

Level Sets of the Takagi Function and Hausdorff Dimension

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September 2, 2007

Abstract

The Takagi function τ is a continuous non-differentiable function on the unit interval defined by T. Takagi in 1903. This paper bounds above by $\alpha = 0.773 < 1$ the box-counting dimension and Hausdorff dimension of all level sets $\tau(x) = y$ of the Takagi function with a method that generalizes to the Takagi-van der Waerden functions — a class of functions F_r with $F_2 = \tau$ — with r even. The argument uses induction that relies upon the self-similarity of $F_r(x)$ up to affine shifts.

Introduction

In 1903 T. Takagi [9] first proposed the function τ (now called the *Takagi function*), as an example of a continuous nowhere differentiable function.

Definition 0.1. Given the function

$$\ll x \gg := \min_{n \in \mathbb{Z}} |x - n|,$$

we define the *n*th approximation function (of the Takagi function) to be

$$\tau_n(x) := \sum_{k=0}^n \frac{\ll 2^k x \gg}{2^k}.$$

The *Takagi function* is defined to be

$$\tau(x) := \lim_{n \rightarrow \infty} \tau_n(x).$$

The functions τ_n are each continuous, piecewise linear functions with period 1 and converge uniformly to the continuous function τ , as $n \rightarrow \infty$. Takagi originally proved [9] that τ is nowhere differentiable for all $r \geq 2$, even though J.B. Brown and G. Kozłowski [3] have shown τ (and in fact all Takagi-van der Waerden functions) to be Lipschitz of order λ for all $0 < \lambda < 1$. Although nowhere differentiable, the Hausdorff dimension of the graph of the Takagi function τ is 1, shown by P. C. Allaart and K. Kawamura [1]. As expected of a nowhere differentiable continuous function, the level sets of τ , $L_y := \{x \in [0, 1] : \tau(x) = y\}$, vary dramatically in cardinality. For example, the level set L_0 is clearly finite, while both $L_{1/2}$ and $L_{2/3}$ are infinite, with the former countable and the latter uncountable [4, Sect. 7.2.1.3, Problem 82].

Y. Baba [2] calculates the maximum of the function τ to be $y_{\max} = 2/3$ (and the maximum of F_r to be $y_{r,\max} = \frac{2^2}{2 \cdot 2^2 - 2}$ for all even $r \geq 2$), and extends the work of B. Martynov [6] on maxima of the van der Waerden function, proving the Hausdorff dimension of the level set $L_{y_{\max}}$ to be $1/2$ (and that the level set of maximum height for F_r has Hausdorff dimension $1/2$ for even r).

This paper furthers the analysis of level sets of the Takagi function, with emphasis placed on Hausdorff dimension and countability. We establish an upper bound for the Hausdorff dimension of any set L_y by constructing a sequence of δ_n -covers, with $\delta_n = 2^{-n}$, whose cardinalities exhibit a submultiplicativity relation proven via a multiscale analysis of τ . We then bound the cardinalities of the first few covers in the sequence by computer calculation and prove the cardinality of the n th cover is less than $2^{\alpha n}$ for $\alpha = 0.773$. This, along with the submultiplicativity, is enough to run an induction argument, showing that the box-counting dimension, and consequently the Hausdorff dimension, of any level set of τ is at most α . This is our main theorem:

Theorem 0.1. *The box-counting dimension of any level set of the Takagi function L_y is at most $\alpha = 0.773$. Consequently, the Hausdorff dimension $\dim_{\mathcal{H}}(L_y) \leq 0.773$.*

Our method of method of proof is not specific to the Takagi function; it generalizes to any Takagi-van der Waerden function F_r with even $r \geq 2$.

Definition 0.2. Given an integer $r \geq 2$ and the function

$$\ll x \gg := \min_{n \in \mathbb{Z}} |x - n|,$$

we define the n th approximation function (of the r th Takagi-van der Waerden function) to be

$$F_{r,n}(x) := \sum_{k=0}^n \frac{\ll r^k x \gg}{r^k}.$$

The r th Takagi-van der Waerden function is defined to be

$$F_r(x) := \lim_{n \rightarrow \infty} F_{r,n}(x).$$

A multiscale analysis of F_r yields the same submultiplicativity relation for $\delta_n = 2^{-n}$ covers as in the case $r = 2$. We explore this generalization, and uncover that all one needs to obtain an upper bound α for the Hausdorff dimension of level sets of F_r for an arbitrary even $r \geq 2$ is for the cardinalities of the first few δ_n covers to be calculated with n th cover of cardinality less than $2^{\alpha n}$.

1 Geometric Outline

This subsection demonstrates the main ideas of the first section of the paper from a geometric perspective, while saving the details of the full proof for the following subsections, so that the underlying geometry can motivate the reader through the algebra found in the rigorous proof.

To show that an arbitrary level set of τ has Hausdorff dimension strictly less than 1, it is enough to restrict our analysis to level sets of τ restricted to the unit interval because τ has period 1. To find an upper bound on the Hausdorff dimension of an arbitrary level set L_y , it will suffice to find an upper bound on the box-counting dimension as described in the following lemma found in Falconer [7, p. 54].

Lemma 1.1. *Let \mathfrak{N}_δ be the smallest number of sets of diameter at most δ that cover $F \subset \mathbb{R}^n$. Also suppose for any $k \in \mathbb{N}$, F can be covered by \mathfrak{n}_k sets of diameter at most δ_k with $\delta_k \rightarrow 0$ as $k \rightarrow \infty$. If $\delta_{k+1} \geq c\delta_k$ for some $0 < c < 1$, then*

$$\dim_{\mathcal{H}}(F) \leq \liminf_{\delta \rightarrow 0} \frac{\log \mathfrak{N}_\delta(F)}{-\log \delta} \leq \limsup_{\delta \rightarrow 0} \frac{\log \mathfrak{N}_\delta(F)}{-\log \delta} \leq \limsup_{k \rightarrow \infty} \frac{\log \mathfrak{n}_k}{-\log \delta_k}.$$

In particular, when the Hausdorff dimension $\dim_{\mathcal{H}}(F)$ and the box-counting dimension $\dim_B(F)$ exists, one has

$$\dim_{\mathcal{H}}(F) \leq \dim_B(F) \leq \limsup_{k \rightarrow \infty} \frac{\log \mathfrak{n}_k}{-\log \delta_k}.$$

Therefore, to prove that $\dim_{\mathcal{H}}(L_y) \leq \alpha$, it is enough to show that $\limsup_{k \rightarrow \infty} \frac{\log \mathfrak{n}_k}{-\log \delta_k} \leq \alpha$, which also shows that the box-counting dimension $\dim_B(L_y) \leq \alpha$.

