The Likelihood of Cyclic Results in Pairwise Elections

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1 Definitions and Background

Of the numerous methods of tallying votes in an attempt to determine "what the people want", only the Condorcet method will be considered in this paper. The method is named after the Marquis de Condorcet, who is credited with its invention in the 18th century, and is alternatively known as *pairwise voting*. In an election using this method, the winning candidate is the one who, when compared in turn with each of the other candidates, wins the most pairwise votes. For example, if in pairwise elections Alice would lose against Bob, Alice would win against Charlie, and Bob would win against Charlie, then the final ranking should be Bob then Alice then Charlie.

We will exclusively consider elections in which there are three alternatives to choose from. For such an election, a *voter profile* lists the number of voters of each type. Each *voter type* corresponds to one of the six possible rankings of three candidates:

Type 1	A $\succ$ B $\succ$ C
Type 2	A $\succ$ C $\succ$ B
Type 3	C $\succ$ A $\succ$ B
Type 4	C $\succ$ B $\succ$ A
Type 5	B $\succ$ C $\succ$ A
Type 6	B $\succ$ A $\succ$ C

Any particular profile can now be indicated by the vector $(n_1, n_2, n_3, n_4, n_5, n_6)$, where n_i is the number of voters of Type i. For a total electorate of n voters, a *normalized profile* is a vector $(p_1, p_2, p_3, p_4, p_5, p_6)$ where $p_i = n_i/n$. The normalized profile thus gives the portion of the electorate constituted by voters of each type.

Unfortunately, pairwise voting need not give conclusive election results. If, for example, our voter profile is $(2, 1, 2, 1, 2, 1)$, we see that A beats B, B beats C, and C beats A in pairwise elections, giving the result of A $\succ$ B $\succ$ C $\succ$ A. This outcome is referred to as a *positive cycle*, while the reverse outcome (C $\succ$ B, B $\succ$ A, A $\succ$ C) is a *negative cycle*. The choice of sign comes from one way of tallying pairwise results, in which the outcome is given as a vector (x, y, z). In this vector x is the portion of voters who prefer A to B minus the portion preferring B to A, y is the portion with B $\succ$ C preferences minus that with C $\succ$ B preferences, and z is the portion with C $\succ$ A minus that with A $\succ$ C. If all three terms are positive, the result is a positive cycle, and if all three are negative, the result is a negative cycle.

2 Statement of Problem and Results

In his article, “Geometry, Voting, and Paradoxes” (*Mathematics Magazine*, Vol. 71, No. 4, Oct. 1998), D. G. Saari considers the possible results of pairwise elections in which all voters are divided among three voter types, instead of the full six. He demonstrates that when all voters are of types 1, 3, and 5, or all are of types 2, 4, and 6, the portion of all profiles resulting in cyclic pairwise election outcomes approaches 1/4 when the number of voters approaches infinity. When the voters are constrained to any other three voter types, cyclic outcomes cannot occur. The question I chose to consider was what portion of profiles, when all six voter types are allowed, result in cycles.

For n voters, there are $\binom{n+5}{5}$ possible ways to divide them among the six voter types. Of these, we wish to know what portion correspond to cycles. Because positive and negative cycles are equally likely (simply switch odd voter types with even and the cycle type changes), we will simply double the number that correspond to positive cycles. For each of the three pairwise results in a positive cycle, there is a corresponding inequality that must hold among the number of voters of each type:

$$\begin{array}{ll} \mathrm{A} \succ \mathrm{B} & n_1 + n_2 + n_3 > n_4 + n_5 + n_6 \\ \mathrm{B} \succ \mathrm{C} & n_1 + n_5 + n_6 > n_2 + n_3 + n_4 \\ \mathrm{C} \succ \mathrm{A} & n_3 + n_4 + n_5 > n_1 + n_2 + n_6 \end{array}$$

The number of profiles of n voters resulting in positive cycles is the number of ways these three inequalities can hold when $\sum_{i=1}^{6} n_i = n$.

An even number of voters can produce tied outcomes, so the parity of the size of an electorate has a huge effect on the possible outcomes, especially for relatively small electorates. Indeed, for six or more voters, there are 27 possible election outcomes when the number of voters is even, and only 8 when it is odd. Because cycles require that no ties occur, it is to be expected that cyclic outcomes are more likely for an odd number of voters than for an even number, but that the difference will shrink as the number of voters increases.

For odd n, $\binom{\frac{1}{2}(n+7)}{5}$ voter profiles result in positive cyclic election outcomes. Doubling this, we get the number of profiles resulting in cycles of any kind, and then dividing by $\binom{n+5}{5}$ gives the probability of cycles for n voters for odd n. This simplifies to $\frac{(n-1)(n+7)}{16(n^2+6n+8)}$. For even n, the number of profiles resulting in positive cycles decreases to $\binom{\frac{1}{2}(n+4)}{5}$. Doubling this and dividing by $\binom{n+5}{5}$ gives $\frac{n(n-2)(n-4)}{16(1+n)(3+n)(5+n)}$.

For small n, the parity differences are striking. One or two voters, of course, have zero chance of producing a cyclic outcome. With 3 voters, the probability is $\frac{1}{28}$. For 4, it's again 0. For 5 voters, $\frac{1}{21}$; for 6, $\frac{1}{231}$; for 7, $\frac{7}{132}$; and for 8, $\frac{4}{429}$. Even when the number of voters grows to the relatively large value of 20, the likelihood of a cyclic outcome ($\frac{24}{805}$) is less than half of what it is for 21 voters ($\frac{7}{115}$).

So while the portion of profiles resulting in cyclic outcomes approaches $\frac{1}{16}$ as the number of voters becomes very large, but it approaches much faster for odd numbers of voters than for even.