HOMEOMORPHISM GROUPS OF THE SIERPINSKI CARPET AND SIERPINSKI GASKET

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Abstract. We study the topology of two particular subsets of $\mathbf{R}^2$, the Sierpinski carpet and the Sierpinski gasket. The goal is to determine the homeomorphism groups of these spaces.

1. Introduction

First, we define two subsets of $\mathbf{R}^2$, the Sierpinski carpet and the Sierpinski gasket.

Given $a \in [0, 1]$, let $0.a_1a_2a_3\ldots$ be a base-3 representation of a.

Definition 1. *The set*

$$C := \{(x, y) \in [0, 1] \times [0, 1] : \textit{ x and y admit base-3} \\ \textit{representations such that for no } n \in \mathbf{N} \textit{ are } x_n \textit{ and } y_n \textit{ both } 1.\}$$

is called the Sierpinski carpet.

This set can also be constructed by removing successive open squares from $[0, 1]\times[0, 1]$. The set of all points in $[0, 1]\times[0, 1]$ where both coordinates must have the digit 1 as the first digit of their base-3 expansion is $(\frac{1}{3}, \frac{2}{3}) \times (\frac{1}{3}, \frac{2}{3})$. There are eight smaller squares disjoint from this which correspond to those points for which both coordinates must have the digit 1 as the second digit of their base-3 expansion. After removing the squares corresponding to both coordinates having the digit 1 as the nth digit of their base-3 expansion for all n, we are left with the Sierpinski carpet.

Given $b \in [0, 1]$, let $0.b_1b_2b_3\ldots$ be a base-2 representation of b.

Definition 2. *The set*

$$G := \{(x, y, z)\} \in \mathbf{R}^3 : x + y + z = 1 \textit{ and } x,\ y, \textit{ and } z \\ \textit{admit base-2 representations such that for all } n \in \mathbf{N} \\ \textit{exactly one of } x_n,\ y_n, \textit{ and } z_n \textit{ is } 1\}$$

is called the Sierpinski gasket.

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Geometrically, one can think of the Sierpinski gasket as being constructed from the equilateral triangle with vertices $(0,0,1)$, $(0,1,0)$, and $(1,0,0)$ by removal of open triangular regions. The subset of this triangle of points all of whose coordinates must have base-2 expansion with the first digit 0 is the open triangle with vertices $(0,\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2},0,\frac{1}{2})$, and $(\frac{1}{2},\frac{1}{2},0)$. There are three smaller open triangles disjoint from this consisiting of points all of whose coordinates must have base-2 expansion with second digit 0. After removing triangles corresponding to open triangles of points all of whose coordinates must have base-2 expansion with nth digit 0 for all $n \in \mathbf{N}$, we are left with the Sierpinski gasket.

We investigate the topological structure of the Sierpinski carpet and Sierpinski gasket, leading to a proof that the homeomorphism group of the Sierpinski gasket is the isometry group of the triangle.

2. Definitions

Definition 3. *Let*

$$X_n := \{(x,y) \in C : \textit{ either } x \textit{ or } y \textit{ admits a base-}3 \textit{ representation which terminates in at most } n \textit{ digits}\}.$$

The points of $[0,1]\times[0,1]$ with a terminating base-3 expansion in at most n digits form a grid of size $\frac{1}{3^n}$ on $[0,1]\times[0,1]$. A base-3 expansion which terminates in at most n digits cannot be required to have the digit 1 as x_m for $m \geq n$ since, by definition of terminating in n digits, x_m can be 0. Thus X_n consists of the points in the grid of size $\frac{1}{3^n}$ on $[0,1]\times[0,1]$ minus those points which are removed in the first $n-1$ stages of constructing the Sierpinski carpet.

Definition 4. *Let*

$$V_n := \{(x,y) \in C : \textit{ both } x \textit{ and } y \textit{ admit a base-}3 \textit{ representation which terminates in at most } n \textit{ digits}\}.$$

The points of $[0,1]\times[0,1]$ for which x and y each admit a base-2 expansion which terminates in at most n digits form a lattice in $[0,1]\times[0,1]$ with horizontal and vertical spacing $\frac{1}{2^n}$. If a point of this lattice is not removed in the first $n-1$ steps of constructing the Sierpinski carpet, it is in V_n. Observe that the points of V_n are exactly the points which are vertices of a square formed by X_n.

