Momentum Matching for the Pendulum on Cart Problem

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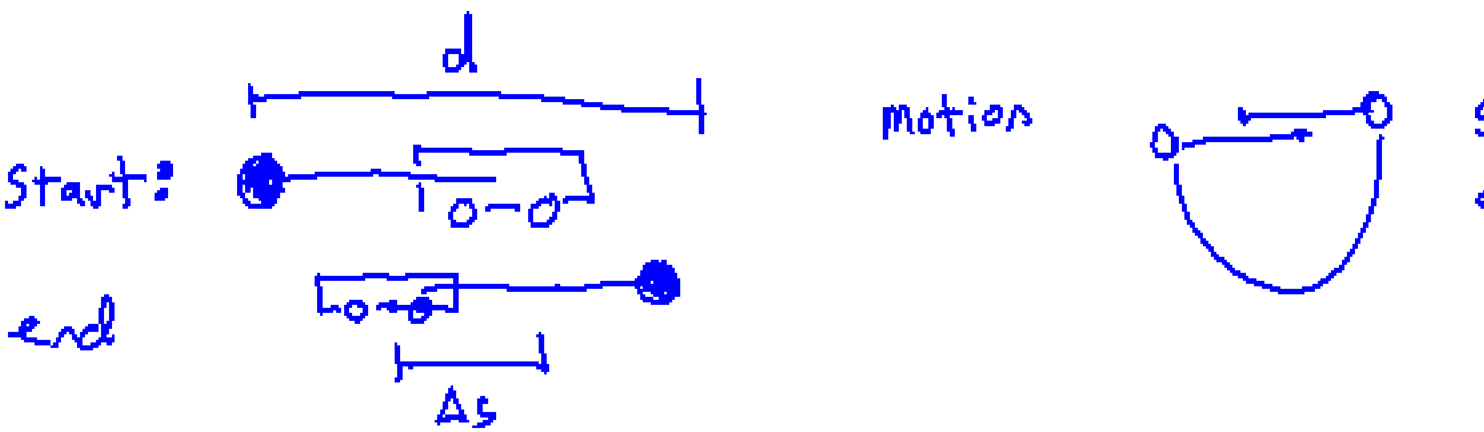
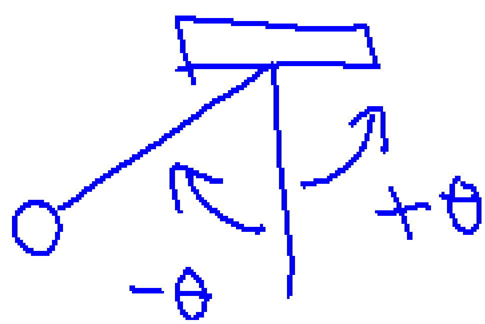
1 Problem Description



There is a pendulum with mass m_1 that is connected with a cart of mass m_2. The object is to move the cart a distance ΔS,



when the pendulum position moves from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$. The coordinate system that will be used is:

1.1 Problem Setup

We first need to set up the equations of motion for this problem. So we need to set up the lagrangian of the system. The lagrangian of a general system is: $KE - U = L$

And for our system:

$$L = \frac{1}{2}\dot{\theta}^2 l^2 m_2 + \dot{s}\dot{\theta} l \cos\theta m_2 + \frac{1}{2}\dot{s}^2 (m_1 + m_2) - m_2 lg\cos(\theta). \tag{1}$$

In order to acquire the equations of motion:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial s} = 0$$

2 Controller

In order to meet our goal, there are two possible controls that could be possible: controlling the pendulum arm or controlling the cart.

2.1 Pendulum Arm Controller

If we were to control the pendulum arm:

$$\ddot{\theta} l^2 m_2 + \ddot{s}\cos\theta m_2 + g\sin\theta m_2 = u \tag{2}$$

$$\ddot{\theta} l^2 \sin\theta m_2 + \ddot{s}(m_1 + m_2) = 0 \tag{3}$$

However due to the conservation of linear momentum, no matter what controls are applied to the pendulum arm or whatever path θ makes, if $\theta(T)$ends up at $\frac{\pi}{2}$ at some point T, the displacement of s is fixed. Assuming that the system begins from rest:

$$\dot{s} = \frac{-lm_2}{m_1 + m_2}\cos(\theta)\dot{\theta} = \frac{-lm_2}{m_1 + m_2}\frac{d}{dt}\sin(\theta(t)) \tag{4}$$

So if $\theta(0) = -\frac{\pi}{2}$, then the $\Delta s = -2\ \frac{-lm_2}{m_1 + m_2}$.

