# STEINBERG COMPLEXES AND AFFINE DESCENTS

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Abstract. We study the $h$-polynomials of a family of complexes obtained by taking an affine Coxeter complex modulo its coroot lattice. Called Steinberg complexes, their maximal cells are in bijection with the elements of the underlying Weyl group. Their $h$-polynomials have a combinatorial description in terms of a descent-like statistic for Weyl group elements due to Cellini. We prove that the $h$-polynomial of a Steinberg complex is always $\gamma$-nonnegative, a property that implies symmetry and unimodality of the coefficients.

## 1. Introduction

The *Eulerian numbers* form a classical two-dimensional array of positive integers, $\{a_{n,k} : 0 \leq k < n\}$, defined (if you like) by the boundary values $a_{n,0} = a_{n,n-1} = 1$ and the recurrence relation

$$a_{n,k} = (n-k)a_{n-1,k-1} + (k+1)a_{n-1,k}. \tag{1}$$

Some interesting features of this set include the fact that, for fixed $n$, the vector $(a_{n,0}, a_{n,1}, \ldots, a_{n,n-1})$ is symmetric (i.e., $a_{n,k} = a_{n,n-k}$) and unimodal (i.e., $a_{n,0} \leq \cdots \leq a_{n,\lfloor n/2 \rfloor} \geq \cdots \geq a_{n,n-1}$). We can encode the vector $(a_{n,0}, a_{n,1}, \ldots, a_{n,n-1})$ in the *Eulerian polynomial*, which we define as

$$A_{n-1}(t) := \sum_{k=0}^{n-1} a_{n,k} t^k.$$

(This definition varies from the classical definition by a power of $t$.) A well-known (and nontrivial) fact is that the Eulerian polynomials have all real zeroes.

There are many ways to generalize the Eulerian numbers and the Eulerian polynomials. One of these is best understood after reinterpreting $A_{n-1}(t)$ itself. Let $S_n$ denote the set of permutations of $[n] := \{1, \ldots, n\}$, and for any $w \in S_n$, let $\mathrm{d}(w) := |\{i \in [n-1] : w_i > w_{i+1}\}|$, called the *descent* statistic. A straightforward combinatorial result gives that $a_{n,k} = |\{w \in S_n : \mathrm{d}(w) = k\}|$, and thus,

$$A_{n-1}(t) = \sum_{w \in S_n} t^{\mathrm{d}(w)}.$$

As a group, $S_n$ is also well-known as a Coxeter group of type $A_{n-1}$ (hence the notation above). The notion of a descent statistic exists in any finite Coxeter group $W$, so one possible generalization of the Eulerian polynomial is the *$W$-Eulerian polynomial*:

$$W(t) := \sum_{w \in W} t^{\mathrm{d}(w)}.$$

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In fact, the $W$-Eulerian polynomial is more than simply a generalization of the classical Eulerian polynomial; it has topological meaning as well. It is the $h$-polynomial of the Coxeter complex of $W$. (See [11, Section 1] for a good explanation of the connection between the $h$-polynomial and the $W$-Eulerian polynomial.) Some basic Coxeter group theory can be used to show that the $W$-Eulerian polynomial is symmetric, and using the topological interpretation gives unimodality of the $W$-Eulerian polynomial from general results. The first [!?] to undertake a systematic study of the $W$-Eulerian polynomials was Brenti [2], who conjectured that the $W$-Eulerian polynomials, along with the properties just mentioned, are in fact real-rooted. He showed the conjecture holds in all cases but type D, where he verified the conjecture up to $n = 23$.

Recently, in similar contexts both topological and combinatorial, several [1, 5, 9, 11] have studied a property of integral polynomials with nonnegative, symmetric coefficients that is stronger than being unimodal but weaker than having all real zeros. Called $\gamma$-nonnegativity, it means that, for a polynomial $h(t)$ of degree $n$, there exist nonnegative integers $\gamma_i$ such that

$$h(t) = \sum_{0 \leq i \leq n/2} \gamma_i t^i (1+t)^{n-2i}.$$

All the $W$-Eulerian polynomials are $\gamma$-nonnegative. See [11].

There are two main goals for this paper. The first is to present a family of polynomials, indexed by Weyl groups, that can be seen as analogues of the $W$-Eulerian polynomials. Called the *affine $W$-Eulerian polynomials*, these polynomials possess many of the properties of the $W$-Eulerian polynomials already discussed. In particular, we will show they are $\gamma$-nonnegative (Theorem 2.1).

To guide our study, we will present a combinatorial description for the affine $W$-Eulerian polynomials in terms a descent-like statistic for Weyl group elements that we call the *affine descent* statistic. The notion of an affine descent was first studied by Cellini [3] as a means for generating commutative subalgebras of Solomon's descent algebra (the generally noncommutative subalgebra of the group algebra $\mathbb{Q}[W]$ with a basis given by the sums of elements of $W$ with a common descent *set*). We will derive several interesting identities relating both affine and ordinary $W$-Eulerian polynomials, and derive exponential generating functions for each of the infinite families. We conjecture that the affine $W$-Eulerian polynomials are real-rooted. Type B (distinct here from type C) and type D remain to be proved, though we have verified the conjecture up to $n = 500$.

How the $W$-Eulerian polynomials arise is best understood topologically. This leads to the second goal for the paper: to present the construction, first due to Steinberg [10], of a complex obtained by taking an affine Coxeter complex modulo its coroot lattice. For a Weyl group $W$, we call such a complex the *Steinberg complex* of $W$. We will show the affine $W$-Eulerian polynomial is the $h$-polynomial of the Steinberg complex of $W$. While we focus on this connection in the present work, we believe there is much more to be said about Steinberg complexes and their uses.

The paper is presented in two main parts. Section 2 introduces the necessary definitions and presents the affine $W$-Eulerian polynomial as the $h$-polynomial of the Steinberg complex. Section 3 goes through the combinatorial descriptions of the affine $W$-Eulerian polynomials, proving $\gamma$-positivity and deriving exponential generating functions for types A, B, C, and D. The paper concludes with evidence to support a conjecture of real-rootedness for all (affine and non-affine) $W$-Eulerian polynomials.

## 2. Steinberg complexes and their $h$-polynomials

2.1. **The Steinberg complex of a Weyl group.** We assume the reader is familiar with basic definitions for Coxeter groups. Here we follow the notational conventions of [6].

Let $W$ be an irreducible Weyl group with root system $\Phi$, acting on a vector space $V$ with inner product $\langle\cdot,\cdot\rangle$. For any root $\alpha \in \Phi$, let $H_\alpha := \{\lambda \in V : \langle\lambda,\alpha\rangle = 0\}$ be the hyperplane orthogonal to $\alpha$ and let $s_\alpha$ denote the reflection through $H_\alpha$. Fix a set $S = \{s_1, s_2, \ldots, s_n\}$ of reflections that generates $W$. We call these reflections *simple*. The corresponding set of simple roots is denoted $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq \Phi$. For any root system we have a unique highest root $\tilde{\alpha}$. Set $\alpha_0 = -\tilde{\alpha}$, called the lowest root, and let $\Delta_0 = \Delta \cup \{\alpha_0\}$. Whereas $\Delta$ is the vertex set for the Dynkin diagram of $W$, $\Delta_0$ is the vertex set of the extended Dynkin diagram corresponding to the affine Weyl group $\widetilde{W}$. The affine group is generated by the reflections of $S$ along with the reflection through the affine hyperplane $H_{\tilde{\alpha},1} = \{\lambda \in V : \langle\lambda,\tilde{\alpha}\rangle = 1\}$, denoted $s_0 = s_{\tilde{\alpha},1}$. Set $S_0 = S \cup \{s_0\}$.

