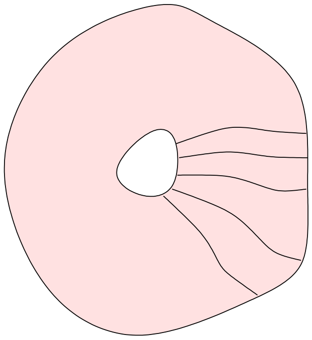
Theorem 1.7. *The Principle of Reciprocity:*

Let D be a topological annulus, let Γ be the family of curves separating the two connected complements of D or separating the single connected complement of D from infinity, and let Υ be the family of curves in D connecting the two connected complements of D or originating from the single connected complement of D and going to infinity. Then

$$\operatorname{mod}(\Upsilon) = \frac{1}{\operatorname{mod}(\Gamma)}.$$

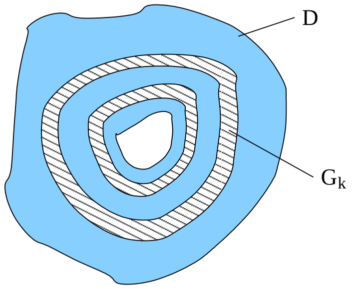
In the case of $\operatorname{mod}(\Gamma) = 0$*, we have* $\operatorname{mod}(\Upsilon) = \infty$ *and vice-versa.*

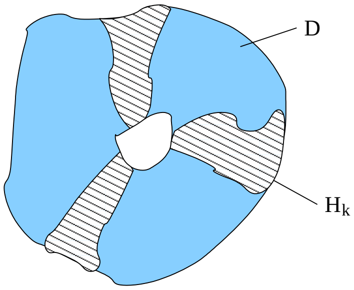




Proof. Follows from the Riemann mapping theorem. (See [Co], pp. 160-161). ☐

Theorem 1.8. *The Grötzsch Principles*





Principle One:

Suppose D is a topological annulus and $\{G_k\}, k \in \mathbb{N}$ is a set of mutually non-intersecting ring domains contained within D. Then if Λ is the curve family separating the two connected complements or D or separating the single connected complement of D from infinity and Γ_k is the curve family separating the complements of G_k for each $k \in \mathbb{N}$, we have

$$\operatorname{mod}(\Lambda) \geq \sum_{k=1}^{\infty} \operatorname{mod}(\Gamma_k)$$

Proof. The statement is clear. Removing curves from Λ increases the number of admissible functions, thereby decreasing the modulus, and removing area from D likewise decreases the modulus. ☐

Principle Two:

Suppose D and Λ are defined as above and $\{H_k\}, k \in \mathbb{N}$ is a set of mutually non-intersecting simply connected domains within D which connect the boundaries of the two connected complements of D (if D has two connected complements) or originate from the single connected complement of D and go to infinity (If D has only one connected complement). Then if for each $k \in \mathbb{N}$, we have that η_k is the family of curves lying in H_k and connecting the boundaries of the two connected complements of D or originating from *the boundary of the single connected complement of D and going to infinity as before, we have*

$$\sum_{k=1}^{\infty} \operatorname{mod}(\eta_k) \leq \frac{1}{\operatorname{mod}(\Lambda)}$$

Proof. Clear. The statement follows directly from the first Grötzsch principle and the principle of reciprocity. □

Theorem 1.9. *Suppose F is an open, simply connected Riemann surface. Then if Γ is a family of all curves originating from the boundary of a given compact ball in F and going to infinity, we have*

$$F\,is \left\{ \begin{array}{c} parabolic \\ hyperbolic \end{array} \right\} iff \left\{ \begin{array}{l} \operatorname{mod}(\Gamma) = 0, \\ \operatorname{mod}(\Gamma) > 0. \end{array} \right.$$

Proof. Follows readily from conformal invariance of modulus and conformal invariance of compactness. □

2. Introduction

Our investigation is based on the following well-known criteria for surface type (the so-called Ahlfors criteria), which are as follows:

Definition 2.1. Let X be a smooth surface and $f\colon X \to \Re$ a C^1-smooth function on X. Then for each $t \in \Re$, we define the set $\Gamma_t := \{x \in X : f(x) = t\}$. Γ_t is called the *t-level line of f.*

Theorem 2.2. *Ahlfors Criterion for Parabolicity:*

Let X be an arbitrary, open, simply-connected Riemann surface, $t_0 \geq 0$ and $f\colon X \to [t_0, \infty)$ a function on X with C^1-smooth, closed level lines $\Gamma_t, t \in [t_0, \infty)$ that fulfill the following properties:

1. *Denote the simply-connected portion of X cut out by the line Γ_t by X_t. Then as $t \to \infty$, the sections X_t grow monotonically and converge to X, and as $t \to t_0$, do not converge to a single point.*
2. *With the possible exception of a finite or unboundedly growing sequence of values $\{t_k\}$, $(k = 1, 2, ...)$, the lines Γ_t vary continuously with t (possibly with the spherical metric).*
3. *In X, excluding some set E, closed with respect to X and containing a countable number of points and a countable number of rectifiable lines, $\nabla(f) \neq 0$.*