2.2 Controlling the Cart

Therefore,our only option is to control the cart.

$$\ddot{\theta} l^2 m_2 + \ddot{s} l\cos(\theta) m_2 + gl\sin(\theta) m_2 = 0 \tag{5}$$

$$\ddot{\theta}\cos(\theta) m_2 - d\theta^2 l\sin(\theta) m_2 + \ddot{s}(m_1 + m_2) = u \tag{6}$$

2.3 Momentum Matching

Based on the behavior of the uncontrolled system, we know that the system displaces by $-2\ \frac{m_2 l}{m_1 + m2}$, and if we substitute $\beta = m_2 l$ and $\gamma = m_1 + m_2$, the uncontrolled system displaces by $\frac{\beta}{\gamma}$. This is because of the conservation of momentum equation. So, to consider whether or not a control exists, so that the system conserves a different linear momentum equation, we need to find a control u such that the momentum is conserved:

$$\dot{\theta}\beta_n \cos(\theta) + \dot{s}\gamma_n = C$$

. with C being a constant.

This has a time derivative of:

$$\ddot{\theta}\cos(\theta)\beta_n - \dot{\theta}^2\sin(\theta)\beta_n + \ddot{s}\gamma_n = 0 \tag{7}$$

Using the controlled equations above:

$$\ddot{\theta} + \frac{\dot{\theta}^2\cos(\theta)\sin(\theta)m_2}{m_1 + \sin(\theta)^2 m_2} + \frac{g\sin(\theta)\gamma}{m_1 l + \sin(\theta)^2\beta} = \frac{u\cos(\theta)}{lm_1 + l\sin(\theta)^2} \tag{8}$$

$$\ddot{s} - \frac{\dot{\theta}^2 l\sin(\theta)m_2}{m_1 + \sin(\theta)^2 m_2} - \frac{g\cos(\theta)\sin(\theta)m_2}{m_1 + \sin(\theta)^2 m_2} = \frac{u}{m_1 + \sin(\theta)^2 m_2} \tag{9}$$

To replace the accelerations one sees a control of the form

$$u = u_0(\theta) + u_1\dot{\theta}^2 \tag{10}$$

Solving for u_0 and $u_1(\theta)$ will result in:

$$u_1 = \frac{l\sin(\theta)(\kappa m_1 + (-l+\kappa)m_2)}{l - \kappa\cos(\theta)^2} \tag{11}$$

$$u_0 = \frac{g\sin(2\theta)(lm_2 - \kappa(m_1 m_2))}{-2l + \kappa + \kappa\cos(2\theta)} \tag{12}$$

For $\kappa < 1$, the denominators are nonzero and the control is bounded. Using this controller, the equations become:

$$\ddot{\theta} = \frac{2(g + d\theta^2\kappa\cos(\theta)\sin(\theta)}{-2l + \kappa + \kappa\cos(2\theta)} \tag{13}$$

$$\ddot{s} = -\frac{2\kappa(d\theta^2 l + g\cos(\theta))\sin(\theta)}{-2l + \kappa + \kappa\cos(2\theta)} \tag{14}$$

If the pendulum arm goes from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$,then the cart will displace a distance of 2κ to $\frac{\pi}{2}$, or in other words,if the initial conditions are as such: $\theta(0) = \frac{-\pi}{2}$; $\dot{\theta}(0) = 0$; $\dot{s}(0) = 0$; there must be a time τ, when $\theta(\tau) = \frac{\pi}{2}$; $\dot{\theta} = \frac{\pi}{2}$;$\dot{s} = 0$; So now, we will use a concept called time-reversal symmetry and symmetry over θ, where $\theta = -\theta$. The concept of time-reversal symmetry means that a system will follow the same path regardless of whether time is positive or negative. Using this, we will show that if the pendulum moves from $\frac{-\pi}{2}$ to 0. Then it implies that the pendulum can also move from 0 to $\frac{-\pi}{2}$. The second symmetry will imply that if the pendulum moves from $\frac{-\pi}{2}$ to 0 then it will also move from $\frac{\pi}{2}$ to 0, thus it with these two symmetries combined, we will show that the pendulum will move from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$