In general, reflecting hyperplanes in $\widetilde{W}$ are of the form $H_{\alpha,k} = \{\lambda \in V : \langle\lambda,\alpha\rangle = k\}$ for any root $\alpha \in \Phi$ and any integer $k \in \mathbb{Z}$. Let $\mathcal{H}$ denote the set of all such hyperplanes. The open regions in $V \setminus (\cup_{H\in\mathcal{H}} H)$ are called *alcoves*. Of particular interest is the alcove $A = \{\lambda \in V : \langle\lambda,\alpha\rangle > 0, \alpha \in \Delta, \langle\lambda,\tilde{\alpha}\rangle < 1\}$, whose closure is the fundamental domain for the action of $\widetilde{W}$ on $V$.

The *affine Coxeter complex* $\Sigma = \Sigma(W)$ is a geometric realization of $\widetilde{W}$. It is a simplicial complex in $V$ cut by the reflecting hyperplanes of $\widetilde{W}$. Cells in $\Sigma$ are of the form $wA_J$, where $w \in \widetilde{W}$ and for any $J \subseteq [0,n] := \{0, 1, \ldots, n\}$,

$$A_J := \{\lambda \in V : \langle\lambda,\alpha_j\rangle = 0 \text{ for } j \neq 0 \in J, \langle\lambda,\alpha_j\rangle > 0 \text{ for } j \neq 0 \in J^c\}$$
$$\text{intersected with } \begin{cases} \{\lambda \in V : 1 - \langle\lambda,\tilde{\alpha}\rangle = 0\} & \text{if } 0 \in J, \\ \{\lambda \in V : 1 - \langle\lambda,\tilde{\alpha}\rangle > 0\} & \text{if } 0 \in J^c, \end{cases}$$

where $J^c := [0,n] \setminus J$. Notice at the extremes that $A_\emptyset = A$ is the fundamental alcove and $A_{[0,n]} = \emptyset$. The vertices of $\Sigma$ are $wA_{\widehat{j}}$, given by maximal proper subsets $\widehat{j} := [0,n] \setminus \{j\}$. If we color vertex $wA_{\widehat{j}}$ with color $j$, we see that the affine Coxeter complex is *balanced*; that is, every face has distinctly colored vertices. (This is clear because every face of the fundamental domain is balanced, and $\widetilde{W}$-action preserves colors.) Specifically, the cell $wA_J$ has color-set $J^c$.

It is well known that $\widetilde{W}$ can be expressed as the semi-direct product of $W$ with the coroot lattice $L = \mathbb{Z}\Phi^\vee$, where

$$\Phi^\vee := \{\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle : \alpha \in \Phi\}$$

is the set of coroots. The projection of the affine Coxeter complex onto the torus $T = V/L$ is what we call the *Steinberg complex* $\overline{\Sigma} = \overline{\Sigma}(W) := \Sigma(W)/L$. The fundamental domain for $\widetilde{W}$ acting on $V$ is, under the projection, the fundamental domain for a $W$-action on $T$. By abuse of language, we will call the open regions in $T \setminus (\cup_{H\in\mathcal{H}} H/L)$ alcoves. We see that these alcoves are in bijection with the elements of $W$.

The faces of $\overline{\Sigma}$ are simply projections of the faces of $\Sigma$. We denote cells of $\overline{\Sigma}$ by $w\overline{A}_J$ for any $w \in W$ with $\overline{A}_J := A_J/L$. It is clear that the Steinberg complex is balanced under the coloring scheme inherited from $\Sigma$.

**Example 1.** *The Steinberg complex $\overline{\Sigma}(A_2)$ is a hexagon, formed by the union of the six equilateral triangles neighboring the origin in $\Sigma(A_2)$, with opposite sides identified. See figure [[whatever figure, and maybe say a little more]]*

**Example 2.** *Although types $B$ and $C$ have different root (and coroot) systems in general, $\overline{\Sigma}(B_2) = \overline{\Sigma}(C_2)$ is a square, composed of the eight 45-degree right triangles neighboring the origin in $\Sigma(B_2) = \Sigma(C_2)$, with opposite sides identified.*

**Remark 1.** *In general, a quick-and-dirty construction for $\overline{\Sigma}(W)$ is to take the closure of the union of alcoves neighboring the origin in $\Sigma(W)$ and identify $H_{\alpha,1}$ and $H_{\alpha,-1}$ for all roots $\alpha$.*

**Remark 2.** *While we defined the Steinberg complex for irreducible $W$, the construction works in general. If $W = W_1 \times W_2$ is a Weyl group that is not irreducible, its coroot lattice is the product of the coroot lattices $L_1 \times L_2$ and the Steinberg complex is nothing but the product of Steinberg complexes: $\overline{\Sigma}(W) = \overline{\Sigma}(W_1) \times \overline{\Sigma}(W_2)$.*

2.2. **The $f$- and $h$-polynomials.** Suppose we have a set of simplices $\Sigma$ that is balanced; that is, the vertices of $\Sigma$ have been given colors from $[0, n]$ so that no two vertices of the same color lie on the same face. This set need not have any additional structure (like being a simplicial complex).

For every $J \subseteq [0, n]$, we let $f_J(\Sigma)$ denote the number of simplices whose vertices have color-set $J$, and define

$$h_J(\Sigma) := \sum_{I \subseteq J} (-1)^{|J \setminus I|} f_I(\Sigma). \tag{2}$$

These numbers, over all subsets, make up the flag $f$- and $h$-vectors of $\Sigma$. Their corresponding generating functions are the flag $f$-polynomial and the flag $h$-polynomial:

$$f(\Sigma; t_0, \ldots, t_n) := \sum_{J \subseteq [0,n]} f_J(\Sigma) \prod_{j \in J} t_j,$$
$$h(\Sigma; t_0, \ldots, t_n) := \sum_{J \subseteq [0,n]} h_J(\Sigma) \prod_{j \in J} t_j.$$

We can see from (2) that

$$h(\Sigma; t_0, \ldots, t_n) = (1 - t_0) \cdots (1 - t_n) f(\Sigma; t_0/(1 - t_0), \ldots, t_n/(1 - t_n)). \tag{3}$$

The generating functions for the ordinary $f$- and $h$-vectors, i.e., the ordinary $f$- and $h$-polynomials, are the following one-variable specializations:

$$f(\Sigma; t) := f(\Sigma; t, \ldots, t) \qquad h(\Sigma; t) := h(\Sigma; t, \ldots, t).$$

Now we obtain the well-known relationship between $f$- and $h$-polynomials from (3), namely

$$h(\Sigma; t) = (1 - t)^{n+1} f(\Sigma; t/(1 - t)).$$

Suppose all the maximal simplices of $\Sigma$ have the same dimension, $n$, and let $\Sigma_{\max}$ denote the set of all such simplices. In many applications (e.g., Cohen-Macaulay complexes), the ordinary or flag $h$-polynomial has nonnegative coefficients. A simple combinatorial way to demonstrate nonnegativity is, if possible, to provide a simplex-to-subset mapping $C : \Sigma_{\max} \to 2^{[0,n]}$ such that

$$\Sigma = \bigcup_{\sigma \in \Sigma_{\max}} \{\sigma_J : C(\sigma) \subseteq J\} \qquad \text{(disjoint union)},$$

where $\sigma_J$ denotes the face of $\sigma$ formed by $J$-colored vertices. For such a decomposition to work, we see that each simplex with color-set $J$ must occur as $\sigma_J$ for a unique $\sigma \in \Sigma_{\max}$, and hence

$$f_J(\Sigma) = |\{\sigma \in \Sigma_{\max} : C(\sigma) \subseteq J\}|.$$

From the inclusion-exclusion formula (2) we get

$$h_J(\Sigma) = |\{\sigma \in \Sigma_{\max} : C(\sigma) = J\}|,$$

and thus nonnegativity of the $h$-polynomial follows since

$$h(\Sigma; t) = \sum_{\sigma \in \Sigma_{\max}} t^{|C(\sigma)|}.$$

**Example 3.** *An illustrative example of this approach is where $\Sigma$ is the ordinary (non-affine) Coxeter complex of type A. Here, maximal simplices correspond to permutations, and the simplex-to-subset map is given by the descent set of a permutation. Thus, the $h$-vector is in this case nothing but the Eulerian polynomial, $A_n(t) = \sum_{w \in A_n} t^{\mathrm{d}(w)}$ as presented in Section 1.*

2.3. **Affine descents.** We will now shift our attention to the $h$-polynomial of the Steinberg complex. Because the Steinberg complex is balanced, we may apply the observations of Section 2.2.

