Schur Positivity Conjectures about Temperley Lieb functions on Jacobi Trudi matrices

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Research conducted under the supervision of Mark Skandera at the University of Michigan, May-August, 2003[1]

Abstract

An overview of REU undergraduate research conducted during the summer of 2003 at the University of Michigan, 2003, under the supervision of Professor Mark Skandera. In particular, we cover the foundation of material up to Skandera's conjecture about the Schur-positivity of "Temperley Lieb functions", which count particular 2-coverable paths in a planar network whose path matrix is a $s_{\lambda,\mu}$ Jacobi-Trudi matrix. Additional observations of Schur-positivity involving products of elements of the Jacobi-Trudi matrix are included.

[1]Supported by the NSF

Contents

1 Introduction

Because the introductory material for this research closely follows [4], most initial notation and terminology used here will be identical to that in [4]; we shall provide a summary of relevant material, though it is recommended that, if the reader is interested in furthering this research, s/he refer to that paper for a slower, more descriptive, and more gentle introduction.

1.1 Networks and Nonintersecting Path Families

1.1.1 The Planar Network G

Let G be a planar network of order n with nonnegative weighted edges and sources and sinks corresponding to an index set I,

$$S_I = \{s_i | i \in I\},$$

$$T_I = \{t_i | i \in I\}$$

Where $I = \{1, \ldots, n\}$. In particular, we shall consider only networks with no crossing edges, and in which the network may be drawn with sources and sinks on the boundary and in the order, counting counterclockwise, $(s_1, \ldots, s_n, t_n, \ldots, t_1)$. We will by convention draw sources on the left and sinks on the right and assume there exist no loops in the graph. Let A be its path matrix, i.e. an $n \times n$ matrix whose i, j entry is the sum of weighted paths from source i to sink j; a weighted path is the product of edge weights of all edges in the path. Let $\Delta_{K,K'}$ denote the K, K' minor of A when $K, K' \subseteq I$. We shall only consider cases where $|K| = |K'|$.

First, recall the definition of determinant on the matrix $[x]_{i,j}$:

$$\sum_{\pi \in S_n} (-1)^{\text{INV}(\pi)} \underbrace{x_{1,\pi(1)} \ldots x_{n,\pi(n)}}_{p} \tag{1}$$

Observation 1 *$|A|$ is the sum of weights of all pairwise-disjoint path families with paths respectively from source i to sink i for $i = 1 \ldots n$.*

More strongly, we have the following observation from [4]

Observation 2 *The product $\Delta_{K,K'} \Delta_{\overline{K},\overline{K'}}$ of minors of A is equal to the weighted sum of all path families $\pi = (\pi_1, \ldots, \pi_n)$ in G with the following properties.*

1. *Each path connects a source in S_K to $T_{K'}$ or a source in $S_{\overline{K}}$ to a sink in $T_{\overline{K'}}$.*

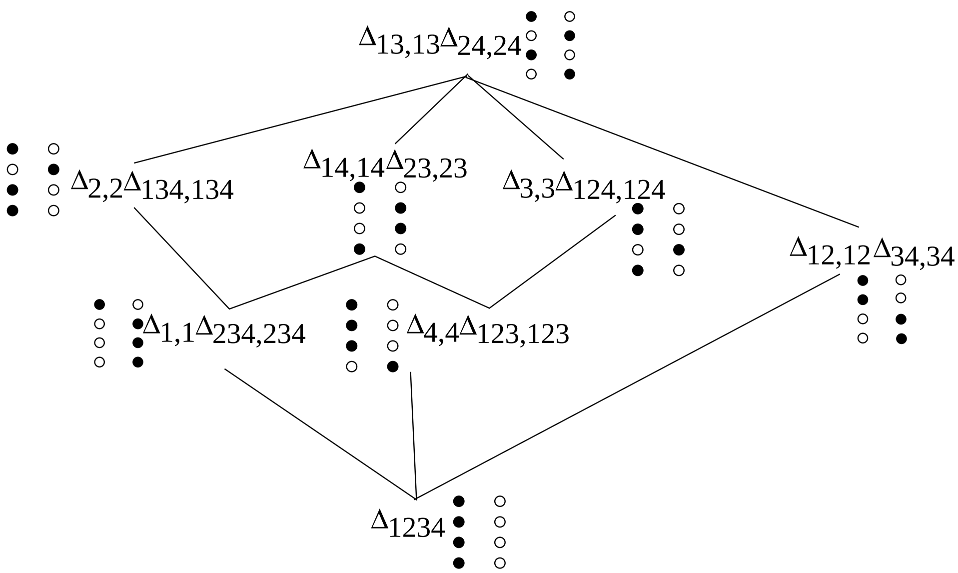


Figure 1: A poset in which products of complementary minors are ordered by nonnegative differences in TNN matrices. Beside each minor is also its corresponding quasi dot diagram.

2. *The paths from S_K to $T_{K'}$ are pairwise vertex disjoint, as are the paths from $S_{\overline{K}}$ to $T_{\overline{K'}}$.*

The pair of conditions above is called a *binary crossing rule.*

We shall also refer to a family of paths satisfying a binary crossing rule as a binary crossing family. A planar network is said to be *2-coverable* if can be covered by a binary crossing but *2-colorable* if there exists a subnetwork of the graph which is 2-coverable; note the distinction. We can denote the covered subgraph corresponding to a binary crossing π as H_π.

It is obvious then that path matrices are TNN; this fact is known as Lindström's Lemma. specific differences of products of nonprinciple minors turn out to be nonnegative in all TNN matrices; we are interested in these differences. Figure 1 shows a poset in which comparable elements have a nonnegative difference. These nonnegativity relations shall become significant in the next section.

1.1.2 Matrix minors and dot diagrams

A convenient visual representation of products of matrix minors is the dot diagram $\mathcal{D}_{K,K'}$.

Definition The *dot diagram* $\mathcal{D}_{K,K'}$ of order n is a pair of columns, each with n black and white vertices or dots, drawn side by side, and such that:

1. In the *left* column, counting down, the i^{th} dot is colored *black* iff $i \in K$.
2. In the *right* column, counting down, the i^{th} dot is colored *black* iff $i \in K'$

We shall also say that a *quasi dot diagram* of order n is the dot diagram $\mathcal{D}_{K,K'}$ with the dots in the right column switched in color. The significance of the quasi dot diagram is primarily its convenience when finding special subsets containing Temperley Lieb functions. Figure 1 shows the quasi dot diagrams corresponding to minor products beside each product.

