GENERALISING CUT AND JOIN AND RECURSIONS OF HURWITZ NUMBERS

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Abstract. The problem addressed in this paper is one of enumerative geometry: How many ramified covering spaces of fixed degree, genus, and ramification profile are there of $\mathbb{P}^1\mathbb{C}$? The number of such covers is called the *Hurwitz number* associated to the cover type. While this is a difficult problem to answer in general, by a cut and join analysis of permutations associated to the ramification profiles, a partial differential equation condition on the generating series of these numbers has been found for the cases in which all but one ramification point is simple; we are able to extend this idea to somewhat more general ramification profiles.

1. Introduction

1.1. **Background / Overview.** In [3], Adolf Hurwitz first posed the problem of counting ramified holomorphic covering spaces of $\mathbb{P}^1\mathbb{C}$. Generally, covering spaces are required to have the same number of preimages over every point in the base space, but a *ramified* covering space relaxes this requirement, allowing a finite subset of the base space (elements of which are called *branch points*) to have fewer preimages. Since counting homeomorphism-equivalent covers immediately allows infinitely many covers, we only count equivalence classes of covers. Similarly, we specify a degree and a genus for the cover to get interesting (finite) results. Finally, we restrict our attention to connected covers, since a disconnected cover is just a disjoint union of lower degree connected ones.

So, how does one go about counting such covering spaces? First we note that the Riemann-Hurwitz formula (which we will state explicitly later) tells us that the degree and genus determine the total ramification, so we need only keep track of the degree, genus, and ramification profiles (or we can go one step further and eliminate the degree, since this information is deducible from the ramification profiles). Now we can use the complex-analytic structure to analyse the local behavior, which we can describe by associating a permutation (called the *monodromy representation*, whose cycle type is the aforementioned ramification profile) to each branch point, in a way we will describe later. It turns out that the effect of a holomorphic homeomorphism on the monodromy representations is a global conjuation, so this leaves the ramification profiles fixed (as conjugation does not alter cycle type). We thus have a well-defined association of a ramification profile to each equivalence class of covering spaces. It is far less obvious that the set of ramification profiles determines this equivalence class, but it turns out to be true, thus giving a bijection between these equivalence classes of covers and a certain subset of the symmetric group on d letters, where d is the degree of the cover. (These cannot be arbitrary since, for example, some would give rise to disconnected covers.) We can then count the number of covers of fixed degree, genus, and ramification type, getting a number called the *Hurwitz number* associated to this type of cover. This question connects to combinatorics as well, since Hurwitz numbers also have an interpretation in terms of factoring permutations.

So, once we have this additional information, how can we use it to actually count coverings? This is still a hard problem, currently studied from different perspectives and for various reasons by algebraic geometers, combinatorialists, and mathematical physicists.

The goal of this paper is to find PDE conditions on a generating series for these Hurwitz numbers, effectively giving a recursion for Hurwitz numbers in terms of Hurwitz numbers of lower genera. We use a cut-and-join approach similar to that used by Goulden and Jackson in [1]. These PDEs will only hold under strong assumptions, however; each PDE will hold only if we are counting a Hurwitz number with a certain sort of ramification profile. Goulden and Jackson already gave such a PDE for the case when one ramification profile is simple (i.e. the monodromy permutation is a 2-cycle). The main results of this paper are two additional PDE conditions – one for when a ramification profile is $(3, 1, \dots, 1)$, and one for when a ramification profile of $(2, 2, 1, \dots, 1)$.

1.2. **Preliminary Definitions.**

Definition 1. An *degree n covering space* of a topological space X is a topological space $\tilde{X}$ equipped with an $n : 1$ map $\pi : \tilde{X} \to X$ such that there exists an open cover $\{U_\alpha\}_{\alpha \in A}$ of X with $\pi^{-1}(U_\alpha)$ homeomorphic to the disjoint union of n copies of U_α for all $\alpha \in A$ (via restriction of π to each copy).

In this paper we will always be considering covering spaces of $\mathbb{P}^1\mathbb{C}$, and so we will be considering holomorphic covering spaces – this means that we require X to have a complex-analytic structure and we require π to be holomorphic.

Definition 2. A Riemann surface equipped with such a projection is called a *ramified covering space* if, given the same setup, $|\pi^{-1}(x)| = n$ for all but finitely many $x \in X$ (so this is a slight relaxation of the covering space condition). These finitely many points are referred to as *branch points*.

We now give these ramified covering spaces some combinatorial structure which we will be able to analyse. We let $\{x_i\}_{i=1}^B$ denote the set of branch points in $\mathbb{P}^1\mathbb{C}$. The local analytic structure of the projection mapping around the preimage of some x_i is such that there is a neighborhood on which the map is equivalent under an analytic change of coordinates to $z \mapsto z^r$ for some integer r (see [2, Chapter 19a]. Now, to get a permutation from each branch point, we denote the branch point by x and choose another point $y \in \mathbb{P}^1\mathbb{C}$ sufficiently close to x so that the local analytic expression applies. Labelling the preimages under the covering map as $\{x_i\}$ and $\{y_j\}$ for x and y, respectively, we examine lifts of a loop around x based at y. We can lift this loop in n ways, one beginning at each y_j. The lift need not return to the same y_j, however – it can end at any one of the preimages. In each $z \mapsto z^{r_i}$ neighborhood of each x_i, we must have precisely r_i preimages of y (since the map is $r_i : 1$ and it hits y), and each of these preimages must be the "translate" by $2\pi/r_i$ radians of another. Each lift of the loop is then a path from one y_j to the next "translate." Now the assignment (initial point of a lift) $\mapsto$ (endpoint of a lift) induces a permutation of the indices. The permutation $\sigma \in S_n$ induced by this method is called the *monodromy representation* at x_i, and the cycle type of the permutation is the *ramification profile*. We note that the cycles of this monodromy representation correspond to preimages of the branch point. Since ramification profiles of branch points winds up being invariant under holomorphic homeomorphisms of the sphere, the ramification profiles of the branch points determine the covering space up to holomorphic homeomorphism. This gives us a bijection

$$\left\{\begin{array}{c}\text{Holomorphic connected covering}\\ \text{spaces of } \mathbb{P}^1\mathbb{C} \text{ up to}\\ \text{holomorphic homeomorphism}\end{array}\right\}$$
$$\updownarrow$$
$$\left\{\begin{array}{c}\text{Certain homomorphisms}\\ \pi_1(\mathbb{P}^1\mathbb{C} - \{x_i\}_{i=1}^B) \to S_n\\ \text{up to labelling}\end{array}\right\}$$

We can measure the "amount" of ramification of a ramified covering space by looking at the local analytic behavior of the lifts of the branch points. The projection (in suitable local coordinates) looks like $z \mapsto z^m$ for some m, so we will call $m-1$ the contribution to the total ramification (since $m = 1$ would not count as ramified). By adding up all these contributions from each lift of each branch point, we get the *total ramification*. An important result which determines the total ramification of such a covering space is the following.

Theorem (Riemann-Hurwitz Formula)**.** *Let X and Y be compact Riemann surfaces with genera g and g', respectively. Let $f : X \to Y$ be a degree-n holomorphic covering mapping between X and Y, with total ramification R. Then*

$$g = \frac{R}{2} + n(g' - 1) + 1$$

Proof. See [2, Chapter 19c]. ∎

Since we will be considering the case where $Y = \mathbb{P}^1\mathbb{C}$, we will always have $g' = 0$, thus giving the relation

$$g = \frac{R}{2} - n + 1 \quad \text{or} \quad R = 2(g + n - 1).$$

2. Cut and Join Analysis

We will be able to count the various types of cut and join interactions induced by left multiplication by 2-cycles, 3-cycles, and two disjoint 2-cycles, which we will then be able to utilize in a condition on a generating series for certain Hurwitz numbers. First, we make note of a convienient tool for reducing the number of cases to check in such cut and join analyses.

Lemma 1. *If an even (respectively, odd) permutation acts on another permutation (which we will denote σ), then the number of cycles in the representation of the image of σ as disjoint cycles will be of the same parity (respectively, of opposite parity) as the number of cycles in σ.*

Proof. We may choose some representation of the acting permutation as a product of 2-cycles and consider each as a cut or a join operation. We denote the number of cuts by C and the number of joins by J, and we denote the number of cycles in σ by N. The number of cycles in the resulting permutation will be $N + C - J$, since each join reduces the number of cycles by one and each cut increases the number of cycles by one. Thus if $C - J$ is even, the parity will be unaffected, while if $C - J$ is odd, the parity will be flipped. But the parity of $C - J$ (in the numerical even / odd sense) is precisely the parity of the acting permutation (in the permutation sense of the word) since the parity of the permutation is the parity of $C + J$, and this is off by an even number from $C - J$. ■

We now give a couple lemmata describing how permutations of certain cycle types act on general permutations. We begin with the simplest case, 2-cycles.