The usual definition of a (right) *descent* of an element $w \in W$ is a generator $s \in S$ such that $\ell(ws) < \ell(w)$, where $\ell(w)$ is the minimal length of an expression for $w$ as a product of the generators in $S$. An equivalent definition involving roots gives a descent at a simple root $\alpha \in \Delta$ if $w(\alpha) < 0$, i.e., if the chamber corresponding to $w$ and the vector $\alpha$ are on opposite sides of the hyperplane $H_\alpha$. The set of descents is

$$\mathrm{D}(w) := \{j \in [n] : w(\alpha_j) < 0\},$$

and the number of descents is denoted $\mathrm{d}(w) = |\mathrm{D}(w)|$.

Cellini [3] introduced a similar notion, that we call an *affine descent*, in order to study certain commutative subalgebras of Solomon's descent algebra (see Remark 3). An affine descent of $w \in W$ is a root $\alpha \in \Delta_0$ such that $w(\alpha) < 0$. The set of affine descents is

$$\widetilde{\mathrm{D}}(w) := \{j \in [0, n] : w(\alpha_j) < 0\},$$

and the number of affine descents is denoted $\widetilde{\mathrm{d}}(w) = |\widetilde{\mathrm{D}}(w)|$. (If we wish to use the length function to define affine descents, there is no problem. The usual descents are also affine descents, and we have an extra descent at $s_0$ if $\ell(ws_0) > \ell(w)$.)

**Remark 3.** *Recall the Solomon descent algebra of $W$ is a subalgebra of the group algebra $\mathbb{Q}[W]$ with basis $\{u_I : I \subset [n]\}$, where $u_I := \sum_{\mathrm{D}(w)=I} w$. In types A and B, the set $\{E_i : 0 \leq i \leq n\}$ where $E_i := \sum_{\mathrm{d}(w)=i} w$ forms a basis for a commutative subalgebra of Solomon's. It was this type of "Eulerian" subalgebra that Cellini sought to emulate in a type-independent way.*

The main goal of this section is to establish the following fact.

**Proposition 1.** *For any nontrivial subset $J \subseteq [0, n]$, we have*

$$f_J(\overline{\Sigma}(W)) = |\{w \in W : \widetilde{\mathrm{D}}(w^{-1}) \subseteq J\}|,$$

*and hence*

$$h_J(\overline{\Sigma}(W)) = |\{w \in W : \widetilde{\mathrm{D}}(w^{-1}) = J\}|.$$

**Corollary 1.** *The h-polynomial of $\overline{\Sigma}(W)$ is the affine descent generating function for the elements of $W$:*

$$h(\overline{\Sigma}(W); t) = \sum_{w \in W} t^{\widetilde{\mathrm{d}}(w)}.$$

Because of the similarity with the classical Eulerian polynomials (and their generalizations to other Coxeter groups) we will refer to $\widetilde{W}(t) := h(\overline{\Sigma}(W); t)$ as the *affine Eulerian polynomial* of $W$. At this point it is also worth mentioning that by looking at the definition, $J = \emptyset$ and $J = [0, n]$ can never be achieved as affine descent sets. Thus, $\widetilde{W}(t)$ is a nonconstant polynomial of degree $n$.

**Remark 4.** *Convention would dictate that our complexes have an empty face of dimension zero. However, we choose to ignore this face because the combinatorics is less interesting. For example, if we include the empty face in our calculations, $f(\overline{\Sigma}(B_2); t) = 1 + 4t + 12t^2 + 8t^3$ and thus $h(\overline{\Sigma}(B_2); t) = 1 + t + 7t^2 - t^3$. Symmetry, unimodality, and even nonnegativity are lost.*

*Proof of Proposition 1.* To begin, fix any nonempty subset $J \subseteq [0, n]$. The $\widetilde{W}$-orbit of $\overline{A}_{J^c}$ in $\overline{\Sigma}(W)$ is precisely all the faces of the Steinberg complex with color-set $J$, and $f_J(\overline{\Sigma})$ is the size of this orbit. The stabilizer of the $\widetilde{W}$-action is generated by all those reflections that fix $A_{J^c}$ or any of its translations via the coroot lattice $L$ (in the affine complex $\Sigma$). Thus we have the stabilizer of $\overline{A}_{J^c}$ under the action of $\widetilde{W} = W \rtimes L$ is the parabolic subgroup $W_{J^c} \rtimes L$, where $W_{J^c} := \langle s_j : j \in J^c \rangle$. Note that because $J$ is nonempty, $W_{J^c}$ is a subgroup of $W$ (not necessarily parabolic) and a finite reflection group in its own right (imagine deleting the vertices of $J$ from the extended Dynkin diagram). Thus cosets of $(W \rtimes L)/(W_{J^c} \rtimes L) \cong W/W_{J^c}$ are in bijection with the set of $J$-colored cells of $\overline{\Sigma}$, and we have

$$f_J(\overline{\Sigma}) = |W/W_{J^c}| = |W|/|W_{J^c}|. \tag{4}$$

Now take $\lambda \in V$ to be dominant, i.e., an interior point of the dominant chamber of $W$. We let $W\lambda := \{w\lambda : w \in W\}$ and let

$$R := \{\mu \in W\lambda : \langle \mu, \alpha_j \rangle > 0 \text{ for } j \in J^c\}.$$

Note that the vectors in $W\lambda$ are in bijection with elements of $W$, and thus $R$ corresponds to some subset of $W$.

We now observe that all the vectors in $R$ are dominant in terms of $W_{J^c}$, and claim that

$$|R| = |W/W_{J^c}|. \tag{5}$$

Consider $W_{J^c}\mu := \{w\mu : w \in W_{J^c}\} \subseteq W\mu = W\lambda$. This set is clearly in bijection with the elements of $W_{J^c}$, but it also corresponds to a subset of $W$. We need to show that the $W_{J^c}\mu$, over all $\mu \in R$, partition $W$ into distinct $W_{J^c}$-cosets.

First of all, every $w \in W$ has an interior point in some coset. To see this, let $\nu \in V$ be an interior point of the chamber corresponding to $w$, and consider $W_{J^c}\nu$. Since $\nu$ is also an interior point of a $W_{J^c}$-chamber, its orbit must contain some $W_{J^c}$-dominant point $\mu \in R$. Hence, $\nu \in W_{J^c}\nu = W_{J^c}\mu$.

Now if $W_{J^c}\mu \cap W_{J^c}\mu' \neq \emptyset$, then we can write $w\mu = w'\mu'$ for some $w, w' \in W_{J^c}$. But then $\mu = w^{-1}w'\mu'$ is clearly in $W_{J^c}\mu'$. There can be only one $W_{J^c}$-dominant point per coset, so it must be that $\mu = \mu'$, and the cosets are disjoint.

Thus, by combining (4) and (5) we have established that

$$\begin{aligned} f_J(\overline{\Sigma}) = |R| &= |\{\mu \in W\lambda : \langle \mu, \alpha_j \rangle > 0 \text{ for } j \in J^c\}|, \\ &= |\{w \in W : \langle \lambda, w^{-1}(\alpha_j)\rangle > 0 \text{ for } j \in J^c\}|, \\ &= |\{w \in W : \widetilde{\mathrm{D}}(w^{-1}) \subseteq J\}|, \end{aligned}$$

and the proposition is proved. □

We now have that the coefficients of the affine Eulerian polynomials are nonnegative. It is relatively straightforward to prove that the coefficients are symmetric as well. Recall that multiplication by the longest element $w_0$ is an involution on $W$ that reverses length, i.e., $\ell(w_0 w) = \ell(w_0) - \ell(w)$. Thus if $\ell(ws) < \ell(w)$,

$$\ell(w_0 ws) = \ell(w_0) - \ell(ws) > \ell(w_0) - \ell(w) = \ell(w_0 w),$$

and descent sets are taken to their complements. (In type A, $w_0$ simply reverses the order of $w$: $w_0 w = w_n \cdots w_2 w_1$.) Alternatively, we know $w_0\lambda = -\lambda$ for any $\lambda \in V$, so $\langle w_0\lambda, \alpha\rangle = -\langle \lambda, \alpha\rangle$ for any root $\alpha$ in $\Delta_0$.