1.2 Temperley Lieb elements and Temperley Lieb functions

In this section we introduce the Temperley Lieb elements and their meaning in our work.

1.2.1 Skandera's $\psi(G)$: Partitioning 2-colorable networks with Temperley Lieb elements

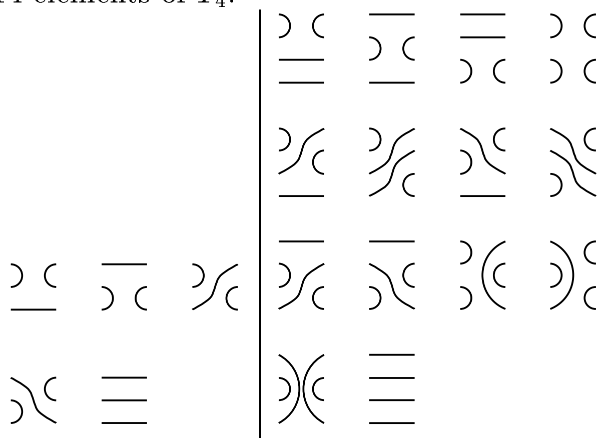
We represent the Temperley Lieb elements as a pair of columns of n vertices with n nonintersecting edges or arcs drawn within the columns and in which each vertex has exactly one neighbor. Figure 2 shows the 5 elements of T_3 and the 14 elements of T_4 represented in this way.

We shall speak of a Temperley Lieb element $r \in T_n$ and an element in $\mathcal{G}_3(2n)$ as the same thing if, when the element in $\mathcal{G}_3(2n)$ has its vertices arranged in two columns of n elements with vertices $\{1, \ldots, 2n\}$ arranged counterclockwise from the top left, all corresponding vertices in the representation are neighbors.

[4] describes a function $\psi : \mathcal{G}_1(n) \longrightarrow \mathcal{G}_3(2n)$ where $\mathcal{G}_1(n)$ is the set of all order n 2-colorable planar networks. $\mathcal{G}_3(2n)$ is the family of graphs on $2n$ vertices labeled $1, \ldots, 2n$, with n noncrossing edges such that each vertex is the endpoint of exactly one edge, given that the vertices $1, \ldots, 2n$ are arranged in increasing order on a straight line and all edges fall on one side of the line. In particular, the distinct elements of $\mathcal{G}_3(2n)$ are in a 1-1 correspondence with the $\frac{1}{n+1}\binom{2n}{n}$ elements of the Temperley-Lieb algebra T_n.

$\psi(G)$ maps a 2-colorable network to an element in $\mathcal{G}_3(2n)$ in the following manner: assuming that G *is* covored by a binary crossing family, an edge exists between vertices i and j in $\psi(g)$ iff knowledge of the color of the i^{th} vertex of G, counting counterclockwise from s_1, implies knowledge of the color of the j^{th} vertex of G, again counting counterclockwise from s_1. See Figure 3 for an example.

Figure 2: Left: the 5 elements of the Temperley Lieb algebra T_3. Right: the 14 elements of T_4.



It can be observed that the path of implication changes direction at any vertex having multiple paths on the side of the incoming path; the path 'bounces off' the vertex in such situations. Because the path of implication follows a zig-zag pattern left and right, the researchers have referred to performing the function as a zig-zag analysis.

ψ^{-1} then is a map which partitions the set of 2-colorable networks corresponding to each Temperley-Lieb element.

Observation 3 *The zig-zag analysis yields the following properties of Temperley Lieb elements.*

1. *If vertices i, j are connected by an edge in $\psi(G)$ and lie on the same side, they must be different colors in G.*

2. *If vertices i, j are connected by an edge in $\psi(G)$ and lie on different sides, they must be the same color in G.*

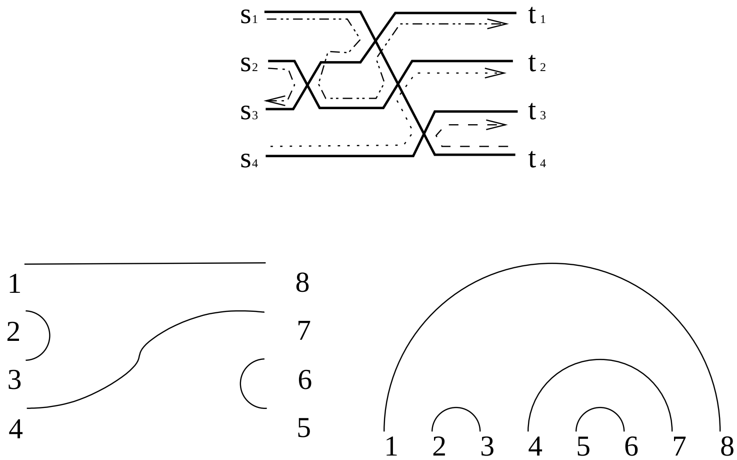
Recalling the dot diagrams, it is important to note here the following points:

- Each particular dot diagram has associated with it a particular set of Temperley Lieb elements such that observation 3 is satisfied.

- Each quasi dot diagram has associated with it all Temperley Lieb elements (i.e. as associated graphs in $\mathcal{G}_3(2n)$ such that each edge of the element connects dots of opposite color.

Immediately we formulate the following definition.

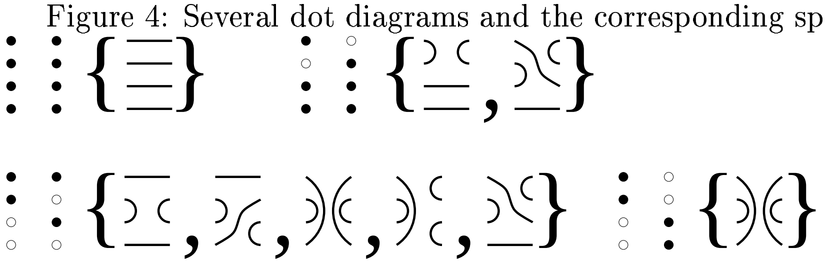
Figure 3: Top: a basic planar network G with arrows of color implication; it is possible, given each start vertex, to deduce the color of the respective end vertex. Bottom: $\psi(G)$, drawn two ways.