Define $N_n(k)$ to be the maximum, taken over all $y \in \mathbb{R}$, of the number of intervals of the form $[\frac{p}{2^n}, \frac{p+1}{2^n}]$ with $p \in \mathbb{Z}$, and $0 \leq p \leq 2^n - 1$ so that for some $x \in [\frac{p}{2^n}, \frac{p+1}{2^n}]$, $\tau(x) + kx = y$. Notice that for any y , the level set L_y can be covered by $N_n(0)$ intervals of length 2^{-n} . Therefore, by the previous lemma, it is enough to prove that

$$\limsup_{n \rightarrow \infty} \frac{\log N_n(0)}{n \cdot \log 2} \leq \alpha.$$

Furthermore, if we let $M_n := \max_{k \in \mathbb{Z}}(N_n(k))$, and if we can show that $M_n \leq 2^{\alpha n}$ for all $n \in \mathbb{N}$, then we will have our result:

$$\limsup_{n \rightarrow \infty} \frac{\log N_n(0)}{n \cdot \log r} \leq \limsup_{n \rightarrow \infty} \frac{\log M_n}{n \cdot \log r} \leq \limsup_{n \rightarrow \infty} \frac{\alpha n \cdot \log 2}{n \cdot \log 2} \leq \alpha.$$

Thus, we have reduced to showing that $M_n \leq 2^{\alpha n}$ for all $n \in \mathbb{N}$.

To bound M_n , we use the self-similarity of the function τ . Notice that for any n , the function $\sum_{k=n}^{\infty} \frac{\llbracket 2^k x \rrbracket}{2^k}$ is a periodic function, with one cycle consisting of a copy of the graph of τ scaled down by a factor of 2^{-n} (see Figure 1). Since

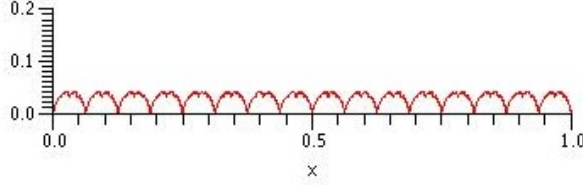


Figure 1: The graph of $\sum_{k=n}^{\infty} \frac{\llbracket 2^k x \rrbracket}{2^k}$ for $n = 4$.

each function $\frac{\llbracket 2^k x \rrbracket}{2^k}$ is a sawtooth function with derivatives ± 1 , the function $\tau_{n-1}(x) := \sum_{k=0}^{n-1} \frac{\llbracket 2^k x \rrbracket}{2^k}$ has constant derivative on intervals $[\frac{p}{2^n}, \frac{p+1}{2^n}]$ with magnitude less than n (see Figure 2).

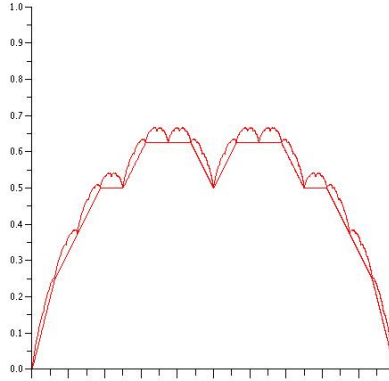


Figure 2: The graph of τ_n and τ for $n = 3$.

Recall we defined the number M_n as the supremum, over all $y \in \mathbb{R}$ and all $k \in \mathbb{Z}$, of the number of intervals of the form $[\frac{p}{2^n}, \frac{p+1}{2^n}]$ containing a solution to $\tau(x) + kx = y$. In particular, as M_{n+m} is the maximum of a bounded collection of integer values, there must be some $y \in \mathbb{R}$ and $k \in \mathbb{Z}$ so that M_{n+m} precisely equals $\#S$ for

$$S := \left\{ \left[\frac{p}{2^{n+m}}, \frac{p+1}{2^{n+m}} \right] \subset \mathbb{R} : 0 \leq p < 2^n, \exists x \in I_p \text{ with } \tau(x) + kx = y \right\}.$$

We know these M_{n+m} elements of S must all fall within at most M_n intervals of form $[\frac{p}{2^n}, \frac{p+1}{2^n}]$, else there would be greater than M_n intervals of the form

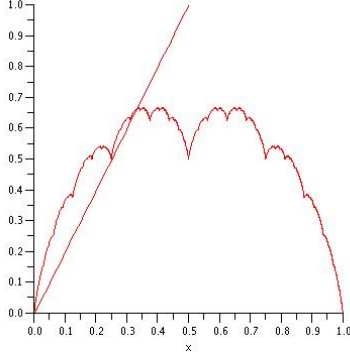


Figure 3: The graphs intersect at solutions of $\tau(x) + kx = y$ for $k = -2$ and $y = 0$.

$[\frac{p}{2^n}, \frac{p+1}{2^n}]$ in which there exists an x such that $\tau(x) + kx = y$, contradicting the definition of M_n . Let $M'_n(m)$ be the maximum number of elements of S that fall within any one interval of the form $[\frac{p}{2^n}, \frac{p+1}{2^n}]$. By counting, we have $M_{n+m} \leq M_n \cdot M'_n(m)$.