Lemma 2. *Cut and join for* 2*-cycles obeys the following rules:*

(1) *Any cycle containing both characters of the acting* 2*-cycle will be cut into two cycles.*
(2) *If the acting* 2*-cycle contains a character from each of two different cycles (of lengths c_i and c_j), the cycles will join into a cycle of length $c_i + c_j$.*

Proof. (1) We first reduce to the case where the 2-cycle is acting on a single cycle. As both characters of the 2-cycle are taken from a single cycle, all other cycles will commute with the 2-cycle and may be moved past the 2-cycle, leaving the 2-cycle acting only on the cycle with which it shares characters. We now simply make our way through each cycle formed. (The indices in the following discussion should be taken modulo n for consistency.) If the acting 2-cycle is $(a_i\ a_j)$ (written with $i < j$), and it is acting on $(a_1\ a_2\ \dots\ a_d)$, a_i will be sent to a_{i+1} and each element will proceed through the cycle until we get to a_{j-1}. This will be sent to a_j by the original permutation, and then the acting 2-cycle redirects it to a_i, closing the cycle. We can similarly trace our way through the cycle beginning with a_j, which will be uninhibited until a_{i-1}, at which point a_{i-1} maps to a_i maps to a_j, closing the cycle. Since the first cycle contains a_i and all characters between a_i and a_j (moving forward through the indices, and not including a_j) and the second cycle contains all characters between a_j and a_i (moving forward through the indices, and not including a_i), these two cycles contain all characters of the original cycle. Thus the cycle type change is $(c_1, \dots, c, \dots, c_r) \mapsto (c_i', c_j', c_1, \dots, \hat{c}, \dots, c_r)$, where $c = c_i + c_j$.

(2) We can first make a similar reduction; as all but two cycles are disjoint and commute with the acting 2-cycle, they can effectively be ignored. We label the two relevant cycles $(a_1\ \dots\ a_d)$ and $(a_1'\ \dots\ a_{d'}')$, and we suppose the transposition is $(a_1\ a_1')$ (this is without loss of generality, as the two cycles have d and d' representations (respectively), each one corresponding to an arbitrary choice of first character, so we may assume that in our labelling we chose the correct characters to begin each cycle). Now we chase the image of each character beginning with a_1. We see that $a_1 \mapsto a_2$, $a_2 \mapsto a_3$, $\dots\ a_{d-1} \mapsto a_d$ without change, but then $a_d \mapsto a_1 \mapsto a_1'$ since the 2-cycle is involved. Next we have $a_1' \mapsto a_2'$, $a_2' \mapsto a_3'$, $\dots\ a_{d'-1}' \mapsto a_{d'}'$, and then $a_{d'} \mapsto a_1' \mapsto a_1$, closing the cycle. Thus the multiplication yields a $d + d'$ cycle in place of the two former cycles, so the cycle type change is of the form $(c_1, \dots, c_i, \dots, c_j, \dots c_r) \mapsto (c', c_1, \dots, c_r)$ where $c' = c_i + c_j$.

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We now move on to 3-cycles.

Lemma 3. *Cut and join for 3-cycles obeys the following rules:*

(1) *Whenever the acting 3-cycle contains characters from three distinct cycles, the cycles join.*
(2) *Whenever the acting 3-cycle contains two characters from one cycle and one character from another cycle, the cycle sharing two characters with the 3-cycle is cut, and then one of the resulting cycles is joined to the cycle containing the other character of the acting 3-cycle.*
(3) *Whenever the acting 3-cycle shares all its characters with one cycle, the cycle type is either unaffected (if the order of the characters is the same in both) or the cycle splits into three (if the order of the characters is inverse).*

Proof. We first note that since a 3-cycle can always be written as a product of two 2-cycles, we can think of a 3-cycle as a composition of two cut and / or join operations. Moreover, if there is a cut involved, we may assume that it happens first, since it will involve exactly two of the three characters in the 3-cycle (call them a and b) and we can write the 3-cycle as $(c\ a\ b) = (c\ a)(a\ b)$. The order of composition then has the cut operation first.

(1) We label the three relevent cycles as $(a_1\ \dots\ a_d)$, $(a'_1\ \dots\ a'_{d'})$, and $(a''_1\ \dots\ a''_{d''})$, we suppose without loss of generality that the acting 3-cycle is $(a_1\ a'_1\ a''_1) = (a_1\ a'_1)(a'_1\ a''_1)$. First $(a'_1\ a''_1)$ joins $(a'_1\ \dots\ a'_{d'})$ and $(a''_1\ \dots\ a''_{d''})$ and then $(a_1\ a'_1)$ joins their product with $(a_1\ \dots\ a_d)$ so that the three cycles are joined into a single cycle.
(2) We label the the cycles involved as $(a_1\ \dots\ a_i\ \dots\ a_d)$ and $(a'_1\ \dots\ a'_{d'})$, and we have two cases: the 3-cycle is either $(a_1\ a_i\ a'_1)$ or $(a_1\ a'_1\ a_i)$. This is again without loss of generality, as in the 2-cycle case. Now we trace through the cycles:
Case 1: $(a_1\ a_i\ a'_1)$
We may write $(a_1\ a_i\ a'_1) = (a'_1\ a_1\ a_i) = (a'_1\ a_1)(a_1\ a_i)$ and apply the lemma on 2-cycle cut and join. The 2-cycle $(a_1\ a_i)$ acts first, cutting $(a_1\ \dots\ a_i\ \dots\ a_d)$ into $(a_1\ \dots\ a_{i-1})$ and $(a_i\ \dots\ a_d)$. Then $(a'_1\ a_1)$ acts on the three cycles, joining $(a_1\ \dots\ a_{i-1})$ and $(a'_1\ \dots\ a'_{d'})$ as described above. We thus get a cut and a join, as claimed.
Case 2: $(a_1\ a'_1\ a_i)$
This case is similar to the previous one, except that the other cycle from the first cut will be joined to $(a_1\ a'_1\ a_i)$.
(3) We can immediately reduce to the case of acting on a single cycle, which we will label as $(a_1\ \dots a_i\ \dots a_j\ \dots a_k\ a_d)$, where a_i, a_j, and a_k are the three characters shared with the acting 3-cycle. Here we also have two cases; either the order of the characters in the acting 3-cycle is the same as the order of those characters in the original cycle, or the order is opposite. (This is because there are only two 3-cycles in S_3, and they are inverse.)
Case 1: The order is the same.
In this case, we can write the acting cycle as $(a_i\ a_j\ a_k) = (a_i\ a_j)(a_j\ a_k)$. Then $(a_j\ a_k)$ cuts the cycle into $(a_j\ \dots\ a_{k-1})$ and $(a_k\ \dots\ a_{j-1})$, after which $(a_i\ a_j)$ joins these cycles (since a_i is in the second cycle).
Case 2: The order is opposite.
We can write the 3-cycle as $(a_i\ a_k\ a_j)$ Here we will not even need to decompose the acting 3-cycle into 2-cycles; we need only note that every

time one of a_i, a_j, or a_k is mapped to, the 3-cycle redirects that character to the previous one of the three. We thus get the three cycles $(a_i\ \ldots a_{j-1})(a_j\ \ldots a_{k-1})(a_k\ \ldots a_{i-1})$.

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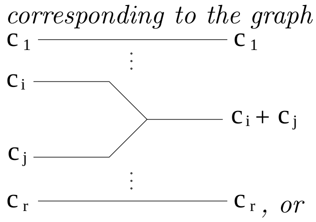
We omit a similar lemma for permutations of cycle type $(2,2)$, and simply refer to Lemma 2 twice. Now that we know how these permutations act, we can begin counting the cut and join coefficients. We start by reviewing the fairly simple case of 2-cycles, although this result is not a new one.