Definition Let $K, K' \subseteq \{1, \ldots, n\}$ with $|K| = |K'|$. Then we say that the K, K' *special subset* $T_{n,K,K'}$ is the subset of T_n such that for all $r \in T_{n,K,K'}$, r satisfies observation 3.

Typically we use a dot diagram to represent the complementary minors; see Figure 4 for several examples. It is here that quasi dot diagrams come in handy. If one were to use quasi dot diagrams, the special subset contains precisely those Temperley Lieb elements with edges joining only dots of opposite color; as can be seen from Figure 4, identifying which elements are in the special subset from the dot diagram alone may be somewhat irritating.

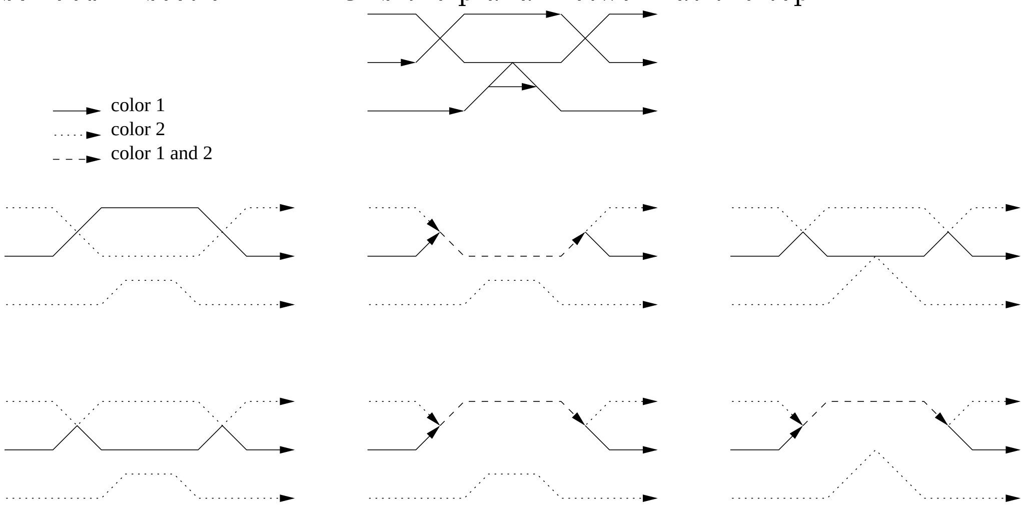
Figure 4: Several dot diagrams and the corresponding special subsets.



1.2.2 The Temperley Lieb Functions

We now define the Temperley Lieb function $r(G)$.

Figure 5: The 6 path families counted by the Temperley Lieb function described in section 1.2.2. G is the planar network at the top.



Definition Let G be a planar network of order n. Let $r \in T_n$. Let $K, K' \subseteq \{1, \ldots, n\}$ be such that observation 3 is satisfied for r. The *Temperley Lieb function* $r(G)$ is the sum of weights of all binary crossing families $\pi = (\pi_1, \ldots, \pi_n)$ which satisfy $\psi(H_\pi) = r$

In other words, $r(G)$ counts the number of *unique* binary crossings of the form r; see figure 5 for an example of (G).

The number of binary coverings of a subnetwork H is in fact 2 to the power of the number of loops generated by Skandera's ϕ cited in [4]; we shall not elaborate upon this loop counting more, but the reader should be aware of this fact.

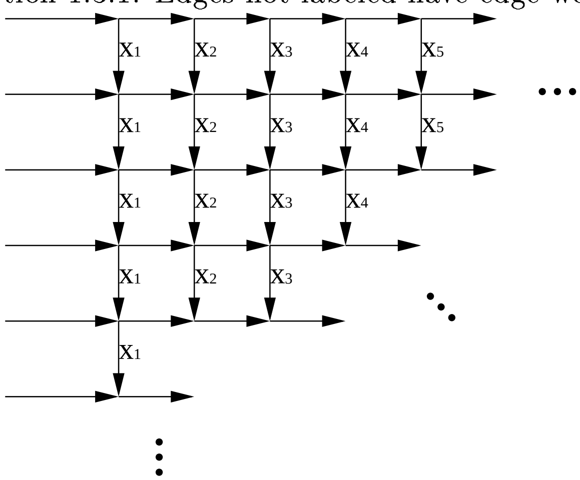
Observation 4 *Let $K, K' \subseteq \{1, \ldots, n\}$. Then the product of minors $\Delta_{K,K'}\Delta_{\overline{K},\overline{K'}}$ is equal to the sum of the temperley Lieb functions corresponding to Temperley Lieb elements in the K, K' special subset.*

Section 2.1 elaborates more upon the actual methods of calculating these functions.

1.3 Schur Positivity and the significance of the Temperley Lieb functions

This section assumes basic familiarity with symmetric functions and exposure to Schur functions before; for a good introduction, see [5].

Figure 6: The planar network $\mathcal{H}$ having as its path matrix H from section 1.3.1. Edges not labeled have edge weight 1.



1.3.1 Schur Function definition and Schur path families

We use the Jacobi-Trudi definition of Schur functions in this paper. First consider the infinite Toeplitz matrix containing the homogeneous symmetric functions as entries:

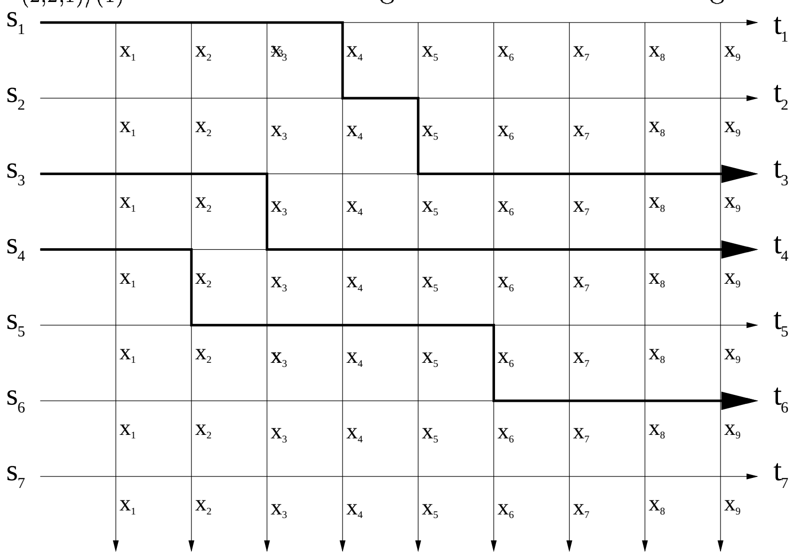
$$H = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & \dots \\ 1 & h_1 & h_2 & h_3 & h_4 & \dots \\ 0 & 1 & h_1 & h_2 & h_3 & \dots \\ 0 & 0 & 1 & h_1 & h_2 & \dots \\ 0 & 0 & 0 & 1 & h_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