Now, on each interval of the form $[\frac{p}{2^n}, \frac{p+1}{2^n}]$, the graph of the function τ consists of the graph of one period of the function τ scaled down by a factor of 2^{-n} and offset by a linear section of constant derivative at most n , as can be seen in Figure 2. Therefore, after scaling by 2^n , to find an upper bound on $M'_n(m)$, it is enough to find the maximum number of intervals of length 2^{-m} of the form $[\frac{p}{2^m}, \frac{p+1}{2^m}]$ within which there exists some x satisfying $\tau(x) + k'x = y'$, for any fixed y' , and k' ; but this is precisely the definition of M_m . Thus, $M'_n(m) \leq M_m$, which motivates proof of the lemma:

Lemma 1.2 (Submultiplicativity Lemma). *For $n, m \in \mathbb{N}$,*

$$M_{n+m} \leq M_n \cdot M_m.$$

Assume we verify by calculation that $M_n \leq 2^{\alpha n}$ for all n with $k \leq n < 2k$ for some integer k and some $\alpha < 1$. Then by inducting on $n \geq k$, we obtain

$$M_n \leq 2^{\alpha n},$$

thus concluding the proof that an arbitrary level set L_y has Hausdorff dimension less than α . We make all the previous arguments rigorous and perform all calculations in the following two sections.

2 Submultiplicativity of Covers of L_y

First, we prove the property of the $(n-1)$ th approximation function τ_{n-1} giving τ the affine self-similarity that will be the crux of our following argument. We shall reference this lemma frequently throughout the rest of the paper.

Lemma 2.1. For any integer p with $0 \leq p \leq 2^n - 1$, the function τ_{n-1} has constant derivative $\frac{d}{dx}\tau_{n-1}(x) = D$ on the interval $[\frac{p}{2^n}, \frac{p+1}{2^n}]$ with $D \in \mathbb{Z}$ and $|D| \leq n$.

Proof. By definition, $\tau_{n-1}(x) = \sum_{k=0}^{n-1} \frac{\ll 2^k x \gg}{2^k}$, with each function $\frac{\ll 2^k x \gg}{2^k}$ having constant derivative with magnitude 1 when restricted to intervals of the form $[\frac{p}{2 \cdot 2^k}, \frac{p+1}{2 \cdot 2^k}]$. For all $k \geq 1$,

$$\left[\frac{p}{2 \cdot 2^k}, \frac{p+1}{2 \cdot 2^k} \right] = \left[\frac{p/r}{2 \cdot 2^{k-1}}, \frac{(p+1)/r}{2 \cdot 2^{k-1}} \right] \subseteq \left[\frac{\lfloor p/r \rfloor}{2 \cdot 2^{k-1}}, \frac{\lfloor p/r \rfloor + 1}{2 \cdot 2^{k-1}} \right].$$

This implies that for some $p' \in \{0, 1\}$,

$$\begin{aligned} \left[\frac{p}{2 \cdot 2^k}, \frac{p+1}{2 \cdot 2^k} \right] &\subseteq \left[\frac{\lfloor p/r \rfloor}{2 \cdot 2^{k-1}}, \frac{\lfloor p/r \rfloor + 1}{2 \cdot 2^{k-1}} \right] \\ &\subseteq \left[\frac{\lfloor \lfloor p/r \rfloor / r \rfloor}{2 \cdot 2^{k-2}}, \frac{\lfloor \lfloor p/r \rfloor / r \rfloor + 1}{2 \cdot 2^{k-2}} \right] \subseteq \dots \subseteq \left[\frac{p'}{2}, \frac{p'+1}{2} \right]. \end{aligned} \quad (1)$$

In particular, for any $k, k' \in \mathbb{N}$ with $k' \leq k$ and each integer $0 \leq p \leq 2^k - 1$, there exists some integer $0 \leq p' \leq 2^{k'} - 1$ such that $[\frac{p}{2 \cdot 2^k}, \frac{p+1}{2 \cdot 2^k}] \subseteq [\frac{p'}{2 \cdot 2^{k'}}, \frac{p'+1}{2 \cdot 2^{k'}}]$.

Consequently, for any $k \leq n - 1$, the function $\frac{\ll 2^k x \gg}{2^k}$ has constant derivative of magnitude 1 when restricted to intervals of the form $[\frac{p}{2 \cdot 2^{n-1}}, \frac{p+1}{2 \cdot 2^{n-1}}]$. Therefore, $\tau_{n-1}(x)$ has constant derivative on intervals of the form $[\frac{p}{2 \cdot 2^{n-1}}, \frac{p+1}{2 \cdot 2^{n-1}}] = [\frac{p}{2^n}, \frac{p+1}{2^n}]$ with integral magnitude at most n . \square

Definition 2.1. For integers $n \geq 1, r \geq 2, k \in \mathbb{Z}$, and each $y \in \mathbb{R}$ set

$$C_n(k; y) := \left\{ p \in \mathbb{Z} : 0 \leq p \leq 2^n - 1; \tau(x) + kx = y \text{ for some } x \in \left[\frac{p}{2^n}, \frac{p+1}{2^n} \right] \right\},$$

and define

$$N_n(k) := \sup_{y \in \mathbb{R}} \#(C_n(k; y)).$$

Conceptually, $N_n(k)$ is the maximal number of closed intervals of the form $[\frac{p}{2^n}, \frac{p+1}{2^n}]$ within which the restriction of the function $\tau(x) + kx$ intersects some fixed horizontal line $y = y_0$.

Definition 2.2. For integers $n \geq 1$, define

$$M_n := \sup_{k \in \mathbb{Z}} N_n(k).$$

Remark 2.1. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, the following properties are easily verified:

1. $N_n(k) \in \mathbb{N}$ with $N_n(k) \leq 2^n$,
2. $N_n(k) \leq N_{n+1}(k) \leq 2N_n(k)$,

3. $N_n(k) = \#C_n(k; y)$ for some $y \in \mathbb{R}$,
4. $M_n = \#C_n(k; y)$ for some $y \in \mathbb{R}$ and $k \in \mathbb{Z}$.