Definition 5. *Let*

$$Y_n := \{(x,y,z) \in G : \textit{ at least one of } x, y, \textit{ and } z \textit{ admits a base-}2 \textit{ representation which terminates in at most } n \textit{ digits}\}.$$

The points of $\{(x, y, z) : x + y + z = 1; x, y, z \geq 0\}$ for which one of x, y, and z admits a base-2 representation which terminates in at most n digits forms a grid on $\{(x, y, z) : x + y + z = 1; x, y, z \geq 0\}$ which partitions the original triangle $\{(x, y, z) : x + y + z = 1; x, y, z \geq 0\}$ into 4^n triangles with sides of length $\frac{\sqrt{2}}{2^n}$. Y_n consists of the points on this grid which are not removed in the first $n-1$ stages of constructing the Sierpinski gasket.

Definition 6. *Let*

$$Z_n := \{(x, y, z) \in G : x,\ y,\ \text{and } z \text{ each admit a base-}2 \text{ representation which terminates in at most } n \text{ digits}\}.$$

The points of $\{(x, y, z) \in \mathbf{N} : x + y + z = 1; x, y, z \geq 0\}$ for which x, y, and z each admit a base-2 expansion which terminates in at most n digits form a lattice in $\{(x, y, z) : x + y + z = 1; x, y, z \geq 0\}$. Z_n consists of the points of this lattice which are not removed in the first $n - 1$ stages of constructing the Sierpinski gasket. For example,

$$Z_0 = \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$$
$$Z_1 = \{(0, 0, 1), (0, 1, 0), (1, 0, 0), (0, \tfrac{1}{2}, \tfrac{1}{2}), (\tfrac{1}{2}, 0, \tfrac{1}{2}), (\tfrac{1}{2}, \tfrac{1}{2}, 0)\}.$$

In particular, the points of Z_n are exactly the points which are the vertex of a triangle formed by Y_n.

3. Preliminaries

Proposition 3.1. *The Sierpinski carpet and Sierpinski gasket are compact.*

Proof. The Sierpinski carpet and Sierpinski gasket are both constructed by removal of open sets from a closed set, so they are closed. They are both also obviously bounded. Thus they are closed, bounded subsets of euclidean space, and hence compact. □

Lemma 3.2. *Let $\{a_i\}$ be a collection of line segments in $\mathbf{R}^n$, with a_i having endpoints x_i and y_i. Let $y_i = x_{i+1}$ for all $i \in \mathbf{N}$. If $\sum_{i\in\mathbf{N}} |a_i| = s$ is finite, then $lim_{n\to\infty} x_n$ exists and there exists a continuous map $f : [0, 1] \to \mathbf{R}^n$ defined by sending 1 to $lim_{n\to\infty} x_n$ and $[\sum_{i=1}^{m-1} \frac{|a_i|}{s}, \sum_{i=1}^{m} \frac{|a_i|}{s}]$ to a_m at unit speed with $\sum_{i=1}^{m-1} \frac{|a_i|}{s}$ being mapped to x_i and $\sum_{i=1}^{m} \frac{|a_i|}{s}$ being mapped to y_i.*

Proof. Since s is finite, for all $\epsilon > 0$, there exists $n \in \mathbf{N}$ such that $\sum_{i=n}^{\infty} |a_i| < \epsilon$. Thus the sequence $\{x_i\}$ is Cauchy. Since $\mathbf{R}^n$ is complete, this means $\{x_i\}$ is convergent. Define a function $f : [0, 1] \to \mathbf{R}^n$ by sending 1 to $lim_{n\to\infty} x_n$ and the interval $[\sum_{i=0}^{m} \frac{|a_i|}{s}, \sum_{i=0}^{m+1} \frac{|a_i|}{s}]$ to

