EQUIDISTRIBUTION OF DESCENT-LIKE STATISTICS FOR THE HYPEROCTAHEDRAL GROUP

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Abstract. This paper proves that two descent statistics, the flag descent and minus descent, are equidistributed over the hyperoctahedral group.

1. Introduction

Statistics such as inversion number, major index, and descent number of a permutation have been well studied over the symmetric group S_n. In past years, these statistics have been generalized to the hyperoctahedral group B_n. In 2001, Adin and Roichman [2] and Adin, Brenti, and Roichman [1], introduced new statistics on B_n, the *negative descent*, the *negative major*, the *flag descent*, and the *flag major*. They proved that the pair of statistics (flag descent, flag major) and (negative descent, negative major) are equidistributed over B_n.

In this paper, we study the flag descent. We show how to generate a permutation in B_n with a desired flag descent number from a permutation in S_n with the appropriate descent number. We also introduce the *minus descent* statistic and prove that it is equidistributed with flag descent number over B_n. This partially answers a question raised in [1] (see Thm. 4.4, Cor. 4.5, and the discussion thereafter), of whether there exists a bijective proof of the equidistribution of (flag descent, flag major) and (negative descent, negative major).

2. Terminology

First, we need to define the terms and variables we will use in this paper. We have attempted to remain consistent as much as possible with the notation used in [1].

We define the *symmetric group* S_n to be the set of all bijections $[n] \to [n]$, where $[n] = \{1, 2, \ldots, n\}$. We represent each permutation in S_n as $\sigma = [\sigma_1, \ldots, \sigma_n]$, where $\sigma_i = \sigma(i)$.

We define B_n to be the set of all bijections $\pi : \pm[n] \to \pm[n]$ such that

$$\pi(-a) = -\pi(a)$$

where $\pm[n] = \{-n, \ldots, -1, 1, \ldots, n\}$. B_n is known as the *hyperoctahedral group* of rank n. For $\pi \in B_n$, we write $\pi = [\pi_{-n}, \ldots, \pi_{-1}, \pi_1, \ldots, \pi_n]$ to mean that $\pi(i) = \pi_i$. Note that each π can be identified by only $[\pi_1, \ldots, \pi_n]$. From here on we will distinguish between these two representations by calling $[\pi_{-n}, \ldots, \pi_{-1}, \pi_1, \ldots, \pi_n]$ a *signed-symmetric* permutation and representing it $\widetilde{\pi}$. We will call $[\pi_1, \ldots, \pi_n]$ a *signed* permutation and represent it with simply π. Thus, while $\widetilde{\pi} = [-2, 1, -1, 2]$ and $\pi = [-1, 2]$ represent the same bijection in B_n, we call $[-2, 1, -1, 2]$ signed-symmetric and $[-1, 2]$ signed.

For some permutation w (in B_n or S_n), we say a *descent* occurs at i if $w(i) > w(i + 1)$. The *descent set* $D(w)$ lists all the descents of w; that is, $D(w) = \{i : w(i) > w(i + 1)\}$. We let $\operatorname{des}(w)$ denote the cardinality of $D(w)$. We also define the *Eulerian polynomial* as the generating function for this statistic over the symmetric group:

$$S_n(t) = \sum_{\sigma \in S_n} t^{\mathrm{des}(\sigma)}.$$

Let $\mathrm{des}_{\mathrm{A}}(\pi)$ count the number of descents of a signed permutation; i.e., $\mathrm{des}_{\mathrm{A}}(\pi) = \mathrm{des}(\pi)$. For example, $\mathrm{des}_{\mathrm{A}}([3,-2,-4,1]) = 2$ (remember that a signed permutation only lists the values $\pi(1), \ldots, \pi(n)$ of π in B_n). Also define

$$\varepsilon_1(\pi) = \begin{cases} 1, & \text{if } \pi(1) < 0, \\ 0, & \text{otherwise.} \end{cases}$$

Now we can define the *flag descent number* of π,

$$\mathrm{fdes}(\pi) = 2 \cdot \mathrm{des}_{\mathrm{A}}(\pi) + \varepsilon_1(\pi)$$

Note that $\mathrm{fdes}(\pi)$ counts the number of descents of the signed-symmetric permutation, so we can write

$$\mathrm{fdes}(\pi) = \mathrm{des}(\widetilde{\pi})$$

For a permutation w comprised of n integers, let $|w| = [|w(1)|, |w(2)|, \ldots, |w(n)|\ ]$. Let $\mathrm{nsigns}(w) = \#\{i : w(i) < 0\}$. We can now define the *minus descent*,

$$\mathrm{mdes}(w) = \mathrm{des}(|w|) + \mathrm{nsigns}(w).$$

As an example, if $w = [-2,-3,-4,1]$, $\mathrm{fdes}(w) = 5$ and $\mathrm{mdes}(w) = 4$.