For each n , the value M_n is computable by inspection from the $N_n(k)$'s due to the following lemma:

Lemma 2.2. *For all $n \in \mathbb{N}$,*

1. $N_n(-k) = N_n(k)$ for all $k \in \mathbb{Z}$;
2. $N_n(k) \leq 2$ for $k \geq n + 1$.

Proof. First, let's show (1). Fix $y \in \mathbb{R}$ and $n, k \in \mathbb{Z}$ with $n \geq 1$. Notice that $\tau(x) + kx = y$ holds for some x if and only if

$$\tau(x) + kx + k(1-x) - k(1-x) = y.$$

Equivalently, since $\tau(1-x) = \tau(x)$ by symmetry,

$$\tau(1-x) - k(1-x) = y - k.$$

Note if $x \in \left[\frac{p}{2^n}, \frac{p+1}{2^n}\right]$ then $1-x \in \left[\frac{2^n-(p+1)}{2^n}, \frac{2^n-(p+1)+1}{2^n}\right]$. Since

$$C_n(k; y) := \left\{ p \in \mathbb{Z} : 0 \leq p \leq 2^n - 1; \tau(x) + kx = y \text{ for some } x \in \left[\frac{p}{2^n}, \frac{p+1}{2^n}\right] \right\},$$

we have just exhibited a bijection between $C_n(k; y)$ and $C_{\tau, n}(-k; y - k)$ defined by $p \mapsto 2^n - (p + 1)$. Therefore

$$\#C_n(k; y) = \#C_{\tau, n}(-k; y - k).$$

Taking suprema of both sides, yields:

$$N_n(k) = N_n(-k).$$

Now we will show (2). Fix $n \in \mathbb{N}$ and some $k \geq n + 1$. The derivative of τ_{n-1} at any point will have magnitude no greater than n by Lemma 2.1. Hence the derivative of $\tau_{n-1} + kx$ is at least 1, with constant derivative on intervals of the form $\left[\frac{p}{2^n}, \frac{p+1}{2^n}\right]$. In particular, the function $\tau_{n-1} + kx$ is strictly increasing.

We wish to show $\#C_n(k; y) \leq 2$ for all $y \in \mathbb{R}$. Fix some $y \in \mathbb{R}$, and let p_0 be the minimum element of $C_n(k; y)$. If $p_0 \geq 2^n - 2$, then there is nothing more to show. Assume that $p_0 \leq 2^n - 3$. Consequently, some $x \in \left[\frac{p_0}{2^n}, \frac{p_0+1}{2^n}\right]$ satisfies $\tau(x) + kx = y$. Since $\max_{x \in \mathbb{R}} \tau(x) = 2/3 < 1$,

$$y = \tau(x) + kx \quad (2)$$

$$= \tau_{n-1}(x) + \sum_{j=n}^{\infty} \frac{\ll 2^j x \gg}{2^j} + kx \quad (3)$$

$$= \tau_{n-1}(x) + 2^{-n} \tau(2^n x) + kx \quad (4)$$

$$< \tau_{n-1}(x) + 2^{-n} \cdot 1 + kx \quad (5)$$

$$\leq \tau_{n-1} \left(\frac{p_0 + 1}{2^n} \right) + \frac{k(p_0 + 1)}{2^n} + 2^{-n} \quad (6)$$

since $\tau_{n-1}(x) + kx$ is strictly increasing and $p_0 + 1 \geq x$.

For all $x > \frac{p_0+2}{2^n}$,

$$\tau(x) + kx \geq \tau_{n-1}(x) + kx \quad (7)$$

$$\geq \tau_{n-1} \left(\frac{p_0 + 1}{2^n} \right) + \frac{k(p_0 + 1)}{2^n} + 1 \cdot \left(x - \frac{p_0 + 1}{2^n} \right) \quad (8)$$

$$\geq \tau_{n-1} \left(\frac{p_0 + 1}{2^n} \right) + \frac{k(p_0 + 1)}{2^n} + 1 \cdot 2^{-n}, \quad (9)$$

since the minimum derivative of $\tau_n(x) + kx$ is 1. Combining these two results, we see that for $x > \frac{p_0+2}{2^n}$, we have $\tau(x) + kx > y$. Therefore, if $\tau(x) + kx = y$, then

$$x \in \left[\frac{p_0}{2^n}, \frac{p_0 + 2}{2^n} \right].$$

Thus, $N_n(k) \leq 2$. □

The symmetry property $\tau(x) = \tau(1 - x)$ implies $N_n(0) \geq 2$ for all $n \in \mathbb{N}$. Thus, as a direct consequence of the previous lemma, we have the following result.

Lemma 2.3. *For all integers $r \geq 2$,*

$$M_n = \sup_{0 \leq k \leq n} N_n(k).$$

This lemma reduces calculating a reasonable upper bound of M_n for any $n \in \mathbb{N}$ to a finite problem, given a method of calculating upper bounds of $N_n(k)$ for any $k \in \mathbb{Z}$. The next lemma will demonstrate such a method by rephrasing the problem as a question concerning the distribution of the values $\tau(\frac{p}{2^n})$ for $0 \leq p \leq 2^n$.

Lemma 2.4. *If $\tau(x) + kx = y$ for some $x \in [\frac{p}{2^n}, \frac{p+1}{2^n}]$ with integer $p \in [0, 2^n - 1]$ and integer $k \in [0, n]$, then*

$$y - \left(\frac{n+k}{2^n} + \frac{2}{3} \cdot \frac{1}{2^n} + \frac{kp}{2^n} \right) \leq \tau\left(\frac{p}{2^n}\right) \leq y + \frac{(n-k)}{2^n} - \frac{kp}{2^n}. \quad (10)$$

Remark 2.2. This lemma reduces finding an upper bound of $N_n(k)$ to finding the number of integers p satisfying inequality (10) for fixed r, n , and k . This is computable because $\tau(\frac{p}{2^n}) = \tau_n(\frac{p}{2^n})$ and the n th approximation function is a finite sum.