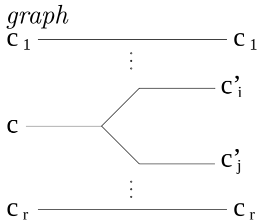
Proposition 1. *Fix a permutation $\sigma \in S_n$, and denote its cycle type by $(c_1, \ldots, c_r)$. The result of left multiplication by a 2-cycle has one of the following cycle types, with the number of each type given:*

(1) *$(c_i + c_j, c_1, \ldots, \widehat{c_i}, \ldots, \widehat{c_j}, \ldots, c_r)$ in $s^\sigma_{c_i,c_j} \cdot c_i \cdot c_j$ ways, where $s^\sigma_{c_i,c_j}$ is the number of ways to choose two distinct cycles of lengths c_i and c_j in σ, corresponding to the graph*



, or

(2) *$(c'_i, c'_j, c_1, \ldots, \widehat{c}, \ldots, c_r)$ in $c/A_{c'_i,c'_j}$ ways, where $c = c'_i + c'_j$ and $A_{c'_i,c'_j}$ is the number of automorphisms of the partition (c'_i, c'_j), corresponding to the graph*



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Proof. (1) From Lemma 2, we know that this cycle type arises when the 2-cycle contains a character from each of two distinct cycles. (This is technically the converse of one of the statements of the lemma, but as we classified all cases the implications hold in both directions.) There are c_i ways to choose a character from a cycle of length c_i and c_j ways to choose a character from a cycle of length c_j, and these are independent choices. The total number of such cycle types that can arise is then the product of these choices times the number of ways to choose cycles of the given lengths, exactly as stated.

(2) Similarly, Lemma 2 gives that we get this cycle type when the 2-cycle contains two characters from a single cycle. In this case, rather than counting the permutations explicitly we may count the number of ways to cut a smaller cycle out of the cycle of length c. For the justification for this, we write the acting 2-cycle as τ and the resulting permutation as η. We may notice that, since $\tau \cdot \sigma = \eta$, $\tau = \eta \cdot \sigma^{-1}$. Thus σ and η uniquely determine τ, and as the possible results for η after acting on σ by a 2-cycle containing two characters from σ are precisely the pairs of cycles of the form $(a_i\ \ldots\ a_{j-1})(a_j\ \ldots\ a_{i-1})$, if we vary η over these permutations

we will be varying τ over all such possible 2-cycles. Now since the pair $(a_i \ \dots \ a_{j-1})(a_j \ \dots \ a_{i-1})$ is determined by either of the cycles, we need only count one of them when they are distinct. So, choosing the cycle of length c'_i, we have c different characters at which we may start it, and these all yield distinct permutations. If we have $c'_i = c'_j$, then after choosing a starting character $c/2$ distinct ways, the other c/2 choices will place the second cycle beginning at one of the characters chosen before. As the cycles are of the same length, these are exactly the same permutations as before, so we have $c/2$ possibilities in this case.

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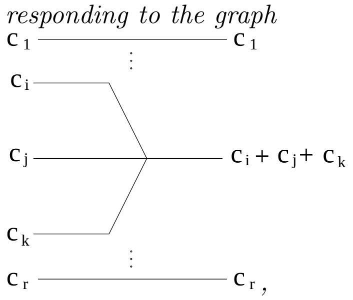
The situation for 3-cycles is more complicated. Here is a similar, new proposition, detailing the case of 3-cycles.

Proposition 2. *Fix a permutation $\sigma \in S_n$, and denote its cycle type by $(c_1, \dots, c_r)$. The result of left multiplication by a 3-cycle has one of the following cycle types, with the number of each type given:*

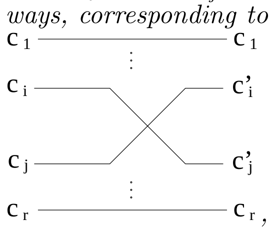
(1) $(c_i + c_j + c_k, c_1, \dots, \widehat{c_i}, \dots, \widehat{c_j}, \dots, \widehat{c_k}, \dots, c_r)$, *in* $2s^{\sigma}_{c_i,c_j,c_k} \cdot c_i c_j c_k$ *ways, corresponding to the graph*



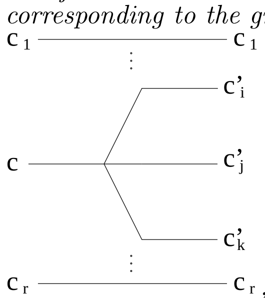
(2) $(c'_i, \widehat{c_i}, c'_j, \widehat{c_j}, c_1, \dots, c_r)$, *where* $c'_i + c'_j = c_i + c_j$, *in*

$$\begin{cases} 0 & c_i = c_j = c'_i = c'_j \\ s^{\sigma}_{c_i,c_j} c_i c_j & c_i = c'_i \text{ or } c_i = c'_j \text{ but } c'_i \neq c'_j \\ 2s^{\sigma}_{c_i,c_j} c_i c_j / A_{c'_i,c'_j} & \text{otherwise} \end{cases}$$

ways, corresponding to the graph

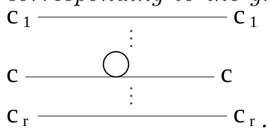


(3) $(c'_i, c'_j, c'_k, c_1, \ldots, \widehat{c}, \ldots, c_r)$, *where* $c'_i + c'_j + c'_k = c$, *in* $s^\sigma_c \cdot c/A_{c_i,c_j,c_k}$ *ways, corresponding to the graph*



, and

(4) $(c_1, \ldots, c, \ldots, c_r)$ *in* $s^\sigma_c(c)(c-1)(c-2)$ *ways if* $c \geq 3$ *and 0 ways if* $c < 3$, *corresponding to the graph*



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The constants in the above proposition are defined as follows:

(1) $s^\sigma_{c_i,c_j,c_k}$ *is the number of ways to choose 3 distinct cycles of lengths* c_i, c_j, *and* c_k,
(2) $s^\sigma_{c_i,c_j}$ *is the number of ways to choose 2 distinct cycles of lengths* c_i *and* c_j,
(3) s^σ_c *is the number of cycles of length* c,
(4) A_{c_i,c_j,c_k} *is the number of automorphisms of the partition* (c_i, c_j, c_k).

Proof. We proceed case by case:

(1) By Lemma 3, the acting 3-cycle contains three elements of three distinct cycles of σ. We can choose the cycles in $s^\sigma_{i,j,k}$ ways, we can choose the characters in c_i, c_j, and c_k ways, and we can choose 2 different orders for the characters. This gives a total of $2s^\sigma_{i,j,k} \cdot c_i c_j c_k$ possibilities.
(2) We want to finish with the same number of cycles we started with, so we must have one cut and one join. As noted above in the proof of Lemma 3, we can assume without loss of generality that the cut comes first. Then we must get the smaller of the two new cycles by cutting it from one of the original cycles, and the other new cycle will then move the other characters. Thus we need only keep track of how we can cut out the smaller of the new cycles. First, if all cycles are the same size, we clearly cannot get any permutation to cut and join appropriately, since it would have to cut a cycle and then join the pieces back together; this would only involve one of the original cycles. We have thus shown the first case. Now if the cycles lengths c_i, c_j, c'_i, c'_j are not all equal, we will have $\max(c_i, c_j) > \min(c'_i, c'_j)$, so we will at least be able to cut the smaller new cycle out of the larger original one. If we want to cut out a cycle of length c'_i, we can choose any character a of the cycle for the first character in the 2-cycle defining the cut, and the second character b is determined by c'_i, the length of the cycle we want to cut out. We then want to join the other cycle we cut with the other of the original cycles; this means the other 2-cycle must begin with b (so that we actually get a 3-cycle out of this) and any character of the other cycle, so we have an additional c_j choices here. Thus the total number of

possibilities is $c_i c_j$. If the larger original cycle is bigger than either of the new cycles, we can in fact cut out either of the new cycles in this way; we thus get twice as many possibilities $(2 \cdot c_i c_j)$ if c_i and c_j are distinct, but if $c_i = c_j$ we can still only cut one length of cycle from the original one, so we still have $c_i c_j$ possibilities. Lastly, if both original cycles are larger than the smaller new one, we can choose to cut the smaller new one from either of the old ones, giving $c_i c_j + c_j c_i = 2c_i c_j$ possibilities. The condition that $c'_i + c'_j = c_i + c_j$ prevents these cases where more possibilities arise from overlapping, so we have all cases accounted for. As before, we multiply all cases by the number of ways to choose the cycles we began with to get the total number of possibilities, giving the result.