as the path matrix of the planar network shown in Figure 6. We shall refer to this planar network as $\mathcal{H}$. Let $\lambda \vdash (m+n)$ and $\mu \vdash m$ with $\lambda_i \leq \mu_i$ for all i. Recall that the λ/μ Jacobi-Trudi matrix J is the $n \times n$ matrix whose i, j entry is the homogeneous symmetric function $h_{\lambda_i - \mu_i + j - i}$:

$$J = \begin{bmatrix} h_{\lambda_1-\mu_1} & h_{1+\lambda_1-\mu_2} & h_{2+\lambda_1-\mu_3} & \dots & h_{n-1+\lambda_1-\mu_n} \\ h_{-1+\lambda_2-\mu_1} & h_{\lambda_2-\mu_2} & h_{1+\lambda_2-\mu_3} & \dots & h_{n-2+\lambda_2-\mu_m} \\ h_{-2+\lambda_3-\mu_1} & h_{-1+\lambda_3-\mu_2} & h_{\lambda_3-\mu_3} & \dots & h_{n-3+\lambda_3-\mu_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_{2-n+\lambda_{n-1}-\mu_1} & h_{3-n+\lambda_{n-1}-\mu_2} & h_{4-n+\lambda_{n-1}-\mu_3} & \dots & h_{1+\lambda_{n-1}-\mu_n} \\ h_{1-n+\lambda_n-\mu_1} & h_{2-n+\lambda_n-\mu_2} & h_{3-n+\lambda_n-\mu_3} & \dots & h_{\lambda_n-\mu_n} \end{bmatrix}$$

In other words, J is a submatrix of the Toeplitz matrix H. We then define the Schur function $s_{\lambda/\mu}$ to be the determinant of J; thus $s_{\lambda/\mu}$ is the weighted

Figure 7: A set of nonintersecting path families counted by the Schur function $s_{(2,2,1)/(1)}$ in 9 variables. Edges not labeled have weight 1.



sum of non-intersecting path families as illustrated in Figure 7. The planar network shown in this figure shall be denoted $\mathcal{H}_{\lambda/\mu}$.

1.3.2 Temperley Lieb functions on the Jacobi-Trudi matrix

Schurly, then, if we have said this much to get this far, there must be significance to the Temperley Lieb functions.

Conjecture 1 *Let $r \in T_n$. Let $s_{\lambda/\mu}$ be a Schur function, and let $\mathcal{H}_{\lambda/\mu}$ be the associated planar network. Then r is Schur-positive (throughout this paper we shall use the term Schur positive whether the value is zero or nonnegative).*

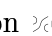
This conjecture would imply that, considering observation 4, the nonnegative differences between comparable elements of the poset in Figure 1 also corresponds to Schur-positive differences when the difference of products of minors is considered over *any* $\mathcal{H}_{\lambda/\mu}$, opening the door to a slew of Schur-positive polynomials. The poset in Figure 1 contains only principle minor products, only a small sliver of the infinitely many possible products. [5, Qu. 2.9.1]

2 Calculation of the Temperley Lieb functions

The original, and primary, calculation of the Temperley Lieb functions is based on isolation of Temperley Lieb structures via special subset union and difference. The secondary calculation is based on a recursive formula in the Bruhat order discovered by Rhoades. We shall illustrate the first by example, considering more complex cases as we move along, and then provide the algorithm.

Let G be a planar network of order 4. First consider (G). The only dot diagram having as a special subset  is $\mathcal{D}_{\{1,2,3,4\},\{1,2,3,4\}}$, or . The $\mathcal{D}_{\{1,2,3,4\},\{1,2,3,4\}}$ minor is, of course, the determinant of the path matrix, and therefore counts all vertex-disjoint path families; it is precisely (G).

Next consider (G). The Temperley Lieb element  is the sole element in the special subset of $\mathcal{D}_{\{2,3,4\},\{1,2,3\}}$, or . Accordingly, *only* and *all* paths of the form  are counted by taking the product of minors $\Delta_{1,4}\Delta_{234,123}$

Although the above two calculations are relatively simple, calculation of Temperley Lieb functions in general is somewhat more difficult; for example, the function (G) cannot be simply calculated by taking a product of complementary minors because there is no dot diagram having  as a sole element in its special subset.

2.1 Isolation via linear combinations of 2-colorable path families

2.1.1 Dot diagram interpretation

Each product of complementary minors $\Delta_{K,K'}\Delta_{\overline{K},\overline{K'}}$ (i.e. each dot diagram $\mathcal{D}_{K,K'}$) counts of course all possible nonintersecting paths from the subset of the sources indexed by K to the subset of sinks indexed by K', and likewise for the complements. Again, as noted in observation 4, we have the important fact that each product counts the sum of the Temperley Lieb functions corresponding to each element found in the K, K' special subset $T_{n,K,K'}$:

$$\Delta_{K,K'}\Delta_{\overline{K},\overline{K'}} = \sum_{r \in T_{n,K,K'}} r(G) \tag{2}$$

where G is a planar network having the path matrix on which $\Delta_{K,K'}\Delta_{\overline{K},\overline{K'}}$ acts.

2.1.2 Linear Combinations of Minors

The fact that (2) holds true lends itself to the possibility of isolating certain functions by means of taking linear combinations of products of minors; and as expected, this works. As Pavlov discovered in [2], even though there is not necessarily a unique linear combination, the Temperley Lieb coefficients must be unique. As later found by [3], there necessarily exists a linear combination of minors.

Consider, as an example of Temperley Lieb isolation, the function (G). is not the sole element of any special subset. However, we can take $\mathcal{D}_{1,2}$ ( ), with special subset { , }, and subtract from it $\mathcal{D}_{1,1}$ ( ), with special subset { }.