Proof.

$$y = \tau(x) + kx = \tau_{n-1}(x) + \sum_{i=n}^{\infty} \frac{\ll 2^i x \gg}{2^i} + kx \quad (11)$$

$$\leq \tau_{n-1}(x) + \frac{1}{2^n} \max(\tau(x)) + kx \quad (12)$$

$$\leq \tau_{n-1}\left(\frac{p}{2^n}\right) + \frac{1}{2^n} \max(\tau(x)) + n \cdot \frac{1}{2^n} + \frac{k(p+1)}{2^n} \quad (13)$$

$$= \tau\left(\frac{p}{2^n}\right) + \frac{1}{2^n} \cdot \frac{2}{3} + \frac{n+k}{2^n} + \frac{kp}{2^n}, \quad (14)$$

because $\max(\tau(x)) = \frac{2}{3}$ for even $r \geq 2$, as calculated by Y. Baba [2]. Also, the inequality (13) is justified because τ_{n-1} has constant derivative on intervals $[\frac{p}{2^n}, \frac{p+1}{2^n}]$ with magnitude at most n by Lemma 2.1. In summary, we have just shown

$$\tau\left(\frac{p}{2^n}\right) \geq y - \left(\frac{n+k}{2^n} + \frac{2}{3} \cdot \frac{1}{2^n} + \frac{kp}{2^n} \right). \quad (15)$$

For the other inequality,

$$y = \tau(x) + kx = \tau_{n-1}(x) + \sum_{i=n}^{\infty} \frac{\ll 2^i x \gg}{2^i} + kx \quad (16)$$

$$\geq \tau_{n-1}(x) + kx \quad (17)$$

But the derivative of $\tau_{n-1} + kx$ on the interval $[\frac{p}{2^n}, \frac{p+1}{2^n}]$ is at least $k - n$ by Lemma 2.1. Since $x \in [\frac{p}{2^n}, \frac{p+1}{2^n}]$,

$$\tau_{n-1}\left(\frac{p}{2^n}\right) + k\left(\frac{p}{2^n}\right) + (k-n)\left(x - \frac{p}{2^n}\right) \leq \tau_{n-1}(x) + kx.$$

Therefore,

$$\tau_{n-1}(x) + kx \geq \tau_{n-1}\left(\frac{p}{2^n}\right) + (k-n)\left(x - \frac{p}{2^n}\right) + k \cdot \frac{p}{2^n} \quad (18)$$

$$\geq \tau_{n-1}\left(\frac{p}{2^n}\right) + (k-n)\frac{1}{2^n} + k \cdot \frac{p}{2^n}, \quad (19)$$

since $k - n < 0$ and $x - \frac{p}{2^n} < \frac{1}{2^n}$. Combining the chain of inequalities before rearranging some terms yields

$$\tau\left(\frac{p}{2^n}\right) \leq y + \frac{(n-k)}{2^n} - \frac{kp}{2^n}. \quad (20)$$

Combining the inequalities from (15) and (20) yield the desired

$$y - \left(\frac{n+k}{2^n} + \frac{2}{3} \cdot \frac{1}{2^n} + \frac{kp}{2^n}\right) \leq \tau\left(\frac{p}{2^n}\right) \leq y + \frac{(n-k)}{2^n} - \frac{kp}{2^n}. \quad (21)$$

□

Given an upper bound for the set $\{N_n(k) : 0 \leq k \leq n\}$, Lemma 2.3 guarantees this upper bound will also bound M_n . Therefore, we can find upper bounds for any M_n for which we can compute the set $\{N_n(k) : 0 \leq k \leq n\}$. These computations provide the base case of the inductive argument of the proof of the main theorem. For the inductive step, we provide the following submultiplicativity relation:

Lemma 2.5 (Submultiplicativity Lemma). *For $n, m \in \mathbb{N}$,*

$$M_{n+m} \leq M_n \cdot M_m.$$

Proof. By Lemma 2.1, τ_{n-1} has constant derivative k_p on $[\frac{p}{2^n}, \frac{p+1}{2^n}]$. For any $x \in [\frac{p}{2^n}, \frac{p+1}{2^n}]$,

$$\begin{aligned} \tau(x) &= \sum_{k=0}^{n-1} \frac{\llcorner 2^k x \gg}{2^k} + \sum_{k=n}^{\infty} \frac{\llcorner 2^k x \gg}{2^k} \\ &= \tau_{n-1}(x) + 2^{-n} \sum_{k=0}^{\infty} \frac{\llcorner 2^k 2^n x \gg}{2^k} \\ &= \tau_{n-1}\left(\frac{p}{2^n}\right) + k_p \left(x - \frac{p}{2^n}\right) + 2^{-n} \sum_{k=0}^{\infty} \frac{\llcorner 2^k 2^n x \gg}{2^k} \\ &= \tau_{n-1}\left(\frac{p}{2^n}\right) + k_p \left(x - \frac{p}{2^n}\right) + 2^{-n} \tau(2^n x) \\ &= \tau_{n-1}\left(\frac{p}{2^n}\right) + k_p \left(x - \frac{p}{2^n}\right) + 2^{-n} \tau(2^n x - p), \end{aligned}$$

with the last step justified since $p \in \mathbb{Z}$. Therefore

$$\tau(x) = 2^{-n} \tau(2^n x') + k_p x' + K_p \quad (22)$$

where $K_p = \tau_{n-1}(\frac{p}{2^n})$ is a constant independent of the choice of $x \in [\frac{p}{2^n}, \frac{p+1}{2^n}]$, and where $x' := x - \frac{p}{2^n}$.