Then if the integral

$$\int_{t_0}^{\infty} \frac{dt}{L(t)}, \tag{2.3}$$

where

$$L(t) = \int_{\Gamma_t} |\nabla(f)(z)| ds \tag{2.4}$$

diverges, it follows that X *is parabolic.*

Theorem 2.5. *Ahlfors S-Criterion for Hyperbolicity:*

Let X *be an arbitrary open simply connected Riemann surface,* $S \subseteq X$ *a subset of* X *topologically equivalent to a rectangular half-strip, and* $f\colon S \to [0,1]$ *a function on* S *with smooth, open level lines* $P_t, t \in [0,1]$ *that fulfill the following properties:*

1. *Each line* P_t *originates from the interior of* X *and goes to infinity.*
2. *Within the set* S*, with the exclusion of some set* E*, closed with respect to* S *and containing a countable number of points and a countable number of rectifiable lines, the lines* P_t *have continuous rotational tangents, and* $\nabla(f) \neq 0$*.*

Then if the integral

$$\int_0^1 \Lambda(t)dt < \infty \tag{2.6}$$

where

$$\Lambda(t) = \int_{P_t} |\nabla(f)(z)|ds \tag{2.7}$$

converges, it follows that F *is hyperbolic.*

3. Half-Strips

We primarily make use of two criteria originally formulated by Volkovisky [Vo], although one of his formulations is incorrect and we have here revised it as well as updated his notation.

Definition 3.1. When we say we *glue* two points x and x^*, it means that we identify them as the same point. A *gluing function* f is a function for which we glue x and $f(x)$. Let S be a strip or half-strip. We call f the *gluing function of* S if f is a gluing function that maps from one infinite edge of S to the other.

Theorem 3.2. *Suppose* S *is a half-strip with gluing function* $\phi\colon [0,\infty) \to [0,\infty)$*. Call* $\phi(x) =: x^*$*. Fix some* $x_0 \geq 0$*. Then if we call* $\Delta(x) := x^* - x = \phi(x) - x$*, the divergence of the integral*

$$\int_{x_0}^{\infty} \frac{\min(1, \phi'(x))dx}{1 + \Delta^2(x)} \tag{3.3}$$

is sufficient to guarantee that S *is of parabolic type.*

Proof. We may assume without loss of generality that S has width 1. Then for all $x > x_0$ we define Γ_x as the segment connecting the points $(x, 0)$ and $(x^*, 1)$ and we define $f\colon S \to [0,\infty)$ as the function that takes the value x on Γ_x. We denote by $\ell(x)$ the Euclidean length of the segment Γ_x. Then

$$|\nabla(f)(x)| \leq \ell(x) \cdot \max\left[1, \frac{1}{\phi'(x)}\right] < \ell(x) \cdot \left(1 + \frac{1}{\phi'(x)}\right),$$

and clearly

$$\int_{\Gamma_x} ds = \ell(x) = \sqrt{1 + \Delta^2(x)},$$

hence

$$\int_{\Gamma_x} |\nabla(f)(x)| ds \leq [1 + \Delta^2(x)] \cdot \left(1 + \frac{1}{\phi'(x)}\right).$$

Thus, by the Ahlfors parabolicity criterion, the divergence of the integral

$$\int_{x_0}^{\infty} \frac{dx}{[1 + \Delta^2(x)] \cdot \left(1 + \frac{1}{\phi'(x)}\right)}$$

is sufficient to show parabolicity of S, and this is equivalent to the divergence of the integral

$$\int_{x_0}^{\infty} \frac{\min(1, \phi'(x)) dx}{1 + \Delta^2(x)}.$$

□

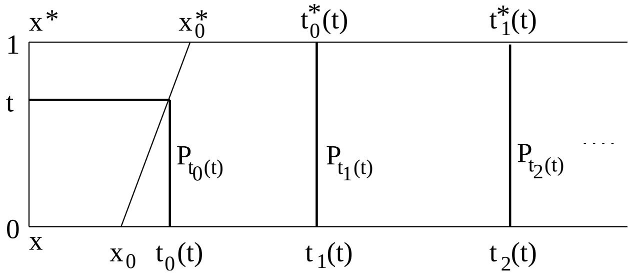
Note in particular that if for all $x > x_0$ we have $\phi'(x) \geq 1$, then the divergence of the integral

$$\int_{x_0}^{\infty} \frac{dx}{\Delta^2(x)} \tag{3.4}$$

is sufficient to show parabolicity.