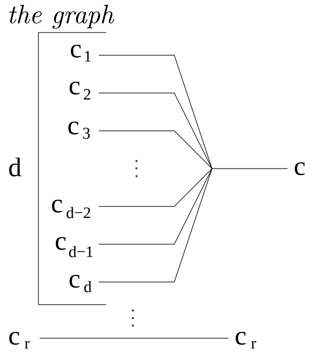
(3) Rather that work directly with the permutations, we will count possible cuts. It is clear that both operations must be cuts by the number of cycles. Similarly to the 2-cycle case, we can write the acting 3-cycle as τ and the resulting permutation as η. We may notice that, since $\tau \cdot \sigma = \eta$, $\tau = \eta \cdot \sigma^{-1}$. Thus σ and η uniquely determine τ, and as the possible results for η after acting on σ by a 3-cycle operating as two cuts are exactly those of the form $(a_{c_i}\ \dots\ a_{c_i+c_j-1})(a_{c_i+c_j}\ \dots\ a_{c_i+c_j+c_k-1})(a_{c_i+c_j+c_k}\ \dots\ a_{c_i-1})$, we may vary η over these permutations to vary τ over all such possible 3-cycles.
We can first reduce to the case of a single cycle, as the rest of the permutation is not involved. Now, without loss of generality, choose the length c'_i to cut out first. Then if $\sigma = (a_1\ \dots\ a_c)$, we have c ways of choosing a cycle of length c'_i, based on where it begins. Next, we can consider the length of the cycle containing the next character, for which we have $|\{c'_j, c'_k\}|$ choices (i.e., as many choices as distinct cycle lengths). After this we have only one option left for the remaining cycle, so we have no more choices to make. Thus if all c'_i, c'_j, and c'_k are distinct, we have $2c$ possibilities, if two are equal and the third is distinct we may choose to begin with the distinct one to give c possibilities, and if all three are equal we have $c/3$ possibilities; we can see this last case since in choosing the first cycle we overcount, and we must divide by the number of ways we could have chosen the cycle to get the number of cases, and after the initial choice the cycle type is determined. The possibilities are thus as posited.

(4) By Lemma 3, we get this sort of cut and join when we have a 3-cycle acting with characters all in one cycle and in the same order that they appear in that cycle. The number of ways to choose three characters from a cycle of length c is $(c)(c-1)(c-2)$, and their order is determined by the fact that we want the order to be the same as the order in the permutation.

■

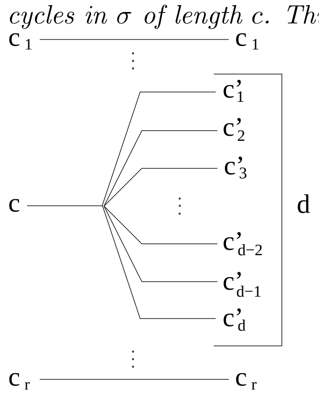
From our analysis of 3-cycles, we get generalizations to cycles of arbitrary length in certain situations. Although we cannot use these coefficients to compute Hurwitz numbers without the intermediate cases, this is at least a step toward computing some more general Hurwitz numbers.

Proposition 3. *Fix a permutation $\sigma \in S_n$, and denote its cycle type by $(c_1, \ldots, c_r)$. The number of d-cycles that can act on σ so that the resulting permutation has $r-d+1$ cycles (the smallest number of possible cycles, corresponding to $d-1$ join operations), joining $c_1, \ldots, c_d$ is $s^{\sigma}_{c_1,\ldots,c_d}(d-1)!c_1 \cdot \ldots \cdot c_d$, where $s^{\sigma}_{c_1,\ldots,c_d}$ is the number of ways to choose distinct cycles of lengths $c_1, \ldots, c_d$. This corresponds to the graph*



Proof. The proof here is a direct generalization of the one for the 3-cycle case. In order to join all d of the cycles, we will need to choose an element of each cycle, otherwise some of the cycles would remain isolated. We can therefore choose characters a_i such that a_i is a character of a cycle of length c_i and no two a_i come from the same cycle. Then for any bijection ϕ of the set of a_is, this gives us a d-cycle $(a_{\phi(1)} \ \ldots \ a_{\phi(d)})$. This will actually join the cycles, as we can see by expanding the d-cycle into $(a_{\phi(1)} \ a_{\phi(2)})(a_{\phi(2)} \ a_{\phi(3)}) \ldots (a_{\phi(d-1)} \ a_{\phi(d)})$ where we can see that each 2-cycle will be a join operation. Now the counting proceeds as before, except we can arrange the characters in $(d-1)!$ orders (since any order will work, giving $d!$ possible orders, and any d-cycle has d representations, so we are left with $d!/d = (d-1)!$ arrangements), so we have $s^{\sigma}_{c_1,\ldots,c_d}(d-1)!c_1 \cdot \ldots \cdot c_d$. ■

Proposition 4. *Fix a permutation $\sigma \in S_n$, and denote its cycle type by $(c_1, \ldots, c, \ldots, c_r)$. The number of d-cycles that can act on σ so that the resulting permutation has $r+d-1$ cycles (the largest number of possible cycles, corresponding to $d-1$ cut operations), cutting c is $s^{\sigma}_{c}c(d-1)!/A_{c_1,\ldots,c_d}$, where $A_{c_1,\ldots,c_d}$ is the number of automorphisms of the partition $(c_1, \ldots, c_d)$ and s^{σ}_{c} is the number of cycles in σ of length c. This corresponds to the graph*

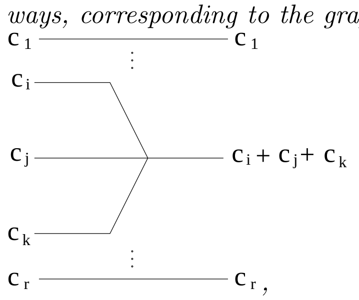


Proof. The result is of the same form as the one from the 3-cycle case, but the proof given there does not generalize. We begin as the proof there, considering the number of ways to divide the permutation into cycles. We are no longer assured that either all cycles are the same length or we have a cycle of unique length, though, so instead we just form the set of all possible divisions of the original cycle into smaller ones. Upon fixing a starting character (giving c choices) this is the set of all strings $c_1 c_{\phi(2)} \dots c_{\phi(d)}$ varying over the $(d-1)!$ bijections ϕ of $\{2, \dots, d\}$, and two such strings are equal if and only if one is the result of applying one of the bijections by which we are dividing out. Since we may choose the cycle of length c in s^σ_c ways, the number of possibilities is precisely $s^\sigma_c c(d-1)!/A_{c_1,\dots,c_d}$. ■

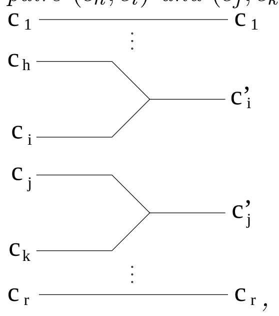
Finally, we count cut and join coefficients for two disjoint 2-cycles, the other new combinatorial result.

Proposition 5. *Fix a permutation $\sigma \in S_n$, and denote its cycle type by $(c_1, \dots, c_r)$. The result of left multiplication by a $(2,2)$-cycle has one of the following cycle types, with the number of each type given:*

(1) $(c_i+c_j+c_k, c_1, \dots, \widehat{c_i}, \dots, \widehat{c_j}, \dots, \widehat{c_k}, \dots, c_r)$, *in* $s^\sigma_{c_i,c_j,c_k} \cdot c_i c_j c_k (c_i+c_j+c_k-3)$ *ways, corresponding to the graph*



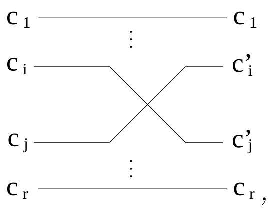
(2) $(c_h+c_i, c_j+c_k, c_1, \dots, \widehat{c_h}, \dots, \widehat{c_i}, \dots, \widehat{c_j}, \dots, \widehat{c_k}, \dots, c_r)$ *in* $s^\sigma_{(c_h,c_i),(c_j,c_k)} \cdot c_h \cdot c_i \cdot c_j \cdot c_k$ *ways, where* $s^\sigma_{(c_h,c_i),(c_j,c_k)}$ *is the number of ways to choose the two pairs* (c_h, c_i) *and* (c_j, c_k) *from* σ*, corresponding to the graph*



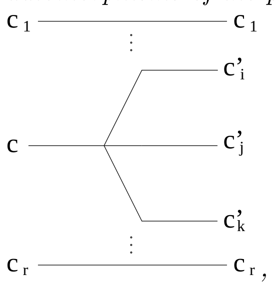
(3) $(c'_i, \widehat{c_i}, c'_j, \widehat{c_j}, c_1, \dots, c_r)$, *where* $c'_i + c'_j = c_i + c_j$, *in (letting* $a = \max(c_i, c_j)$, $b = \min(c_i, c_j)$, $c = \min(c'_i, c'_j)$, $d = \max(c'_i, c'_j)$*)*

$$\begin{cases} s^\sigma_{a,b} ab(c-1) & \text{if } a = b = c = d \\ s^\sigma_{a,b} ab(a-2) & \text{if } a \neq b = c \neq d \\ s^\sigma_{a,b} ab(a+b-c-3) & \text{if } b > c \\ s^\sigma_{a,b} ab(a-b-2+(b-1)/A_{c,d}) & \text{if } c > b \end{cases}$$