$[x_m, x_{m+1}]$ at unit speed for all $m \in \mathbf{N}$. f is obviously continuous at all points other than 1, so we check continuity at 1. Let $\epsilon > 0$. Now choose $n \in \mathbf{N}$ such that $s - \sum_{i=1}^{n} |a_i| < \epsilon$. Then $\sum_{i=n+1}^{\infty} |a_i| < \epsilon$. $\rho(x_i, lim_{n\to\infty} x_n) < \epsilon$ for all $i > n$, and therefore $\rho(r, lim_{n\to\infty} x_n) < \epsilon$ for all r in a_i where $i > n$. Let $\delta = 1 - f^{-1}(x_{n+1})$. Then whenever $1 - t < \delta$, $\rho(f(t), lim_{n\to\infty} x_n) < \epsilon$, so f is continuous. □

4. Connectedness

Proposition 4.1. *The Sierpinski carpet is path-connected.*

Proof. Let $z = (x, y) \in C$. Let $x_0 = 0$ and $y_0 = 0$. Let $x_i \in [0, 1]$ be the largest number with base-3 expansion terminating in i digits and $x_i \leq x$. Let $y_i \in [0, 1]$ be the largest number with base-3 expansion terminating in i digits and $y_i \leq y$. Let $z_i = (x_i, y_i)$. We will construct a path from z to z_0. Let a_i be the line segment from (x_i, y_i) to (x_i, y_{i+1}). Let b_i be the line segment from (x_i, y_{i+1}) to (x_{i+1}, y_{i+1}). Since (x_i, y_i) and (x_{i+1}, y_{i+1}) are in V_n, they are vertices of squares in X_n, which means that a_i and b_i are contained in X_n. Thus all of the paths $\{a_i\}$ and $\{b_i\}$ are in C. Since the endpoints agree, the segments can be composed in the order $a_0, b_0, a_1, b_1, a_2, b_2, \ldots$. The lengths of a_i and b_i are bounded by $2(\frac{1}{3})^i$, so the total length of the path given by the sequence $a_0, b_0, a_1, b_1, a_2, b_2, \ldots$ is bounded by $2\sum_{i=1}^{\infty} 2(\frac{1}{3})^i$, which is finite. Since numbers with a terminating base-3 expansion are dense in [0,1], $lim_{n\to\infty} z_n = z$. Thus by lemma 3.2 there is a path from z_0 to z. Given $v, w \in C$, we can construct a path from v to z_0 and z_0 to w, so by concatenation this gives a path from v to w. Thus, C is path-connected. □

Proposition 4.2. *The Sierpinski gasket is path-connected.*

Proof. Let $w = (a, b, c) \in G$. Let $w_0 = (1, 0, 0)$. Let $w_i = (x_i, y_i, z_i)$ be the point of Z_i with minimal x-coordinate and $x_i \geq a$. If there are multiple such points, choose the one with greatest y-component as w_i. Let a_i be the line segment from w_{i-1} to w_i. Since w_{i-1} and w_i are both in Z_i, they are vertices of triangles formed by Y_i, which means that a_i is contained in Y_i. Thus a_i is contained in G. The length of a_i is at most $\frac{1}{2^i}$, so $s = \sum_{i=1}^{\infty} |a_i| \leq \sum_{i=1}^{\infty} \frac{1}{2^i} = 1$. Thus the a_i can be concatenated to give a path of finite length. Since the set of points whose coordinates all admit terminating base-2 expansions is dense in G, $lim_{n\to\infty} w_n = w$. Then by lemma 3.2, there is a path from w_0 to w. Given $u, v \in G$, we can construct a path from u to w_0 and w_0 to v, so by concatenation this gives a path from u to v. Thus, G is path-connected. □

5. Local Cut Points

Definition 7. *Let X be a path-connected topological space. A point x in X is called a local cut point if there exists a path-connected open neighborhood U of x such that $U \backslash \{x\}$ is not path-connected.*

Lemma 5.1. *Given a path-connected open subset U of the Sierpinski carpet (resp., gasket), for any points x and y in U, there exists a path f from x to y in U such that for all t in $(0,1)$, $f(t)$ is contained in $\bigcup_{n\in\mathbf{N}} X_n (\bigcup_{n\in\mathbf{N}} Y_n)$.*