Theorem 3.5. *Suppose S is a half-strip, $S = \{(t, x) \colon 0 \leq t \leq 1, 0 \leq x\}$ with gluing function $\phi \colon [0, \infty) \to [0, \infty)$. Call $\phi(x) =: x^*$ and $\Delta(x) := x^* - x = \phi(x) - x$. Suppose further that there exists some $x_0 \geq 0$ such that $\Delta(x) \geq \Delta(x_0) > 0$ for all $x \geq (x_0)$. Then we construct the level line P_t as a sequence of cross-sections $P_{t_k(t)}$ of S at each $\{t_k(t)\}, (k = 1, 2, ...)$ defined as follows:*

$$t_0(t) = x_0 + t \cdot \Delta(x_0), t_{k+1}(t) = t_k^*(t) = \phi(t_k)(t). \tag{3.6}$$



Then the convergence of the integral

$$\int_0^1 \sum_{k=0}^{\infty} \frac{1}{\phi'(t_0(t)) \cdot \phi'(t_1(t)) \cdots \phi'(t_k(t))} dt \tag{3.7}$$

is sufficient to show hyperbolicity of S*.*

Proof. Let $f \colon S \to [0,1]$ be the function that takes the value t on the level line P_t. We wish to apply the Ahlfors S-Criterion for hyperbolicity. Note first that

$$\int_{P_t} |\nabla(f)(z)| ds = \sum_{k=0}^{\infty} \int_{P_{t_k(t)}} |\nabla(f)(z)| ds. \tag{3.8}$$

Note further that

$$|\nabla(f)(z)| = \frac{1}{t_k'(t)}$$

on the segment P_{t_k}. By equation 3.6, we have

$$t_0'(t) = \Delta(x_0).$$

Hence

$$\int_{P_{t_k(t)}} |\nabla(f)(z)| ds = \frac{1}{\Delta(x_0)} \cdot \frac{t_0'(t)}{t_k'(t)} \int_{P_{t_k(t)}} ds \leq \frac{1}{\Delta(x_0)} \cdot \frac{t_0'(t)}{t_k'(t)},$$

with inequality only for $k = 0$. It follows that

$$\int_{P_t} |\nabla(f)(z)| ds \leq \frac{1}{\Delta(x_0)} \sum_{k=0}^{\infty} \frac{t_0'(t)}{t_k'(t)}.$$

From equation 3.6, it follows that

$$t_k'(t) = \phi'(t_{k-1}(t)) \cdot t_{k-1}'(t) = \ldots = \phi'(t_{k-1}(t)) \cdot \phi'(t_{k-2}(t)) \cdots \phi(t_0(t)) \cdot t_0'(t).$$

Thus

$$\frac{t_0'(t)}{t_k'(t)} = \frac{1}{\phi'(t_0(t)) \cdot \phi'(t_1(t)) \cdots \phi'(t_{k-1})},$$

and hence the convergence of

$$\int_0^1 \sum_{k=0}^{\infty} \frac{1}{\phi'(t_0(t)) \cdot \phi'(t_1(t)) \cdots \phi'(t_k(t))} dt$$

is sufficient for hyperbolicity of S. $\square$

Corollary 3.9. *Suppose that in the previous theorem, $\phi'(x) \searrow 1$. Then the convergence of the series*

$$\sum_{k=1}^{\infty} \frac{1}{\Delta_k} \tag{3.10}$$

where

$$\Delta_k = x_k - x_{k-1},\ x_k = \phi(x_{k-1})$$

is sufficient to show hyperbolicity of S.

Proof. Note first that as $\phi'(x) \searrow 1$, equation 3.7 converges with the convergence of the series

$$\sum_{k=0}^{\infty} \frac{1}{\phi'(t_0(t)) \cdot \phi'(t_1(t)) \cdots \phi'(t_k(t))}$$

for any $t \in [0, 1]$. Note that as $t_0(0) = x_0$, we have

$$\Delta_k = t_k(0) - t_{k-1}(0),$$

and for $k > 2$ we have

$$\Delta_k = \int_{t_{k-2}(0)}^{t_{k-1}(0)} \phi'(x)dx \leq \phi'(t_{k-2}(0))\Delta_{k-1}.$$

Hence

$$\frac{1}{\phi'(t_{k-2}(0))} \leq \frac{\Delta_{k-1}}{\Delta_k},$$

and therefore

$$\frac{1}{\phi'(t_0(0)) \cdot \phi'(t_1(0)) \cdots \phi'(t_k(0))} \leq \frac{\Delta_1}{\Delta_{k+2}}.$$

It follows that the convergence of equation 3.10 is sufficient for hyperbolicity of S. □

Question 3.11. Suppose S is a half-strip with gluing function $\phi(x) = x + x^\alpha$ for a fixed $\alpha \in [0, 1]$. We know that for $\alpha = 0$, S is parabolic, and that for $\alpha = 1$, S is hyperbolic.

Indeed, if $\alpha = 0$, then we have $\phi(x) = x+1$. Then $\phi'(x) = 1$ and equation 3.4 gives

$$\int_0^\infty dx$$

which obviously diverges. Hence S is parabolic.

If $\alpha = 1$, then $\phi'(x) = 2$, so for equation 3.7 we have

$$\int_0^1 \sum_{k=0}^{\infty} \frac{1}{2^k} = 1.$$

Hence S is hyperbolic.