In any case we see

$$\widetilde{\mathrm{D}}(w_0 w) = [0, n] \setminus \widetilde{\mathrm{D}}(w),$$

leading to the following observation (and the same arguments show $\mathrm{D}(w_0 w) = [n] \setminus \mathrm{D}(w)$).

**Observation 1.** *The affine Eulerian polynomials are symmetric, i.e., if $W$ has rank $n$,*

$$\widetilde{W}(t) = t^n \widetilde{W}(1/t).$$

Using (4) and Stembridge's Coxeter package for Maple [14], we can explicitly compute the affine Eulerian polynomials for all the exceptional groups. Also included in the table below are the first few type B and type D polynomials. It is easy to observe that all the polynomials listed are not only symmetric but also unimodal. In fact, they are real-rooted.

2.4. **The $\gamma$-vector.** Although we will ultimately conjecture that the affine Eulerian polynomials have all real zeroes, we can now only prove a slightly weaker-condition, one that guarantees both the symmetry and unimodality of the $h$-vector. After Brändén [1] and Gal [5], we can see that the polynomials

$$\Gamma_n = \{t^i(1+t)^{n-2i} : 0 \le i \le n/2\}$$

form a $\mathbb{Z}$-basis for $\{h(t) \in \mathbb{Z}[t] : h(t) = t^n h(1/t)\}$. Thus for any set of simplices $\Sigma$ with $h$-vector satisfying $h_i = h_{n-i}$, we can define integers $\gamma_i$ by

$$h(\Sigma; t) = \sum_{0 \le i \le n/2} \gamma_i t^i (1+t)^{n-2i}.$$

The sequence $\gamma(\Sigma) = (\gamma_0, \gamma_1, \ldots)$ is called the $\gamma$-vector of $\Sigma$.

It is clear the polynomials in $\Gamma_n$ have symmetric and unimodal coefficients with the same center of symmetry, thus the same is true of any polynomial in their nonnegative span. So, as Brändén and Gal observe, nonnegativity of $\gamma(\Sigma)$ implies symmetric unimodality of the $h$-vector.

| $W$ | $\widetilde{W}(t)$ |
|---|---|
| $B_2$ | $4t + 4t^2$ |
| $B_3$ | $10t + 28t^2 + 10t^3$ |
| $B_4$ | $24t + 168t^2 + 168t^3 + 24t^4$ |
| $B_5$ | $54t + 904t^2 + 1924t^3 + 904t^4 + 54t^5$ |
| $B_6$ | $116t + 4452t^2 + 18472t^3 + 18472t^4 + 4452t^5 + 116t^6$ |
| $D_2$ | $2t + 2t^2$ |
| $D_3$ | $4t + 16t^2 + 4t^3$ |
| $D_4$ | $16t + 80t^2 + 80t^3 + 16t^4$ |
| $D_5$ | $44t + 464t^2 + 904t^3 + 464t^4 + 44t^5$ |
| $D_6$ | $104t + 2568t^2 + 8848t^3 + 8848t^4 + 2568t^5 + 104t^6$ |
| $E_6$ | $351t + 5427t^2 + 20142t^3 + 20142t^4 + 5427t^5 + 351t^6$ |
| $E_7$ | $4064t + 115728t^2 + 710112t^3 + 1243232t^4 + 710112t^5 + 115728t^6 + 4064t^7$ |
| $E_8$ | $157200t + 9253680t^2 + 87417360t^3 + 251536560t^4 + 251536560t^5 + 87417360t^6 + 9253680t^7 + 157200t^8$ |
| $F_4$ | $72t + 504t^2 + 504t^3 + 72t^4$ |
| $G_2$ | $6t + 6t^2$ |
| $H_3$ | $26t + 68t^2 + 26t^3$ |
| $H_4$ | $960t + 6240t^2 + 6240t^3 + 960t^4$ |

TABLE 1. Some affine $W$-Eulerian polynomials.

If $h(t)$ has nonnegative and symmetric coefficients, then apart from $t \in \{0, -1\}$, the zeroes of $h(t)$ occur in pairs $a, 1/a$. If we know for some reason that $h(t)$ has all real zeroes, then $a < 0$ and $h(t)$ can be written as a product of factors of the form

$$(t - a)(t - 1/a) = (1 + t)^2 - (2 + a + 1/a)t \in \mathbb{R}^+\Gamma_2,$$

along with a positive scalar and possibly factors of $t \in \Gamma_2$ and $1 + t \in \Gamma_1$. Because $\Gamma_m\Gamma_n \subseteq \Gamma_{m+n}$, it follows that a real-rooted polynomial with nonnegative symmetric coefficients is $\gamma$-nonnegative.

We have already checked that all the exceptional affine Eulerian polynomials have all real zeroes, and we will see shortly how work of Petersen [7] and Fulman [4] implies that $\widetilde{A}_n(t)$ and $\widetilde{C}_n(t)$ have all real zeroes. Therefore $\gamma$-nonnegativity follows for all but types B and D. Sections 3.3 and 3.4 will establish $\gamma$-nonnegativity in these cases (though not real-rootedness). Furthermore, since a general Steinberg complex is a product of Steinberg complexes for irreducible Weyl groups (see Remark 2), its $h$-polynomial is a product of $\gamma$-nonnegative $h$-polynomials. Thus we will establish the following.

**Theorem 2.1.** *For any Weyl group $W$, the Steinberg complex $\overline{\Sigma}(W)$ has a nonnegative $\gamma$-vector.*

## 3. Combinatorial descriptions and $\gamma$-nonnegativity

Here we provide, for each of the infinite families of irreducible Weyl groups, a combinatorial description of affine descents along with an explicit formula for the exponential generating function of the affine Eulerian polynomials. We will review types A and C, where much is already known, before moving on to types B and D. Much of the work in this section is devoted to proving $\gamma$-nonnegativity for type B and type D.

3.1. **Type A.** In type A, the affine and ordinary Eulerian polynomials are very closely related and easy to describe. Let $S_n$ denote the set of permutations of $[n]$, i.e., all bijections $w : [n] \to [n]$. We write a permutation as a word $w = w_1 w_2 \cdots w_n$, where $w_i := w(i)$. The elements of $S_n$ are in bijection with elements of $W = A_{n-1}$. Here, the simple roots are $\alpha_i = e_{i+1} - e_i$, $i = 1, \ldots, n-1$, and $\alpha_0 = -\tilde{\alpha} = e_1 - e_n$, where $\{e_i : 1 \leq i \leq n\}$ is the set of standard basis vectors in $\mathbb{R}^n$. We evaluate $w(\alpha)$ by $w(e_i - e_j) = e_{w_i} - e_{w_j}$. Thus $w(\alpha_0) = e_{w_1} - e_{w_{n+1}}$, which is positive if and only if $w_{n+1} < w_1$.

Therefore affine descents of type A, also known as cyclic descents (see [7]), are positions $i$ such that $w_i > w_{i+1 \mod n}$. In other words, they are ordinary permutation descents along with $n$ if $w_n > w_1$. For example, $\widetilde{\mathrm{D}}(13425) = \{3, 5\}$ and $\widetilde{\mathrm{d}}(13425) = 2$. We have the following observation [7, 4].

**Observation 2.** *The affine Eulerian polynomial of type A is expressible in terms of the classic Eulerian polynomial:*

$$\widetilde{A}_n(t) = (n+1)tA_{n-1}(t). \tag{6}$$

This relationship shows that many nice properties known of the classical Eulerian polynomials (e.g., $\gamma$-nonnegativity, real-rootedness), hold for the affine case as well.