Conveniently, the rotation equivalence classes of dot diagrams can come in handy when isolating Temperley Lieb elements. If one element is known to be some linear combination of dot diagrams, the clockwise (or counter-clockwise) rotation of that Temperley Lieb element by k dots is equal to a linear combination, with respectively the same coefficients, of the rotations by k dots of the dot diagrams. That is, denoting the rotation of a Temperley Lieb element r or a dot diagram $\mathcal{D}_{K,K'}$ by k dots $\circlearrowright(r, k)$ or $\circlearrowright(\mathcal{D}_{K,K'}, k)$, respectively,

$$r(G) = \sum a_{\mathcal{D}_{K,K'}} \mathcal{D}_{K,K'}$$

implies

$$\circlearrowright(r(G)) = \sum a_{\mathcal{D}_{K,K'}} \circlearrowright(\mathcal{D}_{K,K'})$$

for all $k \in \{1, 2, \ldots\}$

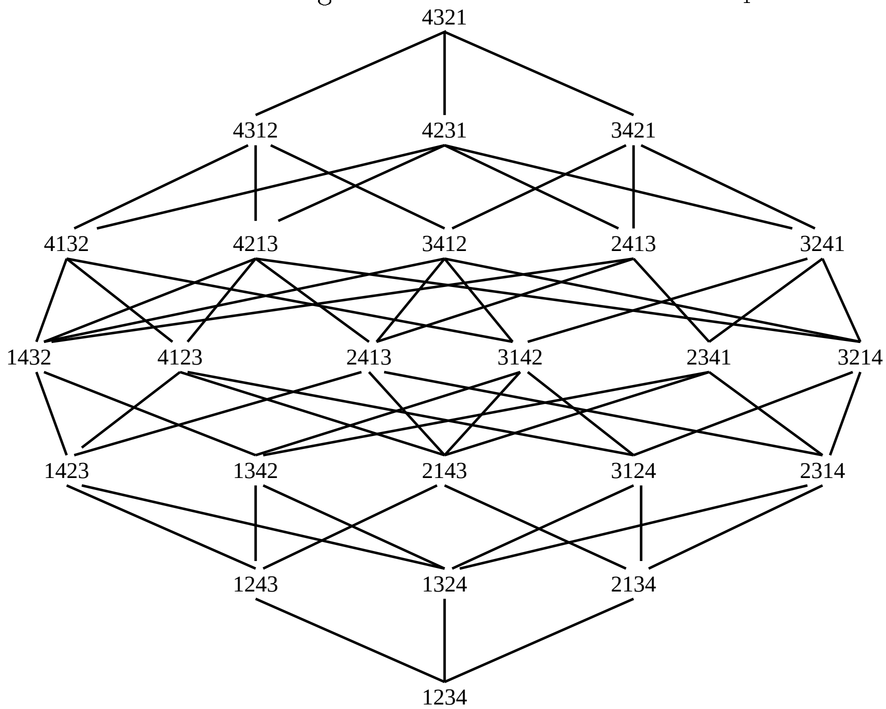
2.2 The Bruhat Order

The Bruhat order is a partial order on all permutations of the set $\{1, \ldots, n\}$. In particular, it is graded; and $i = \{i_1, \ldots, i_n\}$ covers $j = \{j_1, \ldots, j_n\}$ iff there are more inversions in i and there exists a transposition τ such that $i = \tau j$. For clarity, Figure 8 shows the Bruhat order on a permutation representation of S_4.

2.2.1 Temperley Lieb significance

The Bruhat order becomes significant when terms of the form p in equation (1) are considered as elements of the Bruhat order; in general, given a product $x_{1,\pi(1)} \ldots x_{n,\pi(n)}$, we shall look at the element $\{\pi(1), \ldots, \pi(n)\}$ within the context of the Bruhat order and associate with it the coefficient of $x_{1,\pi(1)} \ldots x_{n,\pi(n)}$; these two terms $x_{1,\pi(1)} \ldots x_{n,\pi(n)}$ and $\{\pi(1), \ldots, \pi(n)\}$ may

Figure 8: The Bruhat order on S_4.



be used interchangeably, and the coefficient shall be called the *Temperley Lieb coefficient*.

The Temperley Lieb functions are linear combinations of terms of the form $x_{1,\pi(1)} \dots x_{n,\pi(n)}$ because they are linear combinations of products of complementary minors. More significantly, we find that, associated with each Temperley Lieb function, there is a particular permutation ρ with Temperley Lieb coefficient 1 such that for any $\sigma \not\geq \rho$ by the Bruhat order, σ has a Temperley Lieb coefficient 0.

2.2.2 Criteria for Comparability in the Bruhat Order

We stray for a moment here to cover an important lemma that comes up later; for continuity's sake, the reader may wish to skip this section and refer to it upon reaching section 3.2.

First, let $\pi = (\pi_1, \dots, \pi_n) \in S_n$. We can create a matrix $M_\pi = [a_]i, j$ with

$$a_{i,j} = \begin{cases} 1 & \text{if } \pi_i = j \\ 0 & \text{otherwise} \end{cases}$$

For example,

$$M_{(1,3,4,2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

We now present an important well-known result shared with us by Skandera.

Lemma 1 *Let $\pi, \rho \in S_n$. Then π is incomparable to ρ by the Bruhat order if there exist indices $i_\pi, j_\pi, i_\rho, j_\rho \in \{1, \dots, n\}$ such that*

- *There are more 1's* to the right of and above a_{i_π, j_π} *in M_π than in M_ρ*
- *There are more 1's* to the right of and above a_{i_ρ, j_ρ} *in M_ρ than in M_π*

Equivalently, the differing number of 1's can be found *to the left of and below*, *to the left of and above*, or *to the right of and below*.

2.3 A recursive formula via reduced words in the Bruhat order

We shall provide the recursive algorithm given and justified in [3] in this section.