We then ask, for which values of $\alpha \in [0,1]$ is S parabolic, and for which hyperbolic?

Claim 3.12. *We claim that*

$$S \textit{ is} \left\{ \begin{array}{c} \textit{parabolic} \\ \textit{hyperbolic} \end{array} \right\} \textit{for} \left\{ \begin{array}{l} 0 \leq \alpha \leq \frac{1}{2} \\ \frac{1}{2} < \alpha \leq 1 \end{array} \right.$$

Theorem 3.13. *If S is a half-strip with gluing function $\phi(x) = x + x^{\alpha}$ for some $\alpha \in [0, \frac{1}{2}]$, then S is parabolic.*

Proof. We apply equation 3.4. $\Delta(x) = x^{\alpha}$, and so if the integral

$$\int_1^{\infty} \frac{dx}{x^{2\alpha}}$$

diverges, then S is parabolic. This is true for $\alpha \in [0, \frac{1}{2}]$. □

Theorem 3.14. *If S is a half-strip with gluing function $\phi(x) = x + x^{\alpha}$ for some $\alpha \in (\frac{1}{2}, 1)$, then S is hyperbolic.*

Proof. Let the constant c be defined by

$$c := \left(\frac{2^{\frac{\alpha}{1-\alpha}}}{1-\alpha} \right)^{\frac{1}{\alpha-1}}.$$

Let $x_0 = c$. We wish to show by induction that $x_k = \phi(x_{k-1}) \geq ck^{\frac{1}{1-\alpha}}$ for all $k \in \mathbb{N}$. The induction hypothesis is satisfied by letting $x_0 = c$.

Induction Step:

Assume $x_n \geq cn^{\frac{1}{1-\alpha}}$ for some $n \in \mathbb{N}$. Then

$$x_{n+1} = \phi(x_n) = x_n + x_n^{\alpha} \geq c \cdot (n)^{\frac{1}{1-\alpha}} + c^{\alpha} \cdot (n)^{\frac{\alpha}{1-\alpha}}. \tag{3.15}$$

And from our definition of c, we have

$$(1-\alpha) \cdot c^{\alpha-1} \geq 2^{\frac{\alpha}{1-\alpha}},$$

from which it clearly follows that

$$\begin{aligned} c^{\alpha-1} &\geq \frac{1}{1-\alpha} \cdot \left(1 + \frac{1}{n}\right)^{\frac{\alpha}{1-\alpha}} \\ \equiv c^{\alpha} n^{\frac{\alpha}{1-\alpha}} &\geq c \cdot \frac{1}{1-\alpha} \cdot (n+1)^{\frac{\alpha}{1-\alpha}}. \end{aligned}$$

Applying the mean value theorem, we have:

$$c^{\alpha} n^{\frac{\alpha}{1-\alpha}} \geq c \cdot \left((n+1)^{\frac{1}{1-\alpha}} - n^{\frac{1}{1-\alpha}} \right).$$

Hence, from the previous equation along with equation 3.15, we have that

$$x_{n+1} \geq c \cdot (n+1)^{\frac{1}{1-\alpha}}.$$

Thus, by math induction, we have that $x_k = \phi(x_{k-1}) \geq ck^{\frac{1}{1-\alpha}}$ for all $k \in \mathbb{N}$. It follows that $\Delta_k = x_k^{\alpha} \geq c^{\alpha} k^{\frac{\alpha}{1-\alpha}}$.

Applying equation 3.10, we have that if the expression

$$\sum_{k=1}^{\infty} \frac{1}{\Delta_k} \leq \sum_{k=1}^{\infty} \frac{1}{c^{\alpha} k^{\frac{\alpha}{1-\alpha}}} \tag{3.16}$$

converges, then S is hyperbolic. This sum converges for

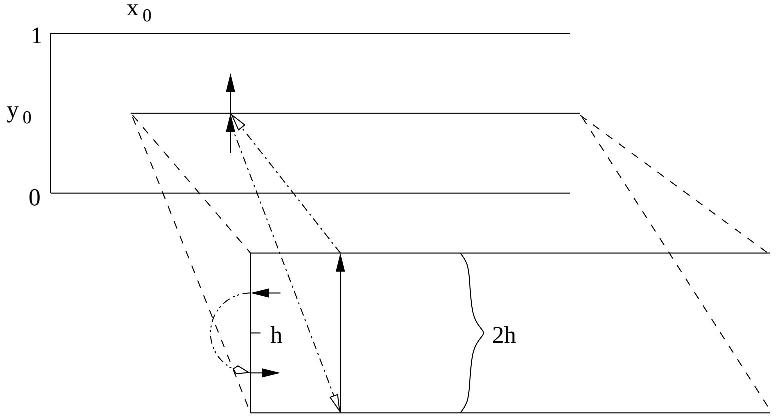
$$\frac{\alpha}{1-\alpha} > 1.$$

Hence S is hyperbolic for $\alpha \in (\frac{1}{2}, 1)$. □

Our claim follows from the previous two theorems and the fact that S is hyperbolic for $\alpha = 1$.