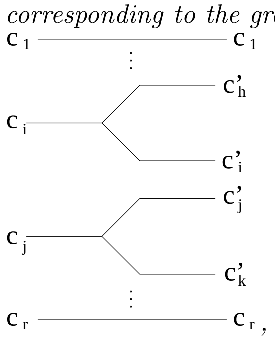
ways, where $s^\sigma_{c_i,c_j}$ *is the number of ways to choose two distinct cycles of lengths* c_i *and* c_j *in* σ *and* $A_{c,d}$ *is the number of automorphisms of the partition* $\{c, d\}$*, corresponding to the graph*



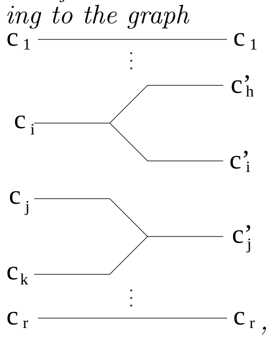
(4) $(c'_i, c'_j, c'_k, c_1, \ldots, \widehat{c}, \ldots, c_r)$, *where* $c'_i + c'_j + c'_k = c$, *in* $s^\sigma_c \cdot c(c - \min(c'_i, c'_j, c'_k) - 2)/A_{c'_i,c'_j,c'_k}$ *ways, where* $A_{c'_i,c'_j,c'_k}$ *is the number of automorphisms of the partition* (c'_i, c'_j, c'_k)*, corresponding to the graph*



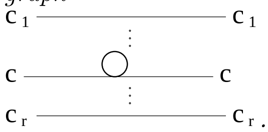
(5) $(c'_h, c'_i, c'_j, c'_k, c_1, \ldots, \widehat{c_i}, \ldots, \widehat{c_j}, \ldots, c_r)$ *in* $s^\sigma_{c_i,c_j} \cdot c_i c_j/(A_{c'_h,c'_i} A_{c'_j,c'_k})$ *ways, where* $A_{c'_i,c'_j}$ *is the number of automorphisms of the partition* (c'_i, c'_j) *and* $s^\sigma_{c_i,c_j}$ *is the number of ways to choose two distinct cycles of lengths* c_i *and* c_j *in* σ*, corresponding to the graph*



(6) $(c'_i, c'_j, c'_k, c_1, \ldots, \widehat{c_i}, \ldots, \widehat{c_j}, \ldots, \widehat{c_k}, \ldots, c_r)$ *in* $s^\sigma_{c_i,c_j,c_k} \cdot c_i c_j c_k/A_{c'_i,c'_j}$ *ways, where* $A_{c'_i,c'_j}$ *is the number of automorphisms of the partition* (c'_i, c'_j)*, corresponding to the graph*



(7) $(c_1, \ldots, c, \ldots, c_r)$ *in* $s_c^\sigma \frac{c}{4} \cdot \sum_{i=1}^{c-1}(c-i-1)(i-1)$ *ways, corresponding to the graph*



Proof. Cases:

(1) We must choose four distinct characters from three cycles, so exactly one cycle will contain two of these characters and the other two will contain exactly one. Suppose it is c_i. Then each 2-cycle has a character of c_i (if both characters of c_i are in the same 2-cycle, we are in the sixth case), so we have $c_i(c_i - 1)$ choices so far, and (picking some order for the 2-cycles) we can fill in the other characters of the 2-cycles with either c_j first or with c_k first. We then have $c_jc_k + c_kc_j$ choices for these other characters, and we then must divide by 2 to compensate for overcounting due to commutativity of the 2-cycles. This gives a total of $c_ic_jc_k(c_i - 1)$ possibilities. While we implicitly assumed that c_i was at least 2 in our counting, the formula gives 0 cases when $c_i = 1$, so it is correct in this case as well. Now repeating this process taking two characters from each of the other cycles and adding the possibilities gives $c_ic_jc_k(c_i + c_j + c_k - 3)$, and we multiply by the number of choices for these cycles.

(2) The number of possibilities for each of the disconnected joins is the product of the cycle lengths, by the 2-cycle cut and join analysis. The choices are independent, so they multiply to give $c_h \cdot c_i \cdot c_j \cdot c_k$ possibilities, which we then multiply by ways to choose those cycles.

(3) Letting $a = \max(c_i, c_j)$, $b = \min(c_i, c_j)$, $c = \min(c'_i, c'_j)$, $d = \min(c'_i, c'_j)$ as above, we clearly must have one cut and one join. Without loss of generality, we can think of the acting cycles as a join followed by a cut, since if a 2-cycle acts as a cut, joining another permutation before the cut operation will leave both characters in the cycle. There are ab ways to join, after which we have a cycle of the form $(||||\ldots|||||\ \ x\ \ |||||\ldots|||||\ \ y)$ where the first 2-cycle was $(x\ \ y)$ and the vertical lines represent regions of unknown characters. The first unknown region is of length $a-1$ (after possibly shifting the cycle around) and the second is of length $b-1$. Now we want to cut out a cycle of length c from this new cycle – this amounts to choosing characters with $c-1$ characters between them. There are $A = a-c-1$ ways of cutting this cycle from the length $a-1$ one and $B = b-c-1$ ways of cutting this cycle from the length $b-1$ one by considering where the starting and ending points must be; starting the first character of the cut from the first character of, say, the A region, once the first character gets to the $(a-c-1)^{\text{st}}$ character, the final character is in the $(a-1)^{\text{st}}$ position, and thus this is the final possible cut. The B case is similar. Of course, both the inital and final characters of the cuts need not come from the same region. The cut can "span" either x or y in $c-1$ ways – the number of characters between the endpoints of the cut. If, however, $b < c$, we can only get $b-1$ possibilities for each, since there are only $b-1$ characters on which to begin or finish. This gives $2 \cdot \min(b-1, c-1)$ possibilities, but these could all have arisen as the initial join since they contain a character from each original cycle, so

we must divide by 2. Now if $c = d$, we notice that we also overcounted since all the cycles spanning x are also cycles of the same length spanning y, so we divide by a further 2. Thus we define $C = \min(b-1, c-1)/A_{c,d}$, so C is the number of possibilities coming from the "edge" cases. There remains one case unaccounted for: when both x and y are in the cycle being cut out. This requires that $b < c-1$, and here we will have a further $c-b-1$ cases, as one can see by considering how we can fit characters with $(b-1)$ characters between them inside the cycle of length c. Defining $D = c-b-1$, we have $ab(\min(A,0) + \min(B,0) + C + \min(D,0))$ cases. The skeptical reader may verify that this agrees with the cases stated in the proposition.

(4) We begin by labelling c'_i, c'_j, and c'_k as l, m, n with $l \geq m \geq n$. We may notice that one of the 2-cycles must cut out a cycle of length n (as this is the only way to get such a cycle). We have c choices for this cut, depending on the choice of initial character. We then have two cycles, one of length n and the other of length $c-n$. We must then cut the cycle of length m from the $c-n$-cycle, and we have $c-n-2$ ways of doing this – we can choose any character of the cycle to begin the cut, except the cut cannot begin or end on the character already used, eliminating two cases. Now we must divide by the appropriate automorphisms to compensate for overcounting, as our choices may not have been unique, and multiply by the number of possibilities for choosing the cycles. This gives the result.

(5) The number of possibilities for each of the disconnected cuts is the cycle over automorphisms of the right side of the graph, by the 2-cycle cut and join analysis. The choices are independent, so they multiply to give $c_i c_j/(A_{c'_h,c'_i} A_{c'_j,c'_k})$ possibilities, which we then multiply by ways to choose those cycles.

(6) The number of possibilities for the disconnected cut and join is the product of each individually; we thus have $c_i c_j c_k / A_{c'_i,c'_j}$ by the 2-cycle cut and join analysis. We then multiply by the ways to choose those cycles.

(7) We have a cut and a join, and we can assume that the cut comes first. Suppose the first 2-cycle cuts out an i-cycle, which can happen in c ways. Then we get an i cycle and a $c-i$ cycle which can be rejoined in $(c-i-1)(i-1)$ ways, using any character of each except those chosen in the first 2-cycle. Now we sum over $1 \geq i \geq c-1$ to get all possibilities, but we have overcounted – we must divide by 2 since cutting out an i-cycle is the same as cutting out a $(c-i)$-cycle, and by a further 2 since all of these joins could also have been initial cuts. We then multiply by the number of c-cycles to get the result.