Let $k \in \mathbb{Z}$ be such that $M_{n+m} = N_{n+m}(k)$, and let y be such that $N_{n+m}(k) = \#C_{n+m}(k; y)$ (i.e. y is the value that maximizes the number of intervals of the form $[\frac{p}{2^{m+n}}, \frac{p+1}{2^{m+n}}]$ that nontrivially intersects the level set L_y). Recall,

$C_{n+m}(k; y)$ is the set of cardinality $N_{n+m}(k)$ containing precisely the integers p such that some $x \in \left[\frac{p}{2^{n+m}}, \frac{p+1}{2^{n+m}}\right]$ satisfies $\tau(x) + kx = y$. Notice that for each $p \in C_{n+m}(k; y)$ the interval $\left[\frac{p}{2^{n+m}}, \frac{p+1}{2^{n+m}}\right]$ must, by definition, be contained within one of the $N_n(k)$ intervals $\left[\frac{p'}{2^n}, \frac{p'+1}{2^n}\right]$ for $p' \in C_n(k; y)$. Let $M'_n(m)$ be the maximum number of the elements of $p \in C_{n+m}(k; y)$ such that $\left[\frac{p}{2^{n+m}}, \frac{p+1}{2^{n+m}}\right]$ is contained within any one fixed interval $\left[\frac{p'}{2^n}, \frac{p'+1}{2^n}\right]$. Concretely,

$$M'_n(m) := \sup_{p' \in C_n(k; y)} \# \left\{ p \in C_{n+m}(k; y) : \left[\frac{p}{2^{n+m}}, \frac{p+1}{2^{n+m}}\right] \subseteq \left[\frac{p'}{2^n}, \frac{p'+1}{2^n}\right] \right\}.$$

By a counting argument, it is apparent that $N_{n+m}(k) \leq N_n(k) \cdot M'_n(m)$. Our next step is to establish, for each $p' \in C_n(k; y)$, a 1-to-1 correspondence between each set

$$\left\{ p \in C_{n+m}(k; y) : \left[\frac{p}{2^{n+m}}, \frac{p+1}{2^{n+m}}\right] \subseteq \left[\frac{p'}{2^n}, \frac{p'+1}{2^n}\right] \right\}$$

and some subset of $C_{r,m}(k'; y')$ for some values k' and y' . This will allow us to conclude $M'_n(m) \leq M_m$. Choose p' such that

$$M'_n(m) = \# \left\{ p \in C_{n+m}(k; y) : \left[\frac{p}{2^{n+m}}, \frac{p+1}{2^{n+m}}\right] \subseteq \left[\frac{p'}{2^n}, \frac{p'+1}{2^n}\right] \right\}.$$

Assume for some $p \in C_{n+m}(k; y)$ we do have $\left[\frac{p}{2^{n+m}}, \frac{p+1}{2^{n+m}}\right] \subseteq \left[\frac{p'}{2^n}, \frac{p'+1}{2^n}\right]$. But $p \in C_{n+m}(k; y)$ by definition implies that there exists some $x \in \left[\frac{p}{2^{n+m}}, \frac{p+1}{2^{n+m}}\right] \subseteq \left[\frac{p'}{2^n}, \frac{p'+1}{2^n}\right]$ such that $\tau(x) + kx = y$. Let \tilde{p} be the unique integer $0 \leq \tilde{p} < 2^m$ so that $p = q \cdot 2^m + \tilde{p}$ for some $q \in \mathbb{Z}$. By (22),

$$y = 2^{-n} \tau(2^n x') + k_p x' + K_p,$$

for $x' = x - \frac{p'}{2^n} \in \left[\frac{\tilde{p}}{2^{n+m}}, \frac{\tilde{p}+1}{2^{n+m}}\right]$. Equivalently,

$$(y - K_p)2^n = \tau(x'') + k_p(x'')$$

for $x'' := 2^n x'$. Therefore $x'' \in \left[\frac{\tilde{p}}{2^m}, \frac{\tilde{p}+1}{2^m}\right]$. But this implies that

$$\tilde{p} \in C_m(k_p; (y - K_p)2^n).$$

Since the division algorithm guarantees that there is a one to one correspondence between such p and \tilde{p} , we can conclude that

$$M'_n(m) \leq \#C_m(k_p; (y - K_p)2^n) \leq M_m.$$

Therefore,

$$M_{n+m} = N_{n+m}(k) \leq N_n(k) \cdot M'_n(m) \leq M_n \cdot M_m.$$

□

3 Calculations and Theorem Bounding Hausdorff Dimension

We will use Submultiplicativity Lemma 2.5 and calculation to bound the Hausdorff dimension of any level set L_y . First, we will explicitly calculate upper bounds for $N_n(k)$ for small n . By Lemma 2.4, if $y \in \mathbb{R}$ and p is an integer $0 \leq p \leq 2^n - 1$ for which some $x \in [\frac{p}{2^n}, \frac{p+1}{2^n}]$ satisfies $\tau(x) + kx = y$, then

$$y - \left(\frac{n+k}{2^n} + \frac{2}{3} \cdot \frac{1}{2^n} + \frac{kp}{2^n} \right) \leq \tau\left(\frac{p}{2^n}\right) \leq y + \frac{(n-k)}{2^n} - \frac{kp}{2^n}. \quad (23)$$

Finding all $p \in C_n(k; y)$ reduces to analyzing the distribution of $\tau(\frac{p}{2^n}) + \frac{kp}{2^n}$. The next lemma expresses $\tau(\frac{p}{2^n})$ in terms of the binary expansion of p .

Definition 3.1. For any integer $m \geq 0$ let $s(m)$ denote the number of ones in the binary representation of m . For any integer $n \geq 0$, let

$$S(n) := \sum_{m=0}^n 2 \cdot s(m).$$

Lemma 3.1. For integers $p, n \geq 0$,

$$\tau\left(\frac{p}{2^n}\right) = (np - S(p))/2^n.$$

Proof. We begin by showing that the derivative of τ_n on the interval $[\frac{p}{2^n}, \frac{p+1}{2^n}]$ is precisely equal to the number of zeroes in the first n digits of the binary representation minus the number of ones in the first n digits of the binary representation of $\frac{p}{2^n}$ (with the choice of representation being the terminating one). Let the binary representation of $x = 0.\beta_1\beta_2\beta_3 \dots \in (\frac{p}{2^n}, \frac{p+1}{2^n})$. Notice that the binary representation of $p = \beta_k\beta_{k+1} \dots \beta_n$ where k is chosen so that $\beta_k = 1$ and $\beta_j = 0$ for all $j < k$. Moreover, $s(p)$ precisely equals the number of 1's among $\beta_1, \beta_2, \dots, \beta_n$. It is apparent from the graph that

$$\frac{d}{dx} \left(\frac{\ll 2^k x \gg}{2^k} \right) = \begin{cases} 1 & : \beta_k = 0 \\ -1 & : \beta_k = 1. \end{cases} \quad (24)$$