4. RIDGES

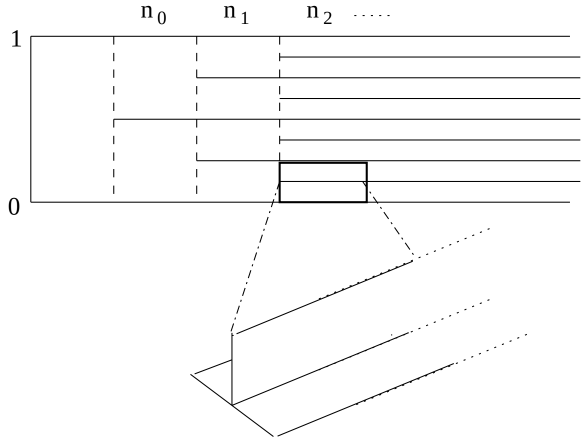
Definition 4.1. Let $S := \{(x, y) \colon 0 \leq y \leq 1, 0 \leq x\}$ be a half-strip. If we cut a slit in S along $y = y_0, x \geq x_0$ with $0 \leq y_0 \leq 1, 0 \leq x_0$ and into this slit we glue another half strip $I := \{(u, v) \colon 0 \leq v \leq 2h, 0 \leq u$ of width $2h$ with the unique length-preserving gluing function as shown:



Furthermore, at the finite edge of the half-strip, we glue the points $(0, h-k)$ and $(0, h+k)$ for all $0 \leq k \leq h$.

Then we call I a *ridge of height* h on S.

Theorem 4.2. *Let* $S := \{(x, y) \colon 0 \leq y \leq 1, 0 \leq x\}$ *be a half-strip of width 1. At each* $x = k, k \in \mathbb{N}$, *we place* 2^{k-1} *new evenly spaced ridges of height* $\frac{1}{2}$ *as shown:*



and we call the resulting surface F. Then F is hyperbolic.

Proof. For each $k \in \mathbb{N}$, let A_k be the rectangle on F with vertices at $(k,0), (k,1), (k+1,1)$, and $(k+1,0)$ with an open edge along the segment connecting $(k,0)$ and $(k,1)$ and all other edges closed. Then $n_k := 2^k - 1$ ridges run through A_k. Let A_0 be the closed rectangle with vertices at $(0,0), (0,1), (1,1)$, and $(1,0)$. Let Γ be the curve family on $F \backslash A_0$ connecting the edges of S at $y = 0$ and $y = 1$. Note that this is also the curve family separating the compact set A_0 from infinity on F. Then for all $k \in \mathbb{N}$, let $\rho_k \colon A_k \to \Re$ be the function that takes the value $\frac{1}{n_{k-1}+1}$ everywhere on A_k. Then let $\rho \colon F \to \Re$ be the function such that $\rho(x) := \rho_k(x)$ for all $x \in A_k$. Then ρ is admissible for Γ:

Consider the extremal curve γ_k^* that runs vertically from $(0,k)$ to $(1,k)$ but avoids all new ridges introduced at $x = k$. Then

$$\int_{\gamma_k^*} \rho \, ds \geq \frac{1}{n_{k-1}+1} \cdot (n_{k-1}+1) = 1.$$

We claim that all curves in Γ have length greater than that of γ_k^* for some $k \in \mathbb{N}$ under the metric ρ.

Indeed, for each $k \in \mathbb{N}$, let γ be an arbitrary curve in Γ originating from the point $(\alpha, 0)$ with $k \leq \alpha < k+1$. Then three cases exist:

(1) Case: γ crosses the first ridge in A_k. If so, it picks up more length than γ_k^*, which does not cross this ridge.

(2) Case: γ travels to $x = k$. In this case, the distance traveled is greater than or equal to that traveled by a straight line and is traveled under the ρ_k metric, which γ_k^* is also subject to. Hence the weight accumulated is at least that accumulated by γ_k^* in reaching the same point.

(3) Case: γ travels to $x = k+1$. As in case 2, the distance traveled is greater than or equal to that of a straight line and is traveled under the rho_k metric, which accumulates more weight than the

ρ_{k+1} metric that γ^*_{k+1} travels under. Hence the weight accumulated by γ is greater than that accumulated by γ^*_{k+1}.

We apply this reasoning inductively after crossing each ridge. The claim follows.