■

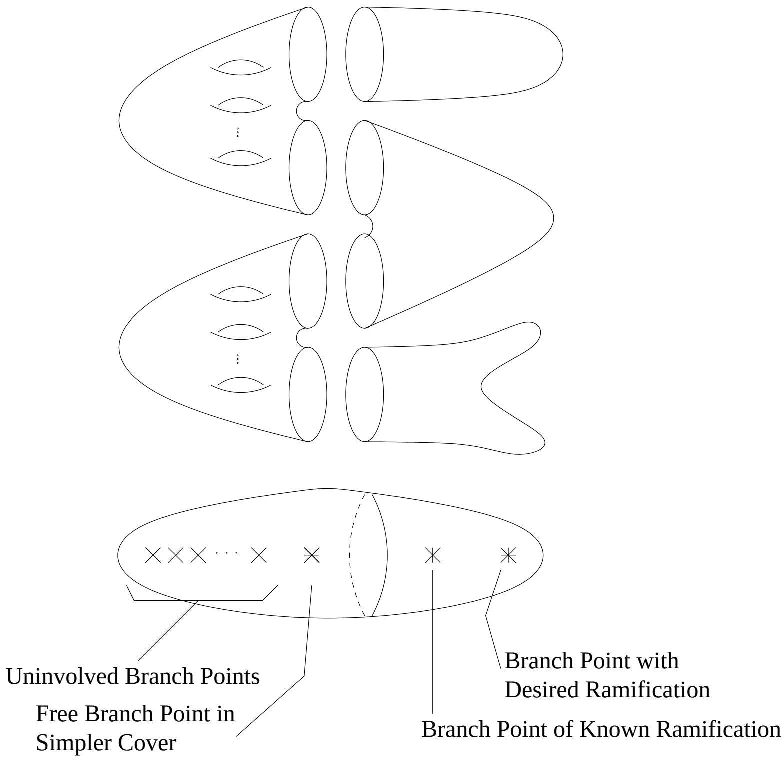
This concludes our combinatorial analysis of cut and join coefficients.

3. PDE Conditions

Now that we have gone through the work of finding these cut and join coefficients, we can use them to get a PDE condition. We consider Hurwitz numbers with one free ramification profile and the others each one of $(2, 1, \ldots, 1), (3, 1, \ldots, 1)$, or $(2, 2, 1, \ldots, 1)$. The general idea is to "cut up" the covering space, viewing it as several pieces. Knowing the Hurwitz number of each piece and the ways in which the pieces can be assembled gives the Hurwitz number we wanted. We make this (somewhat) more precise (although full rigor is beyond the scope of this paper; the justification is the same as for the case of almost all simple branch points as in [1]).

Given a type of ramified covering space (i.e. fixing the degree, genus, and ramification profiles) of $\mathbb{P}^1\mathbb{C}$, we can describe how it can be "built" from covers of lower genera / degree. Taking some such ramified covering space X, we can allow one branch point (which we will denote by x) to have completely free ramification profile (say $[\alpha]$ for $\alpha \in S_n$), and if the other branch points all have ramification profiles $(3, 1, \ldots, 1), (2, 2, 1, \ldots, 1)$, or $(2, 1, \ldots, 1)$ we can procede as follows: taking a loop around x and another branch point in $\mathbb{P}^1\mathbb{C}$, we can shrink the loop, forcing the two branch points together. If we think of the branch points as moving to literally the same point (in some limiting sense) we will get a single branch point whose monodromy representation is precisely the product of the monodromy representations of the original two branch points. The new point will have perhaps a different number of preimages, since a branch point has one preimage per cycle in its ramification profile. Now since locally these covering maps are quite simple, we can extend this to sufficiently small loops around the branch points as well, so that we know the general shape of the preimage of the loop (and its enclosed area). Now this loop, assuming it is sufficiently small, has the same number of preimages as the enclosed branch point. Since preimages correspond to cycles in the ramification profile, we know the number of preimages by knowing how the cycle type of the permutation changes under this multiplication.

At this point we need a way of visualizing this. We imagine that the branch points are lined up on the Riemann sphere with the free branch point farthest right and the one with which we want it to act next to it. We then have the covering space above the sphere, and we draw our loop and its preimages in the covering space. If we think of "breaking off" this piece of the covering space along the preimages of the loop, we will have decomposed the space into at least two pieces (it was originally connected, but each new piece need not be). Now the preimage of this loop and its enclosed area may have many connected components, but we only want to focus on those where something interesting happens – those in which the ramification profile changes occur. This includes any connected component containing multiple preimages either of the loop or of the original point x (corresponding to a cut or a join, respectively), and also those whose ramification profile retains its cycle type but still changes. The other components may be safely ignored for our purposes since they cannot affect the genus and there is only one way they can connect to the rest of the cover. An illustration of this decomposition is given, where the "branching" of the right piece denotes the formation of new preimages of the branch point beneath it. In this example, we have two operations on the cycle type – the middle section on the right represents a cut, and the bottom one represents a join. (Recall that we are coming from the perspective of the final ramification profile, moving back to the one in the simpler cover.)



We now have the "pieces" we wanted, namely the interesting connected components of the preimage of the loop and its enclosed area, and each of these comes with some multiplicity based on how many choices we have for a ramification profile to act in the required way. We now need to figure out how many ways we could attach covering spaces of lower degree and / or genus to these pieces to recover a space of the original form. This will give all possible covering spaces in terms of lower Hurwitz numbers. For this, we can just analyse the geometry of the situation, basically "connecting all the pieces" in all possible ways. We proceed in this way to prove the following PDE conditions.

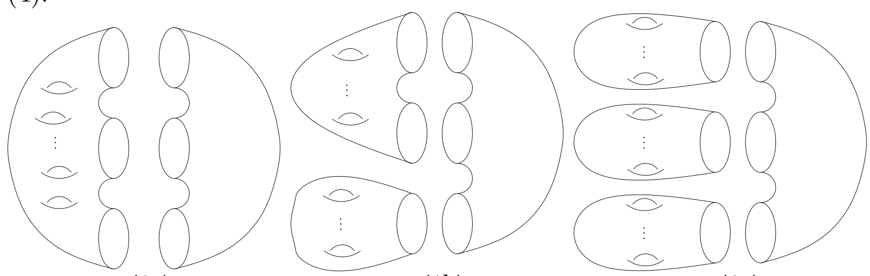
Theorem 4. *Let* $H = \sum H_{\alpha}^{g,r,s,t} p^{\alpha} z^{g} \frac{u^r}{r!} \frac{v^s}{s!} \frac{w^t}{t!}$ *be the exponential generating function with the coefficient* $H_{\alpha}^{g,r,s,t}$ *defined to be the Hurwitz number for genus* g *covers with ramification profiles as follows: one branch point has arbitrary ramification profile* α*,* r *branch points have a 2-cycle as their ramification profile,* s *branch points have a 3-cycle as their ramification profile,* t *branch points have a permutation with cycle type* $(2,2)$ *as their ramification profile, and* g *is the genus of the covering.*

Then we have the partial differential equation

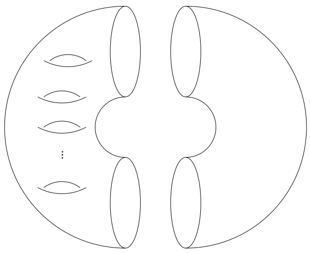
$$\frac{\partial H}{\partial v} = \left(\frac{1}{3!}\right) \sum_{i,j,k\geq 1} 2ijk(p_{i+j+k}z^2 \frac{\partial^3 H}{\partial p_i \partial p_j \partial p_k} + 3p_{i+j+k}z \frac{\partial H}{\partial p_i}\frac{\partial^2 H}{\partial p_j \partial p_k} + p_{i+j+k}\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j}\frac{\partial H}{\partial p_k})$$
$$+ \sum_{\substack{i,j,k,l\geq 1\\ i+j=k+l\\ i=k}} ij(zp_k p_l \frac{\partial^2 H}{\partial p_i \partial p_j} + p_k p_l \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j}) + \left(\frac{1}{4}\right) \sum_{\substack{i,j,k,l\geq 1\\ i+j=k+l\\ i\neq k\\ i\neq l}} 2ij(zp_k p_l \frac{\partial^2 H}{\partial p_i \partial p_j} + p_k p_l \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j})$$
$$+ \left(\frac{1}{3!}\right) \sum_{i,j,k\geq 1} 2(i+j+k)p_i p_j p_k \frac{\partial H}{\partial p_{i+j+k}} + \sum_{i\geq 2} i(i-1)(i-2)p_i z \frac{\partial H}{\partial p_i}$$