2.3.1 Canonical Reduced Words and Reduced Graphs

Let $\pi \in S_n$ have m inversions. Let s_i denote the adjacent transposition $(i\; i+1)$, in cylic notation. Then there exists a unique product of m adjacent transpositions s_i of the form

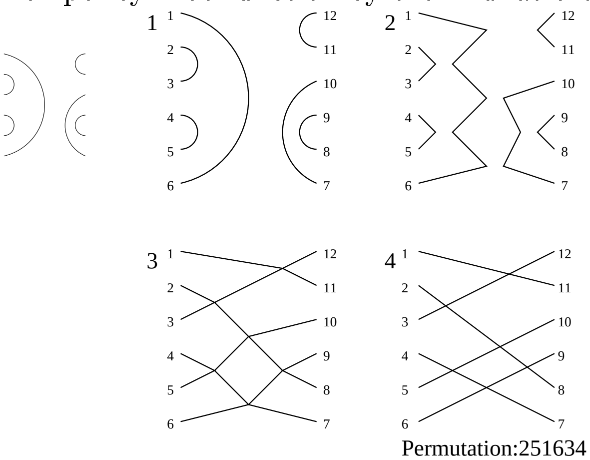
$$(s_{a_1} s_{a_1+1} \dots s_{a_1+b_1}) \dots (s_{a_n} s_{a_n+1} \dots s_{a_n+b_n}) \tag{3}$$

with $a_{i+1} > a_i$ for all i. [**?**]

Curiously enough, these reduced words are unique if and only if the permutation is the lowest permutation with coefficient 1 observed in the previous section; these permutations are the 321-avoiding ones. There is a procedure for finding this associated permutation, given an element $r \in T_n$ (See Figure 9 for an example of this process.):

1. Take the graph $R \in \mathcal{G}_3$ corresponding to r.

Figure 9: The procedure for finding the lowest nonzero permutation of a Temperley Lieb function by the Bruhat order.



2. For each edge in R, given that its endpoints have labels i, j, partition the edge into, without loss of generality,

$$\begin{cases} |i-j|+1 & i,j \leq n \text{ or } i,j > n \\ |i-(2n+1-j)|+1 & i \leq n \text{ and } j > n \end{cases}$$

distinct edges by adding one fewer vertices with a constant vertical offset between new and old vertices between i and j; Arrange the edges into a zig-zag pattern from i to j.

3. For each pair of vertices "pointing" one multi-edge to another, replace the pair of vertices by a new single vertex. In other words, for each pair of vertices having edges that lead away from the line segment between the vertices.

4. To find the permutation, starting at vertex i down on the left, follow the edges right, crossing at every intersection. Then, given the ending vertex j (somewhere between $n+1$ and $2n$), the i^{th} number of the permutation is $(2n+1-j)$.

This procedure produces a graph which contains a reduced decomposition; the series of crossings is a composition of adjacent transpositions, and the number of transpositions is minimal because each transposition increases the number of inversions by exactly 1 since each edge directed right and downward can only meet another edge directed right if the latter is directed upward. The uniqueness of the decomposition follows from the 321 avoidance. Assuming, for example, that a braid relation could occur, we find that one

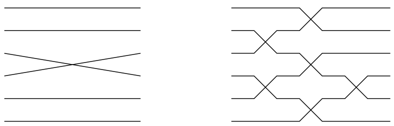


Figure 10: Left: A planar network corresponding to the transposition s_3 in S_6. Right: the reduced planar network of the permutation 251634 from Figure 9

path must be upward, another must be downward, and the last is both, lending itself to a nice contradiction.

We can associate with each adjacent transposition s_i a planar network in which source k has an edge directly to sink k for all $k \neq i, i+1$ with sources $i, i+1$ connected to a vertex m, which is in turn connected to sinks $i, i+1$. Accordingly, any permutation can be realized as a concatenation of such graphs; the i^{th} number of the permutation is merely the label of the sink reached if one starts at the i^{th} source and takes all possible crossings; see Figure 2.3.1 for an example. We shall refer to the planar network corresponding to a permutation π's canonical form as the reduced planar network of π.

2.3.2 The Function r_b

Definition Let H be a 2-coverable planar network of order n. Let $r \in T_n$. We define the function r_b to be

$$r_b(H) = \begin{cases} 1 & H \text{ is of the form } r \\ 0 & \text{Otherwise} \end{cases}$$

This function is exactly the Temperley Lieb function over reduced planar networks; in particular, it cannot be greater than 1 because no loops can occur in Skandera's ϕ when applied to graphs of the form 3. If a loop were to occur, it would need a top portion, which could only happen if $a_i = a_k$ for some i, k.

2.3.3 The Temperley Lieb Function Recursive Algorithm

We move up the Bruhat order with this algorithm, labeling each permutation π with coefficients c_π until we reach the top.

1. Label the permutation found by the procedure in section 2.3.1 **1** and all elements not greater than it **0**.

2. Select any permutation π with the smallest number of inversions and for which the Temperley Lieb coefficient is not known.

3. For each permutation $\rho \leq \pi$, let m_ρ be the number of times the canonical reduced form (3) of ρ appears as a multiset of the reduced form of π.

4. Given the constraint that

$$\sum_{\rho \leq \pi} m_\rho c_\pi = r_b,$$

calculate a value for $c_\pi = r_b - \sum_{\rho<\pi} m_\rho c_\pi$.

5. While not finished, goto step 2

2.4 Combinatorial Interpretations of the Temperley Lieb Functions

The Temperley Lieb functions may sometimes be interpreted combinatorially. The function $\equiv(G)$, for example, is clearly the number of vertex-independent path families from all sources to all sinks. The function $\rtimes(G)$ is also fairly easy to interpret combinatorially; it is simply the 1,3 binary crossing family.

Other functions become somewhat more complicated; given a path matrix $X = [x]_{i,j}$, $\underline{\rtimes}(G)$ is equal to

$$x_{1,2}x_{2,1}x_{3,3} - x_{1,2}x_{2,3}x_{3,1} - x_{1,3}x_{2,1}x_{3,2} + x_{1,3}x_{2,2}x_{3,1}$$

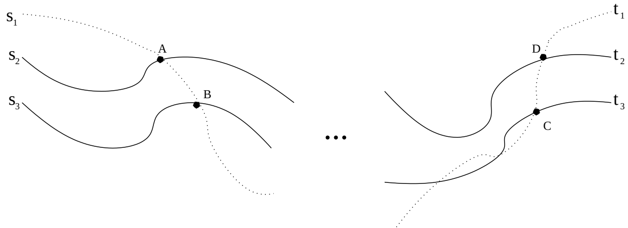
or

$$\underbrace{\underbrace{x_{1,1}x_{2,2}x_{3,3}}_{\text{A}} - \underbrace{x_{1,1}x_{2,3}x_{3,2}}_{\text{B}}}_{\text{C}} - \underbrace{\det(X)}_{\text{D}}$$

where we have

- A: All path families with paths from s_1 to t_1, s_2 to t_2, and s_3 to t_3.
- B: All path families with paths from s_1 to t_1, s_2 to t_2, and s_3 to t_3, in which the last two paths *always* intersect
- C: All path families with paths from s_1 to t_1, s_2 to t_2, and s_3 to t_3, in which the last two paths *never* intersect
- D: All path families with paths from s_1 to t_1, s_2 to t_2, and s_3 to t_3, in which no two paths intersect.