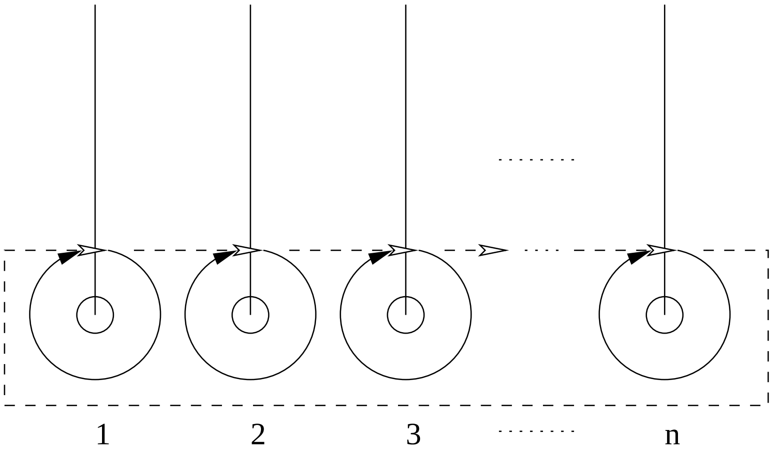
Then ρ is admissible for Γ. And integrating, we have:

$$\begin{aligned}\int_S \rho^2 &= \sum_{k=1}^{\infty}\int_{A_k}\rho_k^2\\ &= \sum_{k=1}^{\infty}(n_k+1)\cdot\left(\frac{1}{n_{k-1}+1}\right)^2\\ &= \sum_{k=1}^{\infty}\frac{1}{2^{k-2}} = 4.\end{aligned}$$

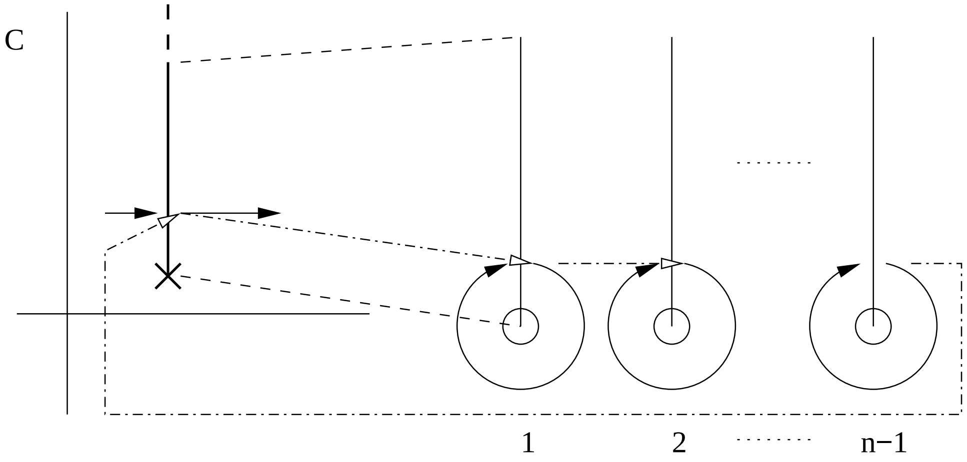
Hence F is hyperbolic. $\square$

5. INSERTS

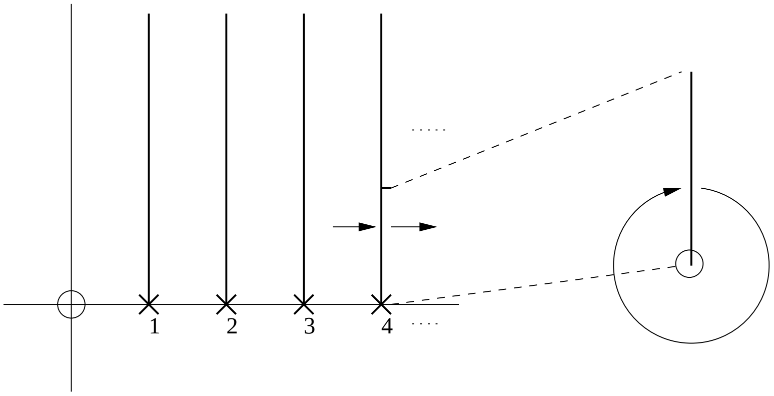
Definition 5.1. The function $z \mapsto z^n, n \in \mathbb{N}$ maps $\mathbb{C}$ to a surface made of n slit planes glued as shown:



Then if we make a slit in $\mathbb{C}$ and glue in $n-1$ planes with this same gluing pattern along this slit, we call this a z^n-*insert*.

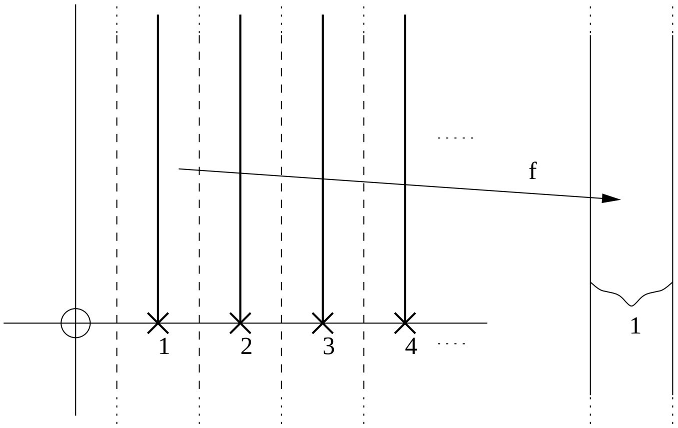


Theorem 5.2. *Suppose we make vertical slits in $\mathbb{C}$ originating from the x-axis at the natural numbers, and into each of these slits we glue a z^2-insert:*



Then the resulting surface is hyperbolic.