Proof. The first sum represents the case when we have a 3-cycle that cuts a cycle of the free ramification profile into three cycles. In this case, the preimage of of a suficiently small loop and its interior will be a connected curve with three disjoint circles as its boundary. Fixing a choice of such a 3-cycle (for which there are $2(i+j+k)$ choices by the cut and join analysis), we have several cases for how the remainder of the curve could look, namely how many connected components it could have. It could have one connected component of genus $g-2$, as reassembling these two pieces would then give a connected genus g curve (See Figure, (1a)). It could also have two connected components, one with two boundary circles and one with a single boundary circle, which have genera adding up to $g-1$ (since one "hole" is formed in this connection), giving the case (1b) shown in the figure. Finally, there could be three connected components (1c), each with a single boundary circle, whose genera add up to g, as no additional "holes" are formed from this connection. We now want to encode these possibilities into our generating series condition. The partial derivative with respect to v on the left side of the equality gives the series $\frac{\partial H}{\partial v} = \sum H_\alpha^{g,r,s,t} p^\alpha z^g \frac{u^r}{r!}\frac{v^{s-1}}{(s-1)!}\frac{w^t}{t!}$, so the term of this series corresponding to the Hurwitz number $H_\alpha^{g,r,s,t}$ is $p^\rho z^g \frac{u^r}{r!}\frac{v^{s-1}}{(s-1)!}\frac{w^t}{t!}$. On the other side, we have an enumeration of the ways we can assemble the pieces of the covering space which we are attaching to the preimage of the loop with its interior. This will be different in each case. In the case with one connected component, the "piece" we need is a single connected component with three boundary circles, with a ramification profile over one branch point containing three cycles whose lengths add up to the cycle we want to add, and one fewer 3-cycle than in the type of cover that the desired Hurwitz number describes (since the other side of the equation has all the v^r terms with one lower power). The rest of the cycles must match the desired one precisely. We also need to count the number of ways to choose the cycles, should there be more than one cycle of the same size. This is precisely what the partial derivative gives us – a generating series in which we have removed the three cycles being joined, and we multiply by p_{i+j+k} so that the variables with the Hurwitz number of the cover we want will match the variables associated to the desired Hurwitz number on the other side of the equality. In this case, we also add two to the power of z since we want to start with a cover that has genus two less than the final cover. Also, the trouble of counting possibilities of choices for the cycles is exactly taken care of when we differentiate out the p_i variables.

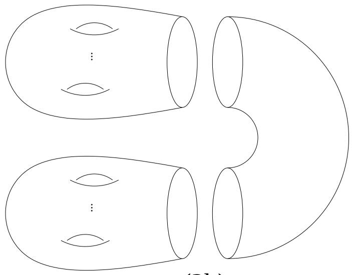
In each case, we will have to add in a p_{i+j+k} variable to balance the variables on each side, and we may have to add a z variables to balance out genera. In the case with two connected components, we have a similar process, although with the added complication of multiple "pieces." Here we will need two covers with genera summing to $g-1$ and degree summing to the degree of the desired cover (although since the degree is encoded in the p variables, we can ignore it and things will work out), one with the free branch point containing a cycle of length i and the other with the free branch point containing cycles of length j and k such that the rest of the cycles in the free branch points together give α and the numbers branch points ramified as 2-cycles, 3-cycles and $(2,2)$ permutations add up to the desired numbers. By multiplying the partial derivatives $\frac{\partial H}{\partial p_i}$ and $\frac{\partial^2 H}{\partial p_j \partial p_k}$, we get exactly this by how series multiply. Now we add one to the z exponent and insert the p_{i+j+k} so that the two sides match, and we note that there are three possible ways we could get this associated picture – one for each place to which the component with the single boundary circle can connect. This gives the 3 coefficient on this term. The last summand is simply the natural next case – where we have three connected components. Here we multiply the three partial derivatives together for reasons similar to the case with two connected components, and we don't need to correct for genus, so we have accounted for all the cases. We multiply the sum by a factor of $\left(\frac{1}{3!}\right)$ to account for repetitions in counting since we are summing over three indices and thus get redundant cases. This takes care of the first cut and join case, and the other cases proceed very similarly. The rest of the terms in the generating series follow from the pictures for the remaining cases by identical reasoning as above and the corresponding cut and join equations derived above. The two summations on the middle line correspond to the same cases (as in Figures 2(a) and 2(b)), and are only broken into two sums to correspond to the different cases in the coefficient for the associated cut and join. The first summation on the last line is for the cases looking like Figure (3), and is straightforward. The one somewhat subtle point that arises is that when the cycle type is unchanged when the branch points are forced together, the lift of the loop with interior has genus one; this is by the Riemann-Hurwitz formula. This comes up in the last summation, corresponding to Figure (4).



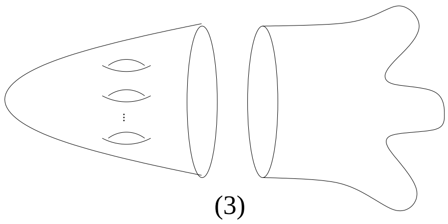
(1a) (1b) (1c)



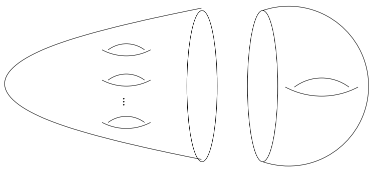
(2a)



(2b)



(3)



(4)

■

Theorem 5. *Let* $H = \sum H_{\alpha}^{g,r,s,t} p^{\alpha} z^{g} \frac{u^r}{r!} \frac{v^s}{s!} \frac{w^t}{t!}$ *be the exponential generating function with the coefficient* $H_{\alpha}^{g,r,s,t}$ *defined to be the Hurwitz number for genus* g *covers with ramification profiles as follows: one branch point has arbitrary ramification profile* α*,* r *branch points have a 2-cycle as their ramification profile,* s *branch points have a 3-cycle as their ramification profile,* t *branch points have a permutation with cycle type* $(2,2)$ *as their ramification profile, and* g *is the genus of the covering. Then we have the partial differential equation*

$$
\begin{aligned}
\frac{\partial H}{\partial w} &= \left(\frac{1}{3!}\right) \sum_{i,j,k\geq 1} ijk(i+j+k-3) p_i p_j p_k \frac{\partial H}{\partial p_{i+j+k}} + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \sum_{i,j,k,l\geq 1} ijkl(p_i p_j p_k p_l \frac{\partial^2 H}{\partial p_{i+j} \partial p_{k+l}}) \\
&+ \sum_{\substack{i,j,k,l\geq 1 \\ i=j=k=l}} ij(k-1)(z p_k p_l \frac{\partial^2 H}{\partial p_i \partial p_j} + p_k p_l \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j}) \\
&+ \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \sum_{\substack{i,j,k,l\geq 1 \\ i+j=k+l \\ i\neq j=k\neq l}} ij(i-2)(z p_k p_l \frac{\partial^2 H}{\partial p_i \partial p_j} + p_k p_l \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j}) \\
&+ \sum_{\substack{i,j,k,l\geq 1 \\ i+j=k+l \\ i\geq j \\ l\geq k \\ j>k}} ij(i+j-k-3)(z p_k p_l \frac{\partial^2 H}{\partial p_i \partial p_j} + p_k p_l \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j}) \\
&+ \sum_{\substack{i,j,k,l\geq 1 \\ i+j=k+l \\ i\geq j \\ l\geq k \\ k>j}} ij(i-j-2)(z p_k p_l \frac{\partial^2 H}{\partial p_i \partial p_j} + p_k p_l \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j})
\end{aligned}
$$

$$+\left(\frac{1}{2}\right)\sum_{\substack{i,j,k,l\geq 1\\ i+j=k+l\\ j<k\leq i\\ j<l\leq i}} ij(j-1)(zp_kp_l\frac{\partial^2 H}{\partial p_i\partial p_j}+p_kp_l\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j})$$

$$+\sum_{\substack{i,j,k\geq 1\\ i=j=k}} i(i+j-2)(p_{i+j+k}z^2\frac{\partial^3 H}{\partial p_i\partial p_j\partial p_k}+p_{i+j+k}z\frac{\partial H}{\partial p_i}\frac{\partial^2 H}{\partial p_j\partial p_k}+p_{i+j+k}\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j}\frac{\partial H}{\partial p_k})$$

$$+\left(\frac{1}{2}\right)\sum_{\substack{i,j,k\geq 1\\ i=j\geq k}} i(i+j-2)(p_{i+j+k}z^2\frac{\partial^3 H}{\partial p_i\partial p_j\partial p_k}+p_{i+j+k}z(\frac{\partial H}{\partial p_i}\frac{\partial^2 H}{\partial p_j\partial p_k}+\frac{\partial H}{\partial p_j}\frac{\partial^2 H}{\partial p_i\partial p_k}+2\frac{\partial H}{\partial p_k}\frac{\partial^2 H}{\partial p_i\partial p_j})$$
$$+p_{i+j+k}\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j}\frac{\partial H}{\partial p_k})$$

$$+\left(\frac{1}{2}\right)\sum_{\substack{i,j,k\geq 1\\ i>j=k}} i(i+j-2)(p_{i+j+k}z^2\frac{\partial^3 H}{\partial p_i\partial p_j\partial p_k}+p_{i+j+k}z(2\frac{\partial H}{\partial p_i}\frac{\partial^2 H}{\partial p_j\partial p_k}+\frac{\partial H}{\partial p_j}\frac{\partial^2 H}{\partial p_i\partial p_k}+\frac{\partial H}{\partial p_k}\frac{\partial^2 H}{\partial p_i\partial p_j})$$
$$+p_{i+j+k}\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j}\frac{\partial H}{\partial p_k})$$