Recall that the exponential generating function for the classical Eulerian polynomials is:

$$A(t,z) := \sum_{n\geq 0} A_{n-1}(t)\frac{z^n}{n!} = \sum_{n\geq 0}\sum_{w\in S_n} t^{\mathrm{d}(w)}\frac{z^n}{n!} = \frac{(1-t)e^{z(1-t)}}{1-te^{z(1-t)}},$$

where $A_{-1}(t) = A_0(t) = 1$. Define the exponential generating function for the affine case:

$$\widetilde{A}(t,z) := \sum_{n\geq 0} \widetilde{A}_{n-1}(t)\frac{z^n}{n!} = \sum_{n\geq 0}\sum_{w\in S_n} t^{\widetilde{\mathrm{d}}(w)}\frac{z^n}{n!},$$

again with $\widetilde{A}_{-1}(t) = \widetilde{A}_0(t) = 1$. By standard manipulations and the identity from (6), we get the following exponential generating function.

**Corollary 2.** *We have*

$$\widetilde{A}(t,z) = \frac{1 + z - tz - te^{z(1-t)}}{1-te^{z(1-t)}}.$$

3.2. **Type C.** Elements of $W = C_n$ can be represented as signed permutations. That is, for any permutation of $[n]$ we have $2^n$ signed permutations obtained by negating some choice of the letters. The simple roots are $\alpha_1 = 2e_1$, $\alpha_i = e_i - e_{i-1}$ for $i = 2, \ldots, n$. The lowest root is $\alpha_0 = -2e_n$. Signed permutations act on basis vectors as before, with the identification $e_{\bar{i}} = -e_i$. Since $w(e_i - e_{i-1}) = e_{w_i} - e_{w_{i-1}}$, an ordinary descent of a signed permutations is a position $i$ such that $w_{i-1} > w_i$ (where $w_0 = 0$). An affine descent corresponds to $w_n > 0$. For example, $23\bar{4}\bar{1}5$ has an ordinary descent in position 3 and an affine descent in position 0. With this description both [7] (using $P$-partition analogues) and [4] (using shuffling techniques) prove the following identity.

**Observation 3.** *We have*

$$\widetilde{C}_n(t) = 2^n tA_{n-1}(t). \tag{7}$$

As with type A, this observation establishes $\gamma$-positivity and real-rootedness of the type C affine Eulerian polynomials. It also leads to a simple expression for the generating function

$$\widetilde{C}(t,z) := \sum_{n\geq 0} \widetilde{C}_n(t)\frac{z^n}{n!}.$$

**Corollary 3.** *We have*

$$\widetilde{C}(t,z) = tA(t,2z) = \frac{t(1-t)e^{2z(1-t)}}{1-te^{2z(1-t)}}.$$

3.3. **Type B $\gamma$-nonnegativity.** As a group, type B is the same as type C. However, their underlying root systems are different. The only change in simple roots is that now $\alpha_1 = e_1$, which leads us to find the lowest root is $\alpha_0 = -e_n - e_{n-1}$. Thus, ordinary descents of type B are the same as ordinary descents of type C, but an affine descent now corresponds to $w_n > -w_{n-1}$ because $w(\alpha_0) = -e_{w_n} - e_{w_{n-1}} = -e_{w_n} + e_{-w_{n-1}}$. For example, $23\bar{4}5\bar{1}$ has ordinary descents in positions 3 and 5, along with an affine descent in position 0.

Recall the flag version of the affine Eulerian polynomial, now seen to be the generating function for affine descent *sets*:

$$\widetilde{B}_n(t_0,t_1,\ldots,t_n) = \sum_{w\in B_n}\prod_{i\in\widetilde{\mathrm{D}}(w)} t_i.$$

We will first show how this polynomial depends only on the underlying unsigned permutations and the location of the signs; $\gamma$-nonnegativity of $\widetilde{B}_n(t)$ will follow. (This approach is the one taken by Stembridge [11, Appendix] in the non-affine case.)

Below, $\chi(S) = 1$ if $S$ is true, 0 otherwise.

**Proposition 2.** *We have*

$$\widetilde{B}_n(t_0,t_1,\ldots,t_n) = \sum_{u\in S_n}(t_1 + t_2^{\chi(u_1>u_2)})\prod_{i=2}^{n-2}(t_i^{\chi(u_{i-1}<u_i)} + t_{i+1}^{\chi(u_i>u_{i+1})})$$
$$\times (t_{n-1}^{\chi(u_{n-2}<u_{n-1})} + (t_0t_n)^{\chi(u_{n-1}>u_n)})(t_n^{\chi(u_{n-1}<u_n)} + t_0^{\chi(u_{n-1}<u_n)}).$$

*Proof.* Fix a permutation $u$ in $S_n$. Let $\sigma\in\mathbb{Z}_2^n$ and let $w = \sigma u$. Clearly the appearance of a descent in position 1 depends only on the sign of $\sigma_1$. For $i = 2,\ldots,n$, we see

$$i\in\widetilde{\mathrm{D}}(w) \Leftrightarrow \begin{cases} u_{i-1} > u_i & \text{and } \sigma_{i-1} = 1, \text{ or} \\ u_{i-1} < u_i & \text{and } \sigma_i = -1. \end{cases}$$

For affine descents we have

$$0\in\widetilde{\mathrm{D}}(w) \Leftrightarrow \begin{cases} u_{n-1} > u_n & \text{and } \sigma_{n-1} = 1, \text{ or} \\ u_{n-1} < u_n & \text{and } \sigma_n = 1. \end{cases}$$

Thus we see that

$$\sum_{\sigma\in\mathbb{Z}_2^n}\prod_{i\in\widetilde{\mathrm{D}}(\sigma u)} t_i = (\underbrace{t_1 + t_2^{\chi(u_1>u_2)}}_{\sigma_1=\pm 1})\prod_{i=2}^{n-2}(\underbrace{t_i^{\chi(u_{i-1}<u_i)} + t_{i+1}^{\chi(u_i>u_{i+1})}}_{\sigma_i=\pm 1})$$
$$\times (\underbrace{t_{n-1}^{\chi(u_{n-2}<u_{n-1})} + (t_0t_n)^{\chi(u_{n-1}>u_n)}}_{\sigma_{n-1}=\pm 1})(\underbrace{t_n^{\chi(u_{n-1}<u_n)} + t_0^{\chi(u_{n-1}<u_n)}}_{\sigma_n=\pm 1})$$

and the proposition follows. ☐

By specializing $t_i = t$, we obtain the following formula, where

$$\overline{\mathrm{pk}}(u) := |\{i \in [n] : u_{i-1} < u_i > u_{i+1},\ \text{with } u_0 = u_{n+1} = 0\}|$$

denotes the number of *exterior peaks* of $u$. In particular, we see that $\widetilde{B}_n(t)$ is $\gamma$-nonnegative.