Figure 11: An illustration of a the generic path family described combinatorially: A and D must exist, and B and C need not exist; but if they do, they must occur between A and D so that the top two paths intersect first.



The total difference then is the set of path families in which the first two *always* intersect and the second two *never* intersect. See Figure 2.4 for an illustration.

3 Differences of products in the Bruhat order

In this section we explore some interesting results about total nonnegativity and Schur positivity of TNN and Jacobi-Trudi matrices, respectively.

3.1 Some nonnegative polynomials on TNN matrices

We now shall move on to a result found by [1]. The proofs provided here shall be similar to, though not exactly, those given by [1]. Let $\pi, \rho \in S_n$. Consider polynomials of the form

$$A = \prod_{i=1}^{n} x_{i,\pi(i)} \tag{4}$$

$$B = \prod_{i=1}^{n} x_{i,\rho(i)} \tag{5}$$

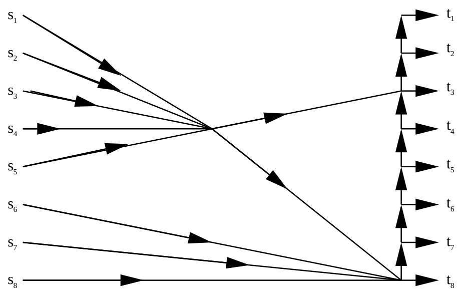
.

Proposition 1 *Let $T = [a]_{i,j}$ be a totally nonnegative $n \times n$ matrix. Then $A - B$ is nonnegative if and only if $\pi \leq \rho$ in the Bruhat order.*

Proof 1

$$\Rightarrow$$

Suppose that π is incomparable to ρ by the Bruhat order. We use the equivalent condition given by lemma (1). Let i_π, j_π be indices such that the representation M_π of the form (2.2.2) has more 1's to the left of or at j_π and



$$\begin{bmatrix} 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Figure 12: A sample planar network with the path matrix to the right.

above or at i_π than the in the same representation M_ρ of the corresponding form; likewise let i_ρ, j_ρ be the corresponding indices in which there are more 1's to the left of or at j_ρ and above or at i_ρ. Now consider the $n \times n$ matrix $T = [t]_{i,j}$ satisfying

$$t_{i,j} = \begin{cases} 2 & i \le i_\pi, j \le j_\pi \\ 1 & i > i_\pi \text{ or } j > j_\pi \end{cases} .$$

T is totally nonnegative because it is the path matrix of the planar network with sources $\{s_1, \ldots, s_n\}$ and sinks $\{t_1, \ldots, t_n\}$ arranged as in the introduction, with an extra vertex v in the center and directed edges

$$(s_k, t_n) \text{ for } k = i_\pi + 1, \ldots, n$$

$$(s_l, v) \text{ for } l = 1, \ldots, i_\pi$$

$$(v, t_n)$$

$$(v, t_{j_\pi})$$

$$(t_p, t_{p-1}) \text{ for } p = 2, \ldots, n.$$

For an example of such a network, see Figure 1.

$$\Leftarrow$$

We need only consider the case in which ρ covers π in the Bruhat order; so suppose ρ covers π. Because T is the path matrix of some planar network, the product A is the number of path families with paths respectively from source i to sink $\pi(i)$ for $i = 1, \ldots, n$, while the product B is the number of path families with paths respectively from source i to sink $\rho(i)$ for $i = 1, \ldots, n$.

Recall that ρ covering π in the Bruhat order equates to ρ equalling the product $\tau\pi$, where ρ is one more transposition than π Thus, we may interpret B to be the number of path families with paths respectively from source i to sink $\pi(i)$, such that the paths ending (resp. beginning) at sinks (resp. sources) with indices transposed by (resp. at locations transposed by) τ intersect.

Then the difference is simply the combinatorial interpretation we now posit:

$A - B$ is the number of path families with paths respectively beginning at sources i and ending at sinks $\pi(i)$ such that the paths to sinks indexed by the elements transposed by τ do not intersect.

This value is clearly nonnegative for all nonnegative matrices.

3.2 Analogous Schur-positive polynomials on the Jacobi-Trudi matrix

Now consider the products p found as addends in (1); in particular, we are concerned with these products within the λ, μ Jacobi Trudi matrix with entries $x_{i,j}$. We consider the permutations $\pi \in S_n$ and $\rho \in S_n$ and the products (4) and (5).

Proposition 2 *$\pi \geq \rho$ in the Bruhat order if and only if $B - A$ is Schur-positive for all $n \times n$ Jacobi-Trudi matrices.*

Proof 2

$$\Rightarrow$$

We show this by proving the case where π covers ρ. First, however, observe that any product of homogeneous symmetric functions is Schur positive. Now suppose π covers ρ. Then there exist $k, l, k \neq l$ such that $\pi(i) = \rho(i)$ for $i \neq k, l$ with $\pi(k) = \rho(l)$ and $\pi(l) = \rho(k)$.

Suppose $k > l$; then $\pi(k) < \rho(k)$. Moreover, observe that the difference of the products is

$$B - A = \left(\prod_{i \in \{1,\ldots,m\} \backslash k \backslash l} k_{i,\pi(i)} \right) \left(k_{l,\rho(l)} k_{k,\rho(k)} - k_{l,\pi(l)} k_{k,\pi(k)} \right) \tag{6}$$

By the above observation, the left factor is clearly Schur positive by the Littlewood-Richardson rule; the right is also known to be Schur positive because it is the minor of a Jacobi Trudi matrix. It follows again that the entire product 6 is Schur-positive. □

$\Leftarrow$

Now suppose that $\pi, \rho \in S_n$ are incomparable by the Bruhat order. Again we use the condition of Bruhat order incomparability given in lemma (1).