3. MAIN RESULT

The following is our main result.

Theorem 1. *We have*

$$\sum_{\pi \in B_n} t^{\mathrm{mdes}(\pi)} = \sum_{\pi \in B_n} t^{\mathrm{fdes}(\pi)}.$$

This theorem will follow from this equally interesting identity:

Theorem 2. *We have*

$$\sum_{\pi \in B_n} t^{\mathrm{fdes}(\pi)} = (1+t)^n S_n(t).$$

First, some lemmas.

Define the *positive subword* $v = v(w)$ of a signed-symmetric permutation w to be the permutation in S_n that remains after all the negative integers are removed. For example, if $w = [-1,2,4,-3,3,-4,-2,1]$, $v = [2,4,3,1]$. Given a permutation $\sigma \in S_n$, define $\widetilde{B}_\sigma = \{\widetilde{\pi} \in B_n$ such that $v = \sigma\}$. Let $\widetilde{B}_\sigma^+ = \{\widetilde{\pi} \in \widetilde{B}_\sigma$ such that $\pi(-n) > 0\}$ and similarly $\widetilde{B}_\sigma^- = \{\widetilde{\pi} \in \widetilde{B}_\sigma$ such that $\pi(-n) < 0\}$.

Lemma 1. *We have*

$$\sum_{\widetilde{\pi} \in \widetilde{B}_\sigma^+} t^{\mathrm{des}(\widetilde{\pi})} = t \cdot \sum_{\widetilde{\pi} \in \widetilde{B}_\sigma^-} t^{\mathrm{des}(\widetilde{\pi})}.$$

Proof. We can build $\widetilde{B}_\sigma$ by taking σ, breaking it into pieces, and fitting in the negative integers. There are $n-1$ places σ can be separated. So there are 2^{n-1} ways to break up σ, given by subsets S of $[n-1]$. Let's arbitrarily break up σ into k parts, $v_1, \ldots, v_k$, according to the subset $S = \{s_1, \ldots, s_{k-1}\}$. Let $v_i = [\sigma(s_{i-1}+1), \ldots, \sigma(s_i)]$ for $i = 2, \ldots, k-1$. We set $v_1 = [\sigma(1), \ldots, \sigma(s_1)]$ and $v_k = [\sigma(s_{k-1}+1), \ldots, \sigma(n)]$.

Now let $\bar{v}_i = [-\sigma(s_i), \ldots, -\sigma(s_{i-1}+1)]$ for $i = 2, \ldots, k$. We set $\bar{v}_1 = [-\sigma(s_1), \ldots, -\sigma(1)]$ and $\bar{v}_k = [-\sigma(n), \ldots, -\sigma(s_{k-1}+1)]$.

Then there are two signed-symmetric permutations which will maintain the separations in σ:

$$v_S^+ = v_1\bar{v}_k v_2 \bar{v}_{k-1} \ldots v_{k-1}\bar{v}_2 v_k \bar{v}_1$$

and

$$v_S^- = \bar{v}_k v_1 \bar{v}_{k-1} v_2 \ldots \bar{v}_2 v_{k-1} \bar{v}_1 v_k.$$

Since both v_S^+ and v_S^- are made from the same $v_1, \ldots, v_k$ and $\bar{v}_1, \ldots, \bar{v}_k$, any descents occurring within these parts will be present in both. The only descents left to count are those occurring in the drop from a positive part to a negative part. For v_S^+, there are k descents created in this manner. For v_S^-, there are only $k-1$.

□

Proof of Theorem 2. First we will show that for all v_S^- in $\widetilde{B}_\sigma$,

$$\operatorname{des}(v_S^-) = 2 \cdot |D(\sigma) \setminus S| + |S|$$

where S is the subset of $n-1$ dictating how to break up σ and $|A|$ denotes the cardinality of some set A.