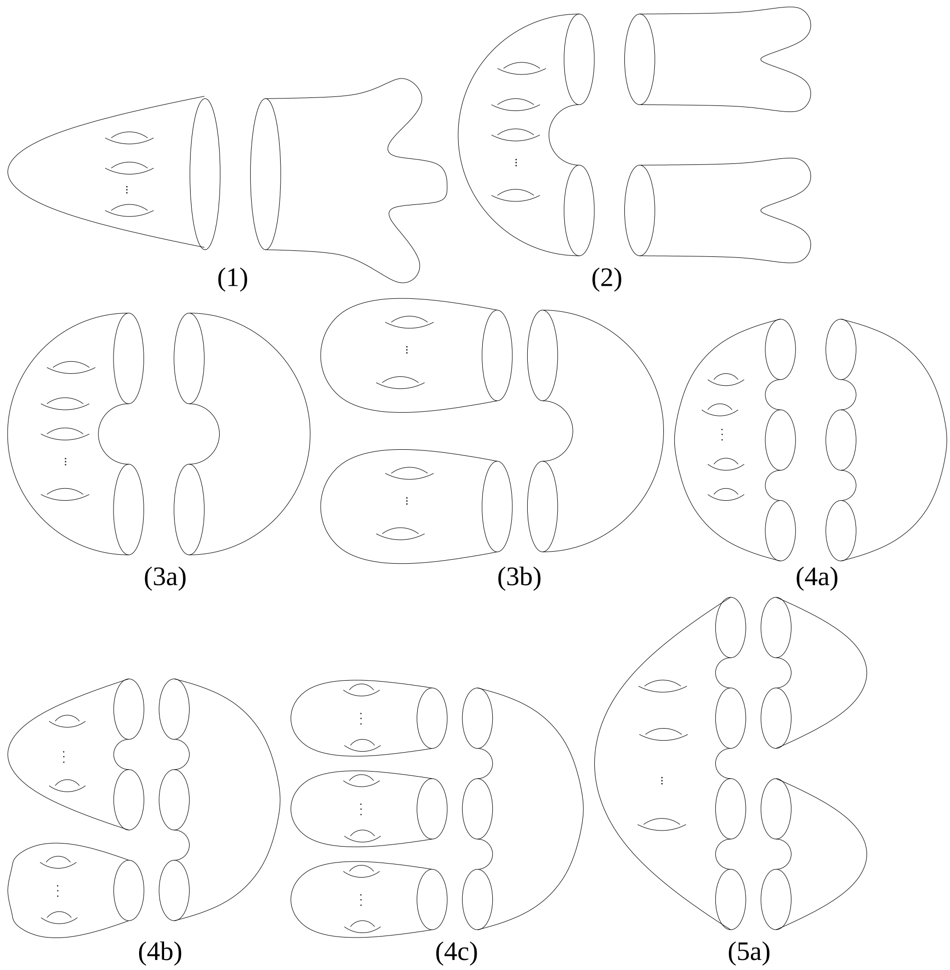
$$+\sum_{\substack{i,j,k\geq 1\\ i>j>k}} i(i+j-2)(p_{i+j+k}z^2\frac{\partial^3 H}{\partial p_i\partial p_j\partial p_k}+p_{i+j+k}z(\frac{\partial H}{\partial p_i}\frac{\partial^2 H}{\partial p_j\partial p_k}+\frac{\partial H}{\partial p_j}\frac{\partial^2 H}{\partial p_i\partial p_k}+\frac{\partial H}{\partial p_k}\frac{\partial^2 H}{\partial p_i\partial p_j})$$
$$+p_{i+j+k}\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j}\frac{\partial H}{\partial p_k})$$

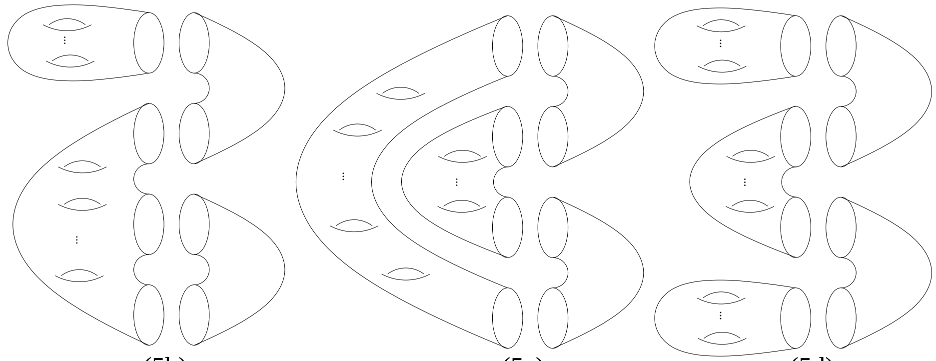
$$+\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\sum_{i,j,k,l\geq 1}(i+j)(k+l)(p_{i+j}p_{k+l}z^2\frac{\partial^4 H}{\partial p_i\partial p_j\partial p_k\partial p_l}$$
$$+2p_{i+j}p_{k+l}z\frac{\partial^3 H}{\partial p_i\partial p_j\partial p_k}\frac{\partial H}{\partial p_l}+p_{i+j}p_{k+l}z\frac{\partial^2 H}{\partial p_i\partial p_l}\frac{\partial^2 H}{\partial p_j\partial p_k}+p_{i+j}p_{k+l}\frac{\partial^2 H}{\partial p_i\partial p_j}\frac{\partial H}{\partial p_k}\frac{\partial H}{\partial p_l})$$

$$+\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\sum_{i,j,k,l\geq 1} ij(k+l)(p_{i+j}p_kp_lz\frac{\partial^3 H}{\partial p_i\partial p_j\partial p_{k+l}}+2p_{i+j}p_kp_l\frac{\partial^2 H}{\partial p_i\partial p_{k+l}}\frac{\partial H}{\partial p_j})$$

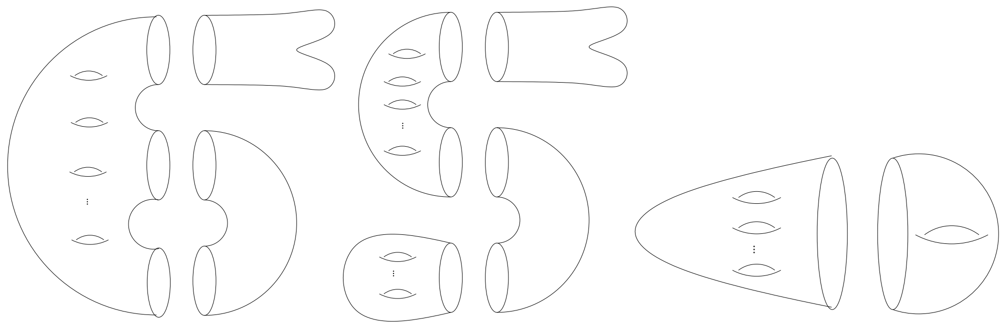
$$+\sum_{i\geq 2}\sum_{j=1}^{i-1}(i-j-1)(j-1)p_iz\frac{\partial H}{\partial p_i}$$

Proof. Although this recurrence looks atrocious, the reasoning behind it is identical to the previous recurrence. Several cases had to broken up into multiple summations to deal with compensating for overcounting this time, but the derivation of each case follows directly from the pictures below for the possible cases.


(1)
(2)
(3a)
(3b)
(4a)
(4b)
(4c)
(5a)



(5b) (5c) (5d)



(6a) (6b) (7)

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