Proof 1 *: For a system to have time-reversal symmetry, there must exist a time T when $\theta(T) = 0$, and it is invariant under the transformations $t -> -t$, and $v -> -v$. Our system is:*

$$\dot{\theta} = V \tag{15}$$

$$\dot{V} = \frac{2(g + V^2\kappa\cos(\theta))\sin(\theta)}{-2l + \kappa + \kappa\cos(2\theta)} \tag{16}$$

a) If we make a change from $t -> -t$ then we must show that if $(\theta(t), V(t))$ is a solution, then so must $(\theta(-t), -V(-t))$. So then

$$\frac{\partial}{\partial t}(\theta(-t), -V(-t)) = (-\dot{\theta}(-t), \dot{V}(-t))$$

Now, if substitute these values into the equation above:

$$-\dot{\theta}(-t) = -V(-t) \tag{17}$$

$$\dot{V}(-t) = \frac{2(g + (-V(-t))^2\kappa\cos(\theta(-t))\sin(\theta(-t))}{-2l + \kappa + \kappa\cos(2\theta(-t))} \tag{18}$$

Because of the nature of the pendulum that can be seen by the above equation, $\theta(t) = \theta(-t)$ *and also in the equation above,* $(-V(-t))$*is squared. Since the* $y = x^2$ *is an even function, The equation becomes:*

$$-\dot{\theta}(-t) = -V(-t) \tag{19}$$

$$\dot{V}(-t) = \frac{2(g + (V(-t))^2\kappa\cos(\theta(t))\sin(\theta(t))}{-2l + \kappa + \kappa\cos(2\theta(t))} \tag{20}$$

So with the time reversal symmetry, if $\theta(\tau) = 0$ *for some time* τ*, when reversed, the system will be invariant under the transformation* $t- > -t$*. It will simply move in the opposite direction, while the acceleration remains invariant.*

b) If the system were to be transformed from $\theta(t) = -\theta(t)$*, then we can deduce that if the pendulum takes the path from* $\frac{-\pi}{2}$ *to* 0*. That it will take the path from* 0 *to* $\frac{\pi}{2}$ *then* $-\dot{\theta} = -V(t)$*. If we replace these in the equation above we get:*

$$-\dot{\theta} = -V \tag{21}$$

$$\dot{V}(t) = \frac{2(g + (-V)^2\kappa\cos(\theta(t))sin(\theta(t))}{-2l + \kappa + \kappa\cos(2\theta(t))} \tag{22}$$

The acceleration remains the same because $V^2 = (-V)^2$*. This shows that if* θ *goes from* $\frac{-\pi}{2}$ *to* 0 *then it will also go from* 0 *to* $\frac{\pi}{2}$

c) Now, we must show that there is at time T *such that* $\theta(T) = 0$*. If you look at the acceleration equations:*

$$\ddot{\theta} = \frac{2(g + \dot{\theta}^2\kappa\cos(\theta)sin(\theta)}{-2l + \kappa + \kappa\cos(2\theta)} \tag{23}$$

$$\ddot{s} = -\frac{2\kappa(d\theta^2 l + g\cos(\theta))sin(\theta)}{-2l + \kappa + \kappa\cos(2\theta)} \tag{24}$$

at $T = 0$ $\ddot{\theta} = 0$*. Now, as T goes from* $\frac{-\pi}{2}$ *to* 0*, the acceleration continues to remain positive. So, by deduction, it can be seen that if a system starts at rest and it continues to have positive acceleration until* $t = 0$*, then it must pass through that point. Therefore by using time reversal symmetry, we have shown that momentum matching is a viable solution, when the pendulum goes from* $-\pi/2$ *to* $\pi/2$*.*

3 Future Problems

We then tried to see if there was a way using this symmetry to get the pendulum to move from $-\pi/2$ to $\pi/2 + k$ although there is symmetry about any two points$K$ and $-K$. The acceleration does not accelerate more than at point $K+-\frac{\pi}{2}$, therefore, we must try to match other functions onto u_1 andu_0.