Sketch of Proof. For every $m \in \mathbf{N}$, let $x_m \in B(x, \frac{1}{2^m}) \cap \bigcup_{n\in\mathbf{N}} X_n$, and let $y_m \in B(y, \frac{1}{2^m}) \cap \bigcup_{n\in\mathbf{N}} X_n$. Let a_i be a path from x_i to x_{i+1} in $U \cap \bigcup_{n\in\mathbf{N}} X_n$, and let b_i be a path from y_i to y_{i+1} in $U \cap \bigcup_{n\in\mathbf{N}} X_n$. There is a path from x_1 to y_1 in $U \cap \bigcup_{n\in\mathbf{N}} X_n$ since both x_1 and y_1 are in some X_n. The $\{a_i\}$ give a path from x_1 to x, and the $\{b_i\}$ give a path from y_1 to y, so composing all these paths gives a path from x to y in $U \cap \bigcup_{n\in\mathbf{N}} X_n$.

For every $m \in \mathbf{N}$, let $x_m \in B(x, \frac{1}{2^m}) \cap \bigcup_{n\in\mathbf{N}} Y_n$, and let $y_m \in B(y, \frac{1}{2^m}) \cap \bigcup_{n\in\mathbf{N}} Y_n$. Let a_i be a path from x_i to x_{i+1} in $U \cap \bigcup_{n\in\mathbf{N}} Y_n$, and let b_i be a path from y_i to y_{i+1} in $U \cap \bigcup_{n\in\mathbf{N}} Y_n$. There is a path from x_1 to y_1 in $U \cap \bigcup_{n\in\mathbf{N}} Y_n$ since both x_1 and y_1 are in some Y_n. The $\{a_i\}$ give a path from x_1 to x, and the $\{b_i\}$ give a path from y_1 to y, so composing all these paths gives a path from x to y in $U \cap \bigcup_{n\in\mathbf{N}} Y_n$. □

Lemma 5.2. *Given a path-connected open subset U of the Sierpinski carpet and points x in $U \cap X_n$ and y in U but not in any X_i with $d(x,y) < \frac{1}{2^{n+1}}$, there exists a path from x to y in U which intersects X_n only at x.*

Sketch of Proof. If, by the construction being used, the path would come to X_n in the mth stage at a point other than x, stop that segment at the intermediary point of $V_{i+1} \backslash V_i$, and branch off of that path at the earliest possible point while still staying in U and moving closer to x. In this way the path will not come to X_n except at x, but the path will still reach x in finite length.

□

Lemma 5.3. *Given a path-connected open subset U of the Sierpinski gasket and points x in $U \cap Y_n$ and y in U and strictly contained in a triangle of Y_n on which x is on the boundary with $d(x,y) < \frac{1}{2^{n+1}}$, there exists a path from x to y in U which intersects Y_n only at x.*

Sketch of Proof. If, by the construction being used, the path would come to Y_n in the mth stage at a point other than x, stop that segment at the intermediary point of $Z_{i+1} \backslash Z_i$, and branch off of that path at the earliest possible point while still staying in U and moving closer to x. In this way the path will not come to Y_n except at x, but the path will still reach x in finite length. □

Proposition 5.4. *The Sierpinski carpet has no local cut points.*

Proof. If x is in the Sierpinski carpet but not in any X_n, then for any path-connected open neighborhood U of x and for any y and z in U different from x, by Lemma 5.1 there exists a path from y to z not containing x, so x is not a local cut point. So let x be in the Sierpinski carpet and in X_n. Let U be a path-connected open neighborhood of x and let y and z be points in U different from x. Let v be a point in $U \cup X_n$ different from x. By Lemma 5.2, there exists paths from y to v and v to z intersecting X_n only at v. Composing these paths gives a path from y to z not containing x, so x is not a local cut point. Hence the Sierpinski carpet has no local cut points. □

Proposition 5.5. *The set of local cut points of the Sierpinski gasket is* $\bigcup_{n \in \mathbf{N}} Z_n$.