**Corollary 4.** *We have*

$$\widetilde{B}_n(t) = \sum_{u \in S_n} \phi(u)(4t)^{\overline{\mathrm{pk}}(u)}(1+t)^{n+1-2\,\overline{\mathrm{pk}}(u)},$$

*where*

$$\phi(u) = \begin{cases} 1 & \text{if } u_{n-2} > u_{n-1} > u_n, \\ 0 & \text{if } u_{n-2} > u_n > u_{n-1}, \\ 1/2 & \text{otherwise.} \end{cases}$$

*Proof.* When we specialize to $t_i = t$, generally we have that for a fixed permutation $u \in S_n$ that the $i$th factor in the product is

$$b_i(t) = \begin{cases} 2t & \text{if } i \text{ is a peak, i.e., } u_{i-1} < u_i > u_{i+1}, \\ 2 & \text{if } i \text{ is a valley, i.e., } u_{i-1} > u_i < u_{i+1}, \\ 1+t & \text{otherwise.} \end{cases}$$

If the permutation $0 \cdot u \cdot 0$ has $k$ peaks, then it will necessarily have $k-1$ valleys, and thus $n+1-2k$ positions with neither a peak nor a valley. If all the factors obeyed this rule, then each permutation $u$ would contribute

$$\frac{1}{2}(4t)^{\overline{\mathrm{pk}}(u)}(1+t)^{n+1-2\,\overline{\mathrm{pk}}(u)}$$

to the total sum. However, the last two factors do not always behave this way because of the affine descent. In the following table we compare the differences between what we assumed the contribution of the last two factors would be and what the contribution of the last two factors actually is for each possible ordering of the last three terms in the permutation $u$.

| | | Assumed $b_{n-1}(t)b_n(t)$ | Actual $b_{n-1}(t)b_n(t)$ |
|---|---|---|---|
| I: | $u_n < u_{n-1} < u_{n-2}$ | $(1+t)^2$ | $2(1+t^2)$ |
| II: | $u_n < u_{n-2} < u_{n-1}$ | $2t(1+t)$ | $2t(1+t)$ |
| III: | $u_{n-1} < u_n < u_{n-2}$ | $4t$ | $4t$ |
| IV: | $u_{n-1} < u_{n-2} < u_n$ | $4t$ | $4t$ |
| V: | $u_{n-2} < u_n < u_{n-1}$ | $2t(1+t)$ | $2t(1+t)$ |
| VI: | $u_{n-2} < u_{n-1} < u_n$ | $2t(1+t)$ | $2t(1+t)$ |

Only the first type runs counter to our expectations. Happily, if $u$ is a permutation satisfying $u_n < u_{n-1} < u_{n-2}$, then we see $u' = u_1 \cdots u_{n-2}u_nu_{n-1}$ satisfies $u_{n-1} < u_n < u_{n-2}$ and swapping $u_n$ and $u_{n-1}$ is a bijection between types I and III. Moreover, since $u$ and $u'$ are identical up to position $n-2$, they have the same contribution from $b_1(t) \cdots b_{n-2}(t)$. For every $u$ of type I, we can count the weight of both $u$ and $u'$ by replacing $b_{n-1}(t)b_n(t) = 2(1+t^2)$ with twice its assumed value, $2(1+t)^2 = 2(1+t^2) + 4t$. Thus, we achieve the desired result. ☐

3.4. **Type D $\gamma$-positivity.** Elements of $W = D_n$ can be represented as even-signed permutations. That is, for any permutation of $[n]$, we have $2^{n-1}$ signed permutations obtained by negating some choice of the letters, with the added condition that we only negate an even number of letters. The simple roots are $\alpha_1 = e_1 + e_2$, $\alpha_i = e_i - e_{i-1}$, for $i = 2, \ldots, n$. The lowest root is $\alpha_0 = -e_n - e_{n-1}$. Thus an ordinary descent of an even-signed permutation is a position $i$ such that $w_{i-1} > w_i$ (where $w_0 = -w_2$) and as with type B, an affine descent corresponds to $w_n > -w_{n-1}$. For example, $3\bar{4}2\bar{1}5$ has ordinary descents in positions 1, 2, and 4, and an affine descent in position 0.

Recall the flag affine Eulerian polynomial of type D:

$$\widetilde{D}_n(t_0, t_1, \ldots, t_n) = \sum_{w \in D_n} \prod_{i \in \widetilde{\mathrm{D}}(w)} t_i.$$

**Proposition 3.** *We have*

$$\widetilde{D}_n(t) = \frac{1}{2} \sum_{u \in S_n} (t_1^{\chi(u_1 > u_2)} + t_2^{\chi(u_1 > u_2)})((t_1 t_2)^{\chi(u_1 < u_2)} + t_3^{\chi(u_2 > u_3)}) \prod_{i=3}^{n-2} (t_i^{\chi(u_{i-1} < u_i)} + t_{i+1}^{\chi(u_i > u_{i+1})})$$
$$\times (t_{n-1}^{\chi(u_{n-2} < u_{n-1})} + (t_0 t_n)^{\chi(u_{n-1} > u_n)})(t_n^{\chi(u_{n-1} < u_n)} + t_0^{\chi(u_{n-1} < u_n)}).$$

*Proof.* First we switch from even signed permutations to all signed permutations. Note that the position of all type D affine descents remain unchanged in a general signed permutation when $\bar{1}$ and 1 are switched, so we can work over all signed permutations at the expense of a factor of 1/2. Thus,

$$\sum_{v \in D_n} \prod_{i \in \widetilde{\mathrm{D}}(v)} t_i = \frac{1}{2} \sum_{w \in B_n} \prod_{i \in \widetilde{\mathrm{D}}(w)} t_i,$$

where we emphasize that although we are summing over type B elements, we are considering *type D* descents.

Now, fix a permutation $u$ in $S_n$. Let $\sigma \in \mathbb{Z}_2^n$ and let $w = \sigma u$. For $i = 1$, we see

$$1 \in \widetilde{\mathrm{D}}(w) \Leftrightarrow \begin{cases} u_1 > u_2 & \text{and } \sigma_1 = -1, \text{ or} \\ u_1 < u_2 & \text{and } \sigma_2 = -1. \end{cases}$$

For $i = 2, \ldots, n$, we see

$$i \in \widetilde{\mathrm{D}}(w) \Leftrightarrow \begin{cases} u_{i-1} > u_i & \text{and } \sigma_{i-1} = 1, \text{ or} \\ u_{i-1} < u_i & \text{and } \sigma_i = -1. \end{cases}$$

For affine descents we have

$$0 \in \widetilde{\mathrm{D}}(w) \Leftrightarrow \begin{cases} u_{n-1} > u_n & \text{and } \sigma_{n-1} = 1, \text{ or} \\ u_{n-1} < u_n & \text{and } \sigma_n = 1. \end{cases}$$

Thus we see that

$$\sum_{\sigma \in \mathbb{Z}_2^n} \prod_{i \in \widetilde{\mathrm{D}}(\sigma u)} t_i = (\underbrace{t_1^{\chi(u_1 > u_2)} + t_2^{\chi(u_1 > u_2)}}_{\sigma_1 = \pm 1})(\underbrace{(t_1 t_2)^{\chi(u_1 < u_2)} + t_3^{\chi(u_2 > u_3)}}_{\sigma_2 = \pm 1}) \prod_{i=3}^{n-2} (\underbrace{t_i^{\chi(u_{i-1} < u_i)} + t_{i+1}^{\chi(u_i > u_{i+1})}}_{\sigma_i = \pm 1})$$
$$\times (\underbrace{t_{n-1}^{\chi(u_{n-2} < u_{n-1})} + (t_0 t_n)^{\chi(u_{n-1} > u_n)}}_{\sigma_{n-1} = \pm 1})(\underbrace{t_n^{\chi(u_{n-1} < u_n)} + t_0^{\chi(u_{n-1} < u_n)}}_{\sigma_n = \pm 1})$$

and the proposition follows. □

By specializing $t_i = t$, we obtain the following formula.

**Corollary 5.** *We have*

$$\widetilde{D}_n(t) = \sum_{u\in S_n} \phi(u)\phi(\overleftarrow{u})(4t)^{\overline{\mathrm{pk}}(u)}(1+t)^{n+1-2\,\overline{\mathrm{pk}}(u)},$$

*where $\phi(u)$ is defined in Corollary 4 and $\overleftarrow{u} = u_n u_{n-1}\cdots u_1$.*

*Proof.* In the proof for type B affine, we examined how, because of the affine descent, the last two factors acted slightly differently from the others. Now for type D affine, we wish to modify the formula for type B affine to accomodate the difference in the first two factors. We compare the the difference bewteen the first two factors from Type B affine and type D affine for each of the possible orderings of the first three elements of a permutation $u$.

| | | Type B contribution | Type D contribution |
|---|---|---|---|
| I: | $u_1 < u_2 < u_3$ | $(1+t)^2$ | $2(1+t^2)$ |
| II: | $u_1 < u_3 < u_2$ | $2t(1+t)$ | $2t(1+t)$ |
| III: | $u_2 < u_1 < u_3$ | $4t$ | $4t$ |
| IV: | $u_2 < u_3 < u_1$ | $4t$ | $4t$ |
| V: | $u_3 < u_1 < u_2$ | $2t(1+t)$ | $2t(1+t)$ |
| VI: | $u_3 < u_2 < u_1$ | $2t(1+t)$ | $2t(1+t)$ |

There is a bijection between permutations of type I and type III via switching $u_1$ and $u_2$. We can combine the contributions from these two kinds of permutations by ignoring type III and replacing the weight for type I with $2(1+t)^2$, which is twice the contribution we would have expected from the type B definition. The result follows. □

3.5. **Identities and generating functions for type B and type D.** Before presenting the formulas for the type B and D generating functions, we need to recall the $\gamma$-nonnegativity formulas for the ordinary type B and type D descent generating functions from [7] and [11] and derive some further identities. In what follows,

$$\mathrm{pk}^\ell(u) := |\{i \in [n-1] : u_{i-1} < u_i > u_{i+1},\ \text{where } u_0 = 0\}|$$

is the number of *left peaks* of $u$.