We need only produce a single $n \times n$ Jacobi-Trudi matrix M_ρ for which an element, i.e. an addend, appears in B applied to $H_{\Lambda/\mu}$ but not in A applied to $H_{\lambda/\mu}$.

Let i_π, j_π be indices such that the representation M_π of the form (2.2.2) has more 1's to the left of or at j_π and *below* or at i_π than the in the same location of the representation M_ρ. Let p be the number of 1's in this location of M_ρ . Now consider the $n \times n$ Jacobi-Trudi matrix $\mathcal{H}_{\lambda/\mu}$, where

$$\lambda = (\underbrace{\lambda_1, \ldots, \lambda_1}_{i_\pi - 1 \text{ times}}, \underbrace{\lambda_2, \ldots, \lambda_2}_{n - i_\pi + 1 \text{ times}})$$

$$\mu = (\underbrace{\mu_1, \ldots, \mu_1}_{j_\pi \text{ times}})$$

Also, let

$$\mu_1 = n - i - 1 + p(j + n - i) \tag{7}$$

$$\lambda_2 = \mu + n \tag{8}$$

$$\lambda_1 = \lambda_2 + j - 2 + p(j + n - i) \tag{9}$$

We shall speak of one element h_i in the matrix being larger or smaller than another based on the subscript i. Let m denote the largest element in the block to the lower left (inclusive) of i_π, j_π.

Lemma 2 *The conditions (7)(8)(9) then yield the following results about the matrix* $\mathcal{H}_{\lambda/\mu}$*:*

- *The lowest element in the matrix, with index* $(n, 1)$*, is* h_1*; this forces the products to be nonzero.*
- *The highest element in the block to the left of and below (inclusive)* i_π, j_π*, in the upper right of the block, is pm less than the lowest element of the block to the left of (inclusive)* j_π *and above (exclusive)* i_π*.*
- *The highest element in the block below and to the left of (inclusive)* i_π*,*j_π*, in the upper right of the block, is pm less than the lowest element in the block below (inclusive)* i_π *and to the right of (exclusive)* j_π*.*

The product A can be written

$$h_{\nu_1} \dots h_{\nu_n}, \tag{10}$$

while B can be written

$$h_{\eta_1} \dots h_{\eta_n}, \tag{11}$$

where $\nu_1, \dots, \nu_n$ and $\eta_1, \dots, \eta_n$ each partition $(i-1)\lambda_1+(n-i+1)\lambda_2-j\mu_1$, and $\eta_i \geq \eta_{i+1}$ and $\nu_i \geq \nu_{i+1}$ for all $i \in 1, \dots, n-1$. Then these products can be written

$$\sum_{z=1}^{t} s_{\lambda_{z_1},\dots,\lambda_{z_n}} \tag{12}$$

for some t where $\lambda_{z_1}, \dots, \lambda_{z_n} \vdash (i-1)\lambda_1 + (n-i+1)\lambda_2 - j\mu_1$ for all z.

Lemma 3 *For each sum (12), the k^{th} part λ_{i_k} of any addend's subscript must be no greater than the sum of the subscripts of factors after $h_{\nu_{k-1}}$ in the product (10). This follows from the Littlewood-Richardson rule.*

Proof 3 *The sum (12) corresponding to the product B contains the term*

$$s_{\eta_1} \dots s_{\eta_n}$$

with subscripts equal to those in (11). This term, however, does not appear in the sum (12) corresponding to A, because

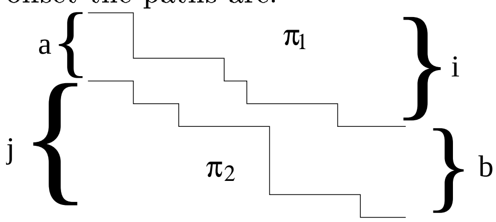
- *The product (10) contains p terms from the block with small elements; therefore the sum of these elements' subscripts is less than mp.*
- *By lemma 2, the smallest element from anywhere outside of the left, lower block has index at least mp greater than the largest of the elements in the left lower block. That is, the sum of the p subscripts in (10) less than or equal to m is smaller than the next highest subscript.*
- *The $(n-p)^{th}$ part λ_{i_k} in any addend $s_{\lambda_{i_1},\dots,\lambda_{i_n}}$ must satisfy lemma (3).*

Thus the difference $A-B$ is not Schur positive in this case. The rest follows.

4 Miscellaneous and Interesting Odds and Ends

This section contains information that relates tangentially to the material above but would stray too far from the material for comfort. Some of these are almost obvious, so only proof ideas shall be supplied.

Figure 13: Paths π_1 and π_2. The brackets show how many vertical edges offset the paths are.



4.1 Schur Function Identity

Let $s_i = h_i$ and $s_j = h_j$ be Schur functions. Then, assuming $j \leq i$ and $b \geq 1$, we find

$$s_i s_j = \underbrace{s_{i+b} s_{j-b}}_{h} + \underbrace{s_{i+b-1,j/(b-1)}}_{k} \tag{13}$$

Consider two paths π_1 and π_2 in the planar network $\mathcal{H}$ from section 1.3.1; refer to Figure 13 for this discussion. The sum of weighted path families π_1 and π_2 is simply $s_i s_j$. The values h and k in equation (13) are the number of path families that intersect and the number of path families that don't intersect, respectively; the former can be obtained by switching the second parts of π_1 and π_2 *at and beyond the point of first intersection*, while the latter is simply the schur function as found by the determinant of the $(i + b - 1, j)/(b - 1)$ Jacobi-Trudi matrix.

4.2 Temperley Lieb Function Rotation

The Temperley Lieb function $r(G)$ applied to a λ/μ Jacobi-Trudi matrix with Young Diagram Y is equal to the Temperley Lieb function $r'((G'))$, where r' is r rotated by 180° and G' corresponds to the skewed Schur function having a corresponding Young Diagram which is Y rotated by 180°.

This result is actually fairly obvious if one considers the function as acting on a subgraph of $\mathcal{H}$., as in Figure 7. Recalling that the function is symmetric, the variables cannot be distinguished from one another; the rest follows.

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