Outline of Proof 5.3. We begin by uniformizing our surface in the following manner. First, we divide the right half-plane into vertical strips as shown:



We then map each of these strips conformally to vertical strips of width 1. It seems possible to show using modulus arguments that the appropriate conformal map, f, will, for points with height x on the edges of the strips in the domain of f and constants C_1, C_2, and α, fulfill the following properties:

$$\text{For}\left\{\begin{array}{c}\text{large}\\ \text{small}\end{array}\right\}x,\text{ we have}\left\{\begin{array}{l}\frac{1}{C_1}e^{\alpha x}\leq f(x)\leq C_1 e^{\alpha x},\\ \frac{1}{C_2}x\leq f(x)\leq C_2 x.\end{array}\right.$$

Then after uniformization, the resulting surface will be two half-planes, call them H_1 and H_2, joined with a gluing function ψ from the edge of H_2 to H_1 that fulfills the previous properties. Call the surface resulting from this gluing H. Let ϕ be the inverse gluing function. Let $R_0 \in \Re$ be a positive number such that for all $R \geq R_0$

$$\frac{1}{C_1}\cdot e^{\cdot\frac{\alpha}{2C_2}R} \geq R \tag{5.4}$$

and

$$e^{\frac{\alpha}{2C_2}R} \geq 2. \tag{5.5}$$

Such a number clearly exists since the exponential function grows faster than any linear function. We construct a sequence $R_n, n \in \mathbb{N}_0$ with the property $R_{n+1} = \psi(|\phi(-R_n)|)$. We then construct a sequence of half-annuli $A_k, k \in \mathbb{N}_0$ thus: for k even, A_k is the half-annulus in H_1 centered at the origin with inner radius $R_{\frac{k}{2}}$ and outer radius $R_{\frac{k+2}{2}}$, and for k odd, A_k is the half-annulus in H_2 centered at the origin with inner radius $|\phi(-R_{\frac{k-1}{2}})|$ and outer radius $|\phi(-R_{\frac{k+1}{2}})|$.

Then for the curve family connecting the edges of the strip $S = \bigcup_{k\in\mathbb{N}} A_k$, we apply the test function $\rho = \frac{1}{|z|\cdot\log\left(\frac{R_{n+1}}{R_n}\right)}$ for $z \in A_{2n}, n \in \mathbb{N}$, $\rho = \frac{1}{|z|\cdot\log\left(\frac{\phi(-R_{n+1})}{\phi(-R_n)}\right)}$ for $z \in A_{2n+1}, n \in \mathbb{N}$. It should be possible to show the admissibility of ρ, perhaps with some constant multiplicative factor to make up for the non-uniformity of ψ and ϕ.

Then we have

$$\int_S \rho^2 = \sum_{k=1}^{\infty} \int_{A_k} \rho^2 = \sum_{k \text{ even}}^{\infty} \frac{2\pi}{\log\left(\frac{R_{n+1}}{R_n}\right)} + \sum_{k \text{ odd}}^{\infty} \frac{2\pi}{\log\left(\frac{\phi(-R_{n+1})}{\phi(-R_n)}\right)}, . \tag{5.6}$$

From the inequality 5.4 and the properties of ψ and ϕ, we have

$$R_{n+1} \geq \frac{1}{C_1} \cdot e^{\frac{\alpha}{C_2} R_n} \geq R_n \cdot e^{\frac{\alpha}{2C_2} R_n},$$

hence along with equation 5.5

$$\log\left(\frac{R_{n+1}}{R_n}\right) \geq \frac{\alpha}{2C_2} R_n \geq 2^{n-1} \frac{\alpha}{C_2} R_0 \tag{5.7}$$

and

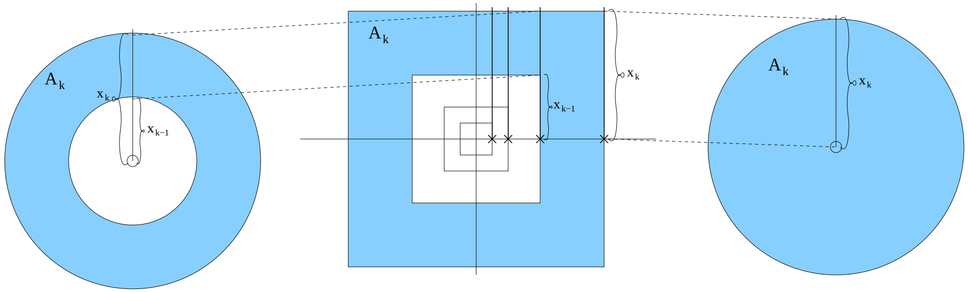
$$\log\left(\frac{\phi(-R_{n+1})}{\phi(-R_n)}\right) \geq 2\log(C_2) \cdot \log\left(\frac{R_{n+1}}{R_n}\right). \tag{5.8}$$

From equations 5.6, 5.7, and 5.8, it follows that $\int_S \rho^2$ converges.