**Proposition 4** ( Petersen, [7] Prop. 4.15)**.** *We have*

$$B_n(t) := \sum_{w\in B_n} t^{\mathrm{d}(w)} = \sum_{u\in S_n} (4t)^{\mathrm{pk}^\ell(u)}(1+t)^{n-2\,\mathrm{pk}^\ell(u)}. \tag{8}$$

**Proposition 5** ( Stembridge, [11] Cor. A.5 )**.** *We have*

$$D_n(t) := \sum_{w\in D_n} t^{\mathrm{d}(w)} = \sum_{u\in S_n} \phi(\overleftarrow{u})(4t)^{\mathrm{pk}^\ell(u)}(1+t)^{n-2\,\mathrm{pk}^\ell(u)}. \tag{9}$$

By combining these formulas with those from previous sections, we can prove several identities.

**Proposition 6.** *We have for $n \geq 1$*

$$\widetilde{B}_n(t) = \widetilde{D}_n(t) + 2ntD_{n-1}(t). \tag{10}$$

*Proof.* We will use previously derived formulas to show that $\widetilde{B}_n(t) - \widetilde{D}_n(t) = 2ntD_{n-1}(t)$.

To begin, we use (9) to write:

$$2ntD_{n-1}(t) = 2nt \sum_{u \in S_{n-1}} \phi(\overleftarrow{u})(4t)^{\mathrm{pk}^\ell(u)}(1+t)^{n-1-2\,\mathrm{pk}^\ell(u)}.$$

Now suppose $n > 3$ (earlier cases are easily checked). To each $u \in S_{n-1}$ we may associate those $n$ permutations $v$ in $S_n$ for which the standardization of the first $n-1$ letters is $u$. If $v' = v_1 \cdots v_{n-1}$, then $\phi(\overleftarrow{v'}) = \phi(\overleftarrow{v})$ and we may now write

$$2ntD_{n-1}(t) = 2t \sum_{v \in S_n} \phi(\overleftarrow{v})(4t)^{\mathrm{pk}^\ell(v')}(1+t)^{n-1-2\,\mathrm{pk}^\ell(v')}.$$

Recall the bijection between those permutations satisfying $\phi(v) = 1$ (i.e., $v_{n-2} > v_{n-1} > v_n$) and those satisfying $\phi(w) = 0$ (i.e., $w_{n-2} > w_n > w_{n-1}$) given by swapping the final two letters. We can see that the permutations paired by this bijection satisfy $\mathrm{pk}^\ell(v') = \mathrm{pk}^\ell(w')$ and $\phi(\overleftarrow{v}) = \phi(\overleftarrow{w})$. Thus we can consider the contribution from such a pair by doubling the contribution from $w$ and ignoring the contribution from $v$. Conveniently, any permutation $v \in S_n$ such that $\phi(v) \neq 1$ satisfies $\mathrm{pk}^\ell(v') = \overline{\mathrm{pk}}(v) - 1$, so now we have

$$\begin{aligned} 2t \sum_{v \in S_n} \phi(\overleftarrow{v})(4t)^{\mathrm{pk}^\ell(v')}(1+t)^{n-1-2\,\mathrm{pk}^\ell(v')} &= 2t \sum_{v \in S_n} 2(1-\phi(v))\phi(\overleftarrow{v})(4t)^{\overline{\mathrm{pk}}(v)-1}(1+t)^{n-1-2(\overline{\mathrm{pk}}(v)-1)} \\ &= \sum_{v \in S_n} (1-\phi(v))\phi(\overleftarrow{v})(4t)^{\overline{\mathrm{pk}}(v)}(1+t)^{n+1-2\,\overline{\mathrm{pk}}(v)}. \end{aligned}$$

On the other hand, using our formulas from Corollaries 4 and 5 we obtain

$$\begin{aligned} \widetilde{B}_n(t) - \widetilde{D}_n(t) &= \sum_{v \in S_n} \phi(v)(4t)^{\overline{\mathrm{pk}}(v)}(1+t)^{n+1-2\,\overline{\mathrm{pk}}(v)} - \sum_{v \in S_n} \phi(\overleftarrow{v})\phi(v)(4t)^{\overline{\mathrm{pk}}(v)}(1+t)^{n+1-2\,\overline{\mathrm{pk}}(v)} \\ &= \sum_{v \in S_n} (1-\phi(\overleftarrow{v}))\phi(v)(4t)^{\overline{\mathrm{pk}}(v)}(1+t)^{n+1-2\,\overline{\mathrm{pk}}(v)}. \end{aligned}$$

Now observing $\overline{\mathrm{pk}}(v) = \overline{\mathrm{pk}}(\overleftarrow{v})$, we get the desired result. □

This strategy works to give a new proof of the following identity due to Stembridge [12].

**Proposition 7** (Stembridge, [12])**.** *We have for* $n \geq 2$

$$B_n(t) = D_n(t) + n2^{n-1}tA_{n-2}(t). \tag{11}$$

*Proof.* ¿From [13] (or [1] or [8]) we have the following formula for $\gamma$-positivity of the classical Eulerian polynomial:

$$A_{n-2}(t) = \frac{1}{2^{n-2}} \sum_{u \in S_{n-1}} (4t)^{\mathrm{pk}(u)}(1+t)^{n-2-2\,\mathrm{pk}(u)},$$

where $\mathrm{pk}(u) := |\{i \in [2, n-1] : u_{i-1} < u_i > u_{i+1}\}|$ is the number of peaks in $u$. Let $n > 2$. Now, for a permutation $v$ in $S_n$, we let $v' = v_2 \cdots v_n$ and write

$$n2^{n-1}tA_{n-2}(t) = 2t \sum_{v \in S_n} (4t)^{\mathrm{pk}(v')}(1+t)^{n-2-2\,\mathrm{pk}(v')}.$$

There is a bijection between those permutations $v$ for which $\phi(\overleftarrow{v}) = 1$ (i.e., $v_1 < v_2 < v_3$) and those $w$ for which $\phi(\overleftarrow{w}) = 0$ (i.e., $w_2 < w_1 < w_3$) given by switching the first two letters of the permutation. For such a pair, we will double the contribution of $w$ and ignore the contribution from $v$. Moreover, we have that if $\phi(\overleftarrow{v}) \neq 1$, then $\mathrm{pk}(v') = \mathrm{pk}^{\ell}(v) - 1$. Thus,

$$2t \sum_{v \in S_n} (4t)^{\mathrm{pk}(v')}(1+t)^{n-2-2\,\mathrm{pk}(v')} = 2t \sum_{v \in S_n} 2(1-\phi(\overleftarrow{v}))(4t)^{\mathrm{pk}^{\ell}(v)-1}(1+t)^{n-2-2(\mathrm{pk}^{\ell}(v)-1)}$$
$$= \sum_{v \in S_n} (1-\phi(\overleftarrow{v}))(4t)^{\mathrm{pk}^{\ell}(v)}(1+t)^{n-2\,\mathrm{pk}^{\ell}(v)}.$$