Consider some $j \in D(\sigma) \setminus S$. So $\sigma(j) > \sigma(j+1)$ and $\sigma(j)$ is not separated from $\sigma(j+1)$. Then there is a descent from $\sigma(j)$ to $\sigma(j+1)$ and also from $-\sigma(j+1)$ to $-\sigma(j)$. Now consider some $j \in S$. Whether $\sigma(j) > \sigma(j+1)$ or $\sigma(j) < \sigma(j+1)$, there will now be one descent from $\sigma(j)$ to the negative part which separates it from $\sigma(j+1)$.

Note that $2 \cdot |D(\sigma) \setminus S| + |S| = \operatorname{des}(\sigma) + |S \triangle D(\sigma)|$.

Also note that there are $\binom{n-1}{0}$ subsets S for which $|S \triangle D(\sigma)| = 0$, $\binom{n-1}{1}$ subsets S for which $|S \triangle D(\sigma)| = 1$, and in general, $\binom{n-1}{k}$ subsets S for which $|S \triangle D(\sigma)| = k$. So we have

$$\sum_{v_S^- \in \widetilde{B}_\sigma^-} t^{\operatorname{des}(v_S^-)} = \binom{n-1}{0} t^{\operatorname{des}(\sigma)} t^0 + \binom{n-1}{1} t^{\operatorname{des}(\sigma)} t^1 + \cdots + \binom{n-1}{n-1} t^{\operatorname{des}(\sigma)} t^{n-1}.$$

From the binomial theorem, we know that $\sum_{k=0}^{n} \binom{n}{k} t^k = (1+t)^n$, so this simplifies to

$$\sum_{v_S^- \in \widetilde{B}_\sigma^-} t^{\operatorname{des}(v_s^-)} = (1+t)^{n-1} t^{\operatorname{des}(\sigma)}$$

By Lemma 1,

$$\sum_{v_S^+ \in \widetilde{B}_\sigma^+} t^{\operatorname{des}(v_S^+)} = t \cdot (1+t)^{n-1} t^{\operatorname{des}(\sigma)}.$$

Since $\widetilde{B}_\sigma = \widetilde{B}_\sigma^- \uplus \widetilde{B}_\sigma^+$, where $\uplus$ indicates disjoint union,

$$\sum_{\widetilde{\pi} \in \widetilde{B}_\sigma} t^{\operatorname{des}(\widetilde{\pi})} = (1+t)^n t^{\operatorname{des}(\sigma)}.$$

Furthermore,

$$B_n = \biguplus_{\sigma \in S_n} \widetilde{B}_\sigma,$$

so

$$\sum_{\pi \in B_n} t^{\mathrm{fdes}(\pi)} = \sum_{\widetilde{\pi} \in B_n} t^{\mathrm{des}(\widetilde{\pi})} = (1+t)^n S_n(t)$$

$\square$

Proof of Theorem 1. By Theorem 2, it will suffice to show that

$$\sum_{\pi \in B_n} t^{\mathrm{mdes}(\pi)} = (1+t)^n S_n(t).$$

Given $\sigma \in S_n$, define $B_\sigma = \{\pi \in B_n \text{ such that } |\pi| = \sigma\}$. Then there will be $\binom{n}{0}$ signed permutations π in B_σ with 0 negative signs, $\binom{n}{1}$ with 1 negative sign, and in general, $\binom{n}{k}$ with k negative signs. Then

$$\sum_{\pi \in B_\sigma} t^{\mathrm{mdes}(\pi)} = t^{\mathrm{des}(\sigma)} \binom{n}{0} t^0 + t^{\mathrm{des}(\sigma)} \binom{n}{1} t^1 + \cdots + t^{\mathrm{des}(\sigma)} \binom{n}{n} t^n.$$

By the binomial theorem, this simplifies to

$$\sum_{\pi \in B_\sigma} t^{\mathrm{mdes}(\pi)} = t^{\mathrm{des}(\sigma)} (1+t)^n.$$

Notice that

$$B_n = \biguplus_{\sigma \in S_n} B_\sigma,$$

so

$$\sum_{\pi \in B_n} t^{\mathrm{mdes}(\pi)} = (1+t)^n S_n(t).$$

$\square$

References

[1] R. Adin, F. Brenti, and Y. Roichman, *Descent numbers and major indices for the hyperoctahedral group*, Adv. in Appl. Math. **27** (2001), 210–224.

[2] R. Adin and Y. Roichman, *The flag major index and group actions on polynomial rings*, European J. Combin. **22** (2001), 431–446.