Hence by the principle of reciprocity, the modulus of the curve family originating from A_0 and going to infinity on S is greater than zero. Then if we remove the compact set A_0 from H, the modulus of the curve family originating from the border of A_0 and going to infinity will also be greater than zero. It follows that H is hyperbolic, and hence that our original surface is hyperbolic.

Theorem 5.9. *Suppose we make vertical slits in $\mathbb{C}$ originating from the x-axis at each $x_n = 2^n$, $n \in \mathbb{N}$ and into each of these slits we glue a z^2-insert. Then the resulting surface is parabolic.*

Proof. For each $k \in \mathbb{N}$, denote by I_k the insert glued at x_k. We begin by constructing a sequence $\{A_m\}$, $m \in \mathbb{N}, n \geq 2$ of square annuli on $\mathbb{C}$ centered at the origin of $\mathbb{C}$, where the square annulus A_k has inner radius x_{k-1} and outer radius x_k. On each $I_n, n \leq k-1$, we let A_k cover the circular annulus centered at the origin of the insert (the base of the slit) with inner radius $x_(k+1)$ and outer radius x_k. On I_k, we let A_k cover the disk centered at the origin of the insert with radius x_k:



Then let Γ_k be the family of all curves connecting the inner edge of A_k with the outer edge of A_k, and let $\rho_k\colon A_k \to [0,\infty]$ be the function that on $\mathbb{C}$ maps the point $z = x+iy$ to $\frac{1}{\max(|x|,|y|)\cdot\log(x_{k-1})}$, that on $I_n \cap A_k, n \leq k-1$ maps the point $z = (r,\theta)$ to $\frac{1}{r\cdot\log(x_{k-1})}$, and that on $I_k \cap A_k$ takes the value zero.

Then ρ_k is admissible for Γ_k. We have that:

$$\int_{\Gamma_k} \rho_k^2 = \int_{A_k\cap\mathbb{C}} \rho_k^2 + \sum_{m=1}^{k-1} \int_{A_k\cap I_m} \rho_k^2.$$

Furthermore,

$$\int_{A_k\cap\mathbb{C}} \rho_k^2 \leq \frac{8}{\log(2)} + \frac{2}{\log^2(2)},$$

and

$$\int_{A_k\cap I_m} \rho_k^2 = \frac{1}{\log(2)}.$$

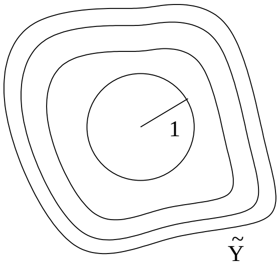
Hence,

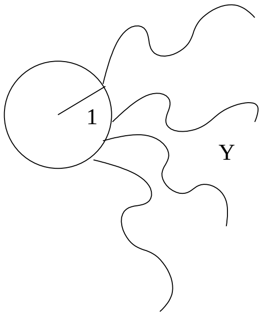
$$\operatorname{mod}(\Gamma_k) \leq \int_{\Gamma_k} \rho_k^2 \leq \frac{k-1}{\log(2)} + \frac{8}{\log(2)} + \frac{2}{\log^2(2)}.$$

Then if we denote by $\tilde{\Gamma}_k$ the family of closed curves traveling around A_k, we have by the principle of reciprocity that

$$\operatorname{mod}(\tilde{\Gamma}_k) \geq \frac{1}{\frac{k-1}{\log(2)} + \frac{8}{\log(2)} + \frac{2}{\log^2(2)}}.$$

Hence by the first Grötzsch principle we have that if we remove the closed unit disk from our surface at the origin and call the family of closed curves separating this disk from infinity $\tilde{\Upsilon}$:





then

$$\operatorname{mod}(\tilde{\Upsilon}) \geq \sum_{k=1}^{\infty} \frac{1}{\frac{k-1}{\log(2)} + \frac{8}{\log(2)} + \frac{2}{\log^2(2)}} = \infty.$$

Hence by the principle of reciprocity the family Υ of open curves originating from the closed unit disk about the origin and going to infinity has modulus 0. Thus the surface is parabolic. ☐

Question 5.10. We pose the following open problem: Suppose that in the above theorem we instead take $x_n = n^2$. What is the type of the resulting surface?

References

[Vo] Volkovyskiĭ, L. I. Investigation of the type problem for a simply connected Riemann surface (Russian). Trudy Mat. Inst. Steklov. 34, (1950). 171 pp.

[Co] Conway, John B. Functions of One Complex Variable 1, Second Edition. Springer-Verlag New York, (1991).

Department of Mathematics, University of Michigan, Ann Arbor, MI, 48109-1109

E-mail address: `charleyc@umich.edu`