On the other hand, from the formulas (8) and (9) this sum is easily seen to be equal to $B_n(t) - D_n(t)$ and the proposition is proved. □

Now we turn to generating functions. We define the type B and D exponential generating functions (ordinary and affine) in the usual way:

$$B(t,z) := \sum_{n \geq 0} B_n(t) \frac{z^n}{n!}, \qquad \widetilde{B}(t,z) := \sum_{n \geq 0} \widetilde{B}_n(t) \frac{z^n}{n!},$$
$$D(t,z) := \sum_{n \geq 0} D_n(t) \frac{z^n}{n!}, \qquad \widetilde{D}(t,z) := \sum_{n \geq 0} \widetilde{D}_n(t) \frac{z^n}{n!},$$

where $B_0(t) = \widetilde{B}_0(t) = D_0(t) = \widetilde{D}_0(t) = 1$. We also have (from [2], for example), the following formulas for $B(t,z)$ and $D(t,z)$ (in fact, $D(t,z)$ follows from (11) once we know $B(t,z)$):

$$B(t,z) = \frac{(1-t)e^{z(1-t)}}{1-te^{2z(1-t)}}, \tag{12}$$

$$D(t,z) = B(t,z) - tzA(t,2z) = B(t,z) - z\widetilde{C}(t,z),$$
$$= \frac{(1-t)e^{z(1-t)}(1-tze^{z(1-t)})}{1-te^{2z(1-t)}}. \tag{13}$$

Our strategy for type B and D affine is to first find a formula for $\widetilde{B}(t,z)$, then use (10) and (13) to obtain $\widetilde{D}(t,z)$.

**Proposition 8.** *We have*

$$\widetilde{B}(t,z) = \frac{1-2t-2t(1-t)ze^{z(1-t)} + te^{2z(1-t)}}{1-te^{2z(1-t)}}. \tag{14}$$

*Proof.* The formula (14) follows from solving the following differential equation, (i.e., integrating),

$$\frac{d}{dz}[\widetilde{B}(t,z)] = 4tB(t,z)D(t,z) - 2t(1+(1-t)z)B(t,z), \tag{15}$$

with initial value $\widetilde{B}_0(t) = 1$ and the formulas (12) and (13).

We will prove (15) holds by comparing coefficients and proving, for $n \geq 1$, the following identity:

$$\widetilde{B}_n(t) = 4t \sum_{k=0}^{n-1} \binom{n-1}{k} B_{n-1-k}(t) D_k(t) - 2tB_{n-1}(t) - (n-1)2t(1-t)B_{n-2}(t). \tag{16}$$

The identity is easily verified for $n = 1, 2$. Let $n \geq 3$.

We begin by observing that each $u \in S_n$ is in the form $v \cdot n \cdot w$, where $w$ is a word of length $k$ and $v$ is a word of length $n-k-1$. Thus, every $u \in S_n$ can be obtained by deciding how many letters occur to the right of $n$, which of the letters in $[n-1]$ will occur to the right of $n$, and the ordering of the $k$ letters after $n$ and the $n-k-1$ letters before. In this way we can write

$$\widetilde{B}_n(t) = \sum_{k=0}^{n-1} \sum_{w \in S_k} \sum_{v \in S_{n-k-1}} \binom{n-1}{k} \phi(u)(4t)^{\overline{\mathrm{pk}}(u)}(1+t)^{n+1-2\,\overline{\mathrm{pk}}(u)}.$$

The total number of exterior peaks in the word $u$ is the number of left peaks in $v$, plus the number of right peaks in $w$, denoted $\mathrm{pk}^r(w) := \mathrm{pk}^\ell(\overleftarrow{w})$, plus one for the peak involving $n$. We may rewrite the formula as

$$\begin{aligned}\widetilde{B}_n(t) &= \sum_{k=1}^{n-1} \sum_{w \in S_k} \sum_{v \in S_{n-k-1}} \binom{n-1}{k} \phi(u)(4t)^{\mathrm{pk}^\ell(v)+\mathrm{pk}^r(w)+1}(1+t)^{n+1-2(\mathrm{pk}^\ell(v)+\mathrm{pk}^r(w)+1)}, \\ &= 4t \sum_{k=1}^{n-1} \sum_{w \in S_k} \sum_{v \in S_{n-1-k}} \\ &\qquad \binom{n-1}{k} \left((4t)^{\mathrm{pk}^\ell(v)}(1+t)^{n-1-k-2\,\mathrm{pk}^\ell(v)}\right)\left(\phi(u)(4t)^{\mathrm{pk}^r(w)}(1+t)^{k-2\,\mathrm{pk}^r(w)}\right),\end{aligned}$$

which equals

$$4t \sum_{k=1}^{n-1} \binom{n-1}{k} \left( \sum_{v \in S_{n-1-k}} (4t)^{\mathrm{pk}^\ell(v)}(1+t)^{n-1-k-2\,\mathrm{pk}^\ell(v)} \right)\left( \sum_{w \in S_k} \phi(u)(4t)^{\mathrm{pk}^\ell(\overleftarrow{w})}(1+t)^{k-2\,\mathrm{pk}^\ell(\overleftarrow{w})} \right).$$

The sum over $v \in S_{n-1-k}$ is precisely the formula (8) for $B_{n-1-k}(t)$. For $k \geq 3$ the value of $\phi(u)$ is the same as $\phi(w)$, and so in these cases the sum over $w \in S_k$ is exactly the formula (9) for $D_k(t)$ (with $w$ and $\overleftarrow{w}$ swapped). Thus $\widetilde{B}_n(t)$ is *almost*

$$4t \sum_{k=0}^{n-1} \binom{n-1}{k} B_{n-1-k}(t) D_k(t).$$

Now we individually examine the cases where $k < 3$. For $k = 2$, we choose the two letters for $w = u_{n-1}u_n$ in any of $\binom{n-1}{2}$ ways. In terms of the larger permutation $u \in S_n$, we will have $u_{n-2} = n$, and either $u_n < u_{n-1}$, or $u_n > u_{n-1}$. The second permutation contributes nothing as $\phi(u) = 0$ in that case. The other permutation will have $\phi(u) = 1$ and the reversed permutation will have no left peaks, so it contributes $(1+t)^2$. Luckily, $D_2(t) = (1+t)^2$, so the sum is correct for $k \geq 2$.

For $k = 1$, in terms of the larger word $u \in S_n$, we will have $u_{n-1} = n$, and thus $\phi(u) = \frac{1}{2}$. There are $(n-1)$ ways to choose the letter appearing to the right of $n$, none of which change the total weight from $u$: $(n-1)2t(1+t)B_{n-2}(t) = (n-1)4tB_{n-2}(t) - (n-1)2t(1-t)B_{n-2}(t)$.

Finally, for $k = 0$, we will have that $u_n = n$, implying $\phi(u) = \frac{1}{2}$. This summand comes out to be $2tB_{n-1}(t) = 4tB_{n-1}(t) - 2tB_{n-1}(t)$. Thus (16), and hence the proposition, is proved. □

Now we use the formula (14) for $\widetilde{B}(t, z)$ along with the identity (10) and the formula (13) for $D(t, z)$ to obtain $\widetilde{D}(t, z)$.

**Proposition 9.** *We have*

$$\widetilde{D}(t,z) = \frac{1-2t+z+t(1-t)z^2-4t(1-t)ze^{z(1-t)}+t(1-z+z^2t(1-t))e^{2z(1-t)}}{1-te^{2z(1-t)}}. \tag{17}$$

3.6. **Real-rootedness conjecture.** One benefit of having an explicit generating function for the polynomials we've studied is that they are rapidly computed.

**Conjecture 1.** *All the (affine and non-affine) Eulerian polynomials have all real zeroes.*

The only open cases for this conjecture are $D_n(t)$, $\widetilde{B}(t)$, and $\widetilde{D}(t)$. The conjecture has confirmed for $D_n(t)$ up to $n = 350$, and for $\widetilde{B}(t)$ and $\widetilde{D}(t)$ up to $n = 500$.

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