Thus, the derivative at x equals precisely the number of zeroes minus the number of ones in the first n terms of the sequence $\{\beta_i\}$. This derivative equals $n - 2s(p)$, and Lemma 2.1 guarantees this value to be the constant derivative over the entire interval $[\frac{p}{2^n}, \frac{p+1}{2^n}]$. Recall that $\tau(\frac{p}{2^n}) = \tau_{n-1}(\frac{p}{2^n})$. Therefore

$$\tau\left(\frac{p}{2^n}\right) = \tau_{n-1}\left(\frac{p}{2^n}\right) \quad (25)$$

$$= \sum_{i=0}^p 2^{-n} \cdot (n - 2s(i)) \quad (26)$$

$$= (np - S(p))/2^n \quad (27)$$

□

Lemma 3.1 applied to Inequality(23) guarantees that if $\tau(x) + kx = y$, then

$$\frac{1}{k+n} \left(y \cdot 2^n - \frac{2}{3} - (n+k) \right) \leq p - \frac{S(p)}{k+n} \leq \frac{1}{k+n} (y \cdot 2^n + n - k). \quad (28)$$

That is, for all $p \in C_n(k; y)$ we will have $p - \frac{S(p)}{k+n}$ fall within the interval

$$\left[\frac{1}{k+n} \left(y \cdot 2^n - \frac{2}{3} - (n+k) \right), \frac{1}{k+n} (y \cdot 2^n + n - k) \right]$$

of length $\frac{1}{k+n} (2n + 2/3)$. Now, if we define the upperbound

$$\mathcal{N}_n(k) := \sup_{x \in \mathbb{R}} \left(\# \left\{ 0 \leq p \leq 2^n - 1 : p - \frac{S(p)}{k+n} \in \left[x, x + \frac{1}{k+n} (2n + 2/3) \right] \right\} \right),$$

then it is clear that

$$\mathcal{N}_n(k) \geq N_n(k). \quad (29)$$

As long as we can compute $S(p) = 2 \cdot \sum_{i=0}^p s(i)$ for all p , we can construct a computer program that can calculate $\mathcal{N}_n(k)$. Since summation is easy, computing $S(p)$ reduces to generating the sequence $s(0), s(1), s(2), \dots$

Lemma 3.2. *For any integer $i \geq 0$,*

$$s(i) + 0 = s(2i); \quad (30)$$

$$s(i) + 1 = s(2i + 1). \quad (31)$$

Proof. If in binary, $i = b_1 \dots b_m$, then $2i = b_1 \dots b_m 0$ and $2i + 1 = b_1 \dots b_m 1$. \square

This suffices, as $s(0) = 0$, $s(1) = 1$, and if we can have $s(m)$ for all $0 \leq m \leq k$, then the lemma gives a formula for $s(m)$ for all $0 \leq m \leq 2k + 1$. Thus, we may implement a computer program to calculate $s(m)$, allowing us to compute $\mathcal{N}_n(k)$ for small n .

n	k	$\mathcal{N}_n(k)$	n	k	$\mathcal{N}_n(k)$
1	0	2	6	2	23
1	1	2	6	3	14
2	0	4	6	4	11
2	1	4	6	5	8
2	2	4	6	6	6
3	0	8	7	0	58
3	1	8	7	1	47
3	2	6	7	2	29
3	3	4	7	3	24
4	0	15	7	4	15
4	1	12	7	5	12
4	2	8	7	6	8
4	3	6	7	7	7
4	4	5	8	0	94
5	0	27	8	1	66
5	1	22	8	2	47
5	2	14	8	3	33
5	3	10	8	4	24
5	4	7	8	5	16
5	5	6	8	6	12
6	0	45	8	7	8
6	1	27	8	8	7

Remark 3.1. For $1 \leq n \leq 19$, calculation shows that

$$\sup_{0 \leq k \leq n} (\mathcal{N}_n(k)) = \mathcal{N}_n(0).$$

Question 3.1. Does $\sup_{0 \leq k \leq n} \mathcal{N}_n(k) = \mathcal{N}_n(0)$ for all $n \in \mathbb{N}$?

Now we can calculate

$$\mathcal{M}_n := \sup_{0 \leq k \leq n} \mathcal{N}_n(k)$$

for all $1 \leq n \leq 19$.

Remark 3.2. $\mathcal{M}_n \geq M_n$ for all $n \in \mathbb{N}$ because of Lemma 2.3 and because $\mathcal{N}_n(k) \geq N_n(k)$.

n	\mathcal{M}_n	$\frac{\log(\mathcal{M}_n)}{n \log(2)}$
1	2	1.00000
2	4	1.00000
3	8	1.00000
4	15	0.97672
5	27	0.95098
6	45	0.91531
7	58	0.83685
8	94	0.81932
9	152	0.80533
10	212	0.77279
11	348	0.76754
12	440	0.73178
13	728	0.73137
14	944	0.70590
15	1488	0.70261
16	2144	0.69163
17	3040	0.68058
18	4864	0.68044
19	6592	0.66771

Now that we have this data for the base case, the induction argument is simple.