Proof. If x is in the Sierpinski gasket but not in any Y_n, then for any path-connected open neighborhood U of x and for any y and z in U different from x, by Lemma 5.1 there exists a path from y to z not containing x, so x is not a local cut point. So let x be in the Sierpinski gasket and in Y_n. Let U be a path-connected open neighborhood of x and let y and z be points in U different from x. If every triangle created by Y_n either encloses both y and z or neither y and z, then let v be a point in $U \backslash \bigcup_{n \in \mathbf{N}} Y_n$ within $\frac{1}{2^{n+1}}$ of y and let w be a point in U in no Y_n within $\frac{1}{2^{n+1}}$ of z. By Lemma 5.3, there are paths from y to v and w to z not containing x, and by Lemma 5.1, there is a path from v to w not containing x. Composing these paths from y to v, v to w, and w to z gives a path from y to z not containing x. If however, y and z are in different triangles of Y_n, any continuous path from y to z must pass through x. It is possible for y and z to be in different triangles of Y_n if and only if x is in Z_n. Thus, a point of G is a local cut point if and only if it is in $\bigcup_{n \in \mathbf{N}} Z_n$. □

6. Homeomorphism Group of Sierpinski Gasket

Proposition 6.1. *The homeomorphism group of the Sierpinski gasket is the dihedral group of the triangle.*

Proof. Let $\{a, b, c\} = Z_1 \backslash Z_0$. These three points have the property that removal of any two of them from the Sierpinski gasket leaves a disconnected space. For if x is the vertex joining the two sides from which a point is removed and l is the length of a side of the original triangle, $G \cap B(x, \frac{l}{2})$ and $G \backslash \overline{B}(x, \frac{l}{2})$ are open sets covering G. They are disjoint because $B(x, \frac{l}{2}) \subseteq \overline{B}(x, \frac{l}{2})$. Now we consider when removal of two points from G leaves a disconnected space. Removal of a point from $G \backslash \bigcup_{n \in \mathbf{N}} Y_n$ has no effect since our paths do not pass through points not in $\bigcup_{n \in \mathbf{N}} Y_n$. Removal of two points from Y_n yields a disconnected space if an only if both of the points are in $Y_0 \backslash Z_0$ and are vertices of a the same triangle with edges contained in Y_n. Any disconnection of G must also disconnect Y_n, since if Y_n were connected and no points were removed from any Y_m for $m \geq n$, any point not in Y_n could be connected to Y_n, which would mean that G is connected. Consequently, the only pairs of points in G whose removal disconnects the space are those points in $Y_0 \backslash Z_0$ which are both vertices of the same same triangle in Y_n. The only points involved in multiple pairs with this property are a, b, and c. Consequently, $\{a, b, c\}$ must be fixed setwise under homeomorphism.

Now observe that $G \backslash \{a, b, c\}$ has three connected components, so these must be permuted setwise under homeomorphism. Call these components A, B, and C. Let h be a fixed homeomorphism of G. Now there exists an isometry i such that A, B, and C are fixed setwise under $i \circ h$. This means that $\overline{A}$, $\overline{B}$, and $\overline{C}$ are also fixed setwise under $i \circ h$. Thus $\overline{A} \cap \overline{B}$, $\overline{A} \cap \overline{C}$, and $\overline{B} \cap \overline{C}$ are fixed setwise under $i \circ h$. But these sets are just $\{a\}$, $\{b\}$, and $\{c\}$, so this means a, b, and c are fixed elementwise under $i \circ h$. Now each of $\overline{A}$, $\overline{B}$, and $\overline{C}$ are homeomorphic to G. Let g be a homeomorphism of $\overline{A}$ and let d, e, and f be the midpoints of the sides of the outer triangle of $\overline{A}$. Since $i \circ h$ fixes two vertices of the outer triangle of $\overline{A}$, the only isometry j of $\overline{A}$ such that $j \circ g$ fixes the three connected components of $\overline{A} \backslash \{d, e, f\}$ setwise is the identity. This same reasoning applies for $\overline{B}$ and $\overline{C}$, and extends to copies in finer partitions.

From this, we conclude that the image of any point of G which is the midpoint of a triangle at any level of partition under homeomorphism is uniquely determined by the isometry i. Since the set of such points is dense in G, the image of any point is uniquely determined by the isometry i. Hence the homeomorphism group is just the isometry group (D_3 or D_6, depending on notation). $\square$

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