Lemma 3.3. *For $r = 2$ and all $n \geq 10$, we have $M_n \leq 2^{\alpha n}$ for $\alpha := 0.773$.*

Proof. This holds for all $10 \leq n \leq 19$, as shown by computer calculation. Now assume for some $k - 1 \geq 19$, that all $n \leq k - 1$ satisfy $M_n \leq 2^{\alpha n}$. By induction, it is enough to show this for $n = k$. But since $k - 1 \geq 19$, it follows $k - 10 \geq 10$. Therefore, by Submultiplicativity Lemma 2.5 we have

$$M_k \leq M_{k-10} M_{10} \leq 2^{\alpha(k-10)} \cdot 2^{\alpha \cdot 10} = 2^{\alpha k}.$$

□

This relationship is strong enough to bound the box-counting dimension and Hausdorff dimension of any arbitrary level set L_y of the Takagi function with the help of the following lemma:

Lemma 3.4. *Let \mathfrak{N}_δ be the smallest number of sets of diameter at most δ that cover $F \subset \mathbb{R}^n$. Also suppose for any $k \in \mathbb{N}$, F can be covered by \mathfrak{n}_k sets of diameter at most δ_k with $\delta_k \rightarrow 0$ as $k \rightarrow \infty$. If $\delta_{k+1} \geq c\delta_k$ for some $0 < c < 1$, then*

$$\dim_{\mathcal{H}}(F) \leq \liminf_{\delta \rightarrow 0} \frac{\log \mathfrak{N}_\delta(F)}{-\log \delta} \leq \limsup_{\delta \rightarrow 0} \frac{\log \mathfrak{N}_\delta(F)}{-\log \delta} \leq \limsup_{k \rightarrow \infty} \frac{\log \mathfrak{n}_k}{-\log \delta_k}.$$

In particular, when the box-counting dimension $\dim_B(F)$ exists, one has

$$\dim_{\mathcal{H}}(F) \leq \dim_B(F) \leq \limsup_{k \rightarrow \infty} \frac{\log \mathfrak{n}_k}{-\log \delta_k}.$$

Proof. See proof of Proposition 4.1 in Falconer [7, p. 54]. \square

Theorem 3.1. *The box-counting dimension and Hausdorff dimension of any level set L_y of the Takagi function is at most $\alpha = 0.773$.*

Proof. By Lemma 3.4, it suffices to show that

$$\limsup_{n \rightarrow \infty} \frac{\log \mathbf{n}_n}{-\log \delta_n} \leq \alpha,$$

when L_y can be covered by \mathbf{n}_n sets of diameter at most δ_n with $\delta_n \rightarrow 0$ as $n \rightarrow \infty$ and $\delta_{n+1} \geq c \cdot \delta_n$ for some $0 < c < 1$. By the definition of $N_n(0)$, if $\delta_n := 2^{-n}$ then we can define $\mathbf{n}_n := N_n(0)$. Lemma 3.3 then guarantees

$$\mathbf{n}_n \leq M_n \leq 2^{\alpha n}.$$

Obviously, $\delta_{n+1} = \frac{1}{2}\delta_n$. Therefore, we have

$$\limsup_{n \rightarrow \infty} \frac{\log \mathbf{n}_n}{-\log \delta_n} \leq \limsup_{n \rightarrow \infty} \frac{\log 2^{\alpha n}}{-\log 2^{-n}} \leq \limsup_{n \rightarrow \infty} \frac{\alpha \log 2^n}{\log 2^n} = \alpha.$$

\square

Corollary 3.1.1. *Every level set of the Takagi function has Lebesgue measure zero.*

4 Generalizing to Takagi-van der Waerden Functions

The method of proof used to bound the Hausdorff dimension of level sets of the Takagi function generalizes for any Takagi-van der Waerden function F_r when r is even. This is due to the fact that Lemma 2.1 generalizes very nicely to Takagi-van der Waerden functions with even r :

Lemma 4.1. *For any integer p with $0 \leq p \leq r^n - 1$, the function $F_{r,n-1}$ has constant derivative $\frac{d}{dx}F_{r,n-1}(x) = D$ on the interval $[\frac{p}{2^n}, \frac{p+1}{2^n}]$ with $D \in \mathbb{Z}$ and $|D| \leq n$.*

Proof. The majority of the proof is identical after replacing 2^k everywhere with r^k . As a result, we obtain $F_{r,n-1}(x)$ has constant derivative on intervals of the form $[\frac{p}{2 \cdot r^{n-1}}, \frac{p+1}{2 \cdot r^{n-1}}]$ with integral magnitude at most n . We now use the evenness of r to finish the proof. Let $r' := r/2 \in \mathbb{N}$. For any $0 \leq p \leq r^n - 1$, we have

$$\left[\frac{p}{r^n}, \frac{p+1}{r^n} \right] = \left[\frac{2p/r}{2 \cdot r^{n-1}}, \frac{2(p+1)/r}{2 \cdot r^{n-1}} \right] \subseteq \left[\frac{\lfloor p/r' \rfloor}{2 \cdot r^{n-1}}, \frac{\lfloor p/r' \rfloor + 1}{2 \cdot r^{n-1}} \right]. \quad (32)$$

\square

Remark 4.1. Equation (32) only works for even r because only if $r' := r/2$ is a natural number can we guarantee that $(p+1)/r' \leq \lfloor p/r' \rfloor + 1$. For example, $r' = 3/2$ and $p = 1$ will provide a counterexample. For odd r , the self-similarity is more complicated; this proof fails and disintegrates the entire analysis that follows.

Next, if we define

$$C_{r,n}(k; y) := \left\{ p \in \mathbb{Z} : 0 \leq p \leq r^n - 1; F_r(x) + kx = y \text{ for some } x \in \left[\frac{p}{r^n}, \frac{p+1}{r^n} \right] \right\},$$

and define $N_{r,n}(k)$ and $M_{r,n}$ analogously, then the rest of the results from Subsection 2 follow almost identically after replacing 2 by r . In any case, we obtain the same submultiplicativity relation.

Lemma 4.2 (Generalized Submultiplicativity Lemma). *For even $r \geq 2$ and for $n, m \in \mathbb{N}$,*

$$M_{r,n+m} \leq M_{r,n} \cdot M_{r,m}.$$

If calculations analogous to those in Subsection 3 are performed for some even $r \geq 2$, then this Generalized Submultiplicativity Lemma 4.2 allows one to bound the Hausdorff dimension of any level set of the function F_r by using a comparable induction argument.

Acknowledgment: This paper was written as an REU project under the mentorship of J.C. Lagarias.

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