Temperley-Lieb Algebra Research Project

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Abstract: Given some special subsets of the Temperley-Lieb algebra, we investigate the properties of this algebra in order to find more meaningful ways to classify the special subsets.

I. Introduction

An interesting topic in modern algebra and combinatorics is the study of classes of totally nonnegative polynomials on n x n matrices. Special cases of these polynomials can be studied by counting path families in planar networks. This problem, in turn, corresponds to comparing various subsets of the Temperley-Lieb algebra, T_n.

To produce the Temperley-Lieb elements, two columns of n vertices each are drawn. Each vertex is connected to another vertex such that no paths cross or touch, forming n paths. This forms a set of Temperley-Lieb elements for each value of n. All of the elements of one of these sets of T_n can be concatenated with any other(s) to produce some element in the same set. The operation is closed and we therefore have an algebra.

For this research project we were given a set of special subsets of T_n that had been generated using dot diagrams containing n solid and n empty dots. To produce the special subsets, the rules for generating the Temperley-Lieb elements were followed, with the addition of the rule that each solid dot must connect to an empty dot and vice versa. This means that if a dot diagram has the positions of its solid and empty dots interchanged, it is still the same diagram. Each of these dot diagrams then produces a particular set of Temperley-Lieb elements, called its special subset.

The purpose of our research was to discover some properties of these special subsets, and ultimately find a better way to describe them than by using the dot diagrams.

II. Left-Right Pairs

One of the first methods we used to look at the special subsets was to identify two elements we together called the Left-Right Pair for each subset. We start with a dot diagram D as described in section I. To produce the Left element of this pair, we used the following steps on a dot diagram of solid and empty dots:

1. For each empty dot with a solid dot immediately following it in the clockwise direction, connect them.
2. Ignore all previously connected dots and repeat step 1.
3. Repeat until each dot is connected to another dot.

To produce the Right element of the Left-Right pair, the same steps are used only with the substitution of counterclockwise for clockwise in step 1. Let L and R be the two elements created by the above algorithm.

To produce a graph G representing the Left-Right Pair, rather than each element separately, we “superimpose” L and R, meaning that G will be composed of all paths in both L and R (i.e. all paths in both L and R will be contained in G.) Using the above three steps followed by the superimposition completes the instructions of what we called the Left-Right Algorithm.

However, even without superimposing the graphs of L and R we are left with two elements that uniquely represent each dot diagram; or equivalently, we have isolated two elements from each special subset to uniquely represent it. A relevant question now is whether the rest of the special subset can be produced given only the Left-Right pairs. To answer this question we use the idea of set partitions.

III. Set Partitions

In order to relate the idea of set partitions to the elements of the Temperley-Lieb algebra, we will first introduce some definitions and notations.

As described in section I, any Temperley-Lieb element is created from a dot diagram, D, of 2n vertices, n solid and n empty. To create the subsets also described in section I, a map must be defined to associate a special subset of T_n with D. Let □ be this map, such that: □(D) ={ x □ T_n□x is compatible with D}.

A set partition is simply another way to view any graph G, and therefore any Temperley-Lieb element. The set partition [2n] = {1, …, 2n} forms blocks of connected components of a graph. For instance, if a graph with six vertices has connected vertices v_1, v_2, v_5, v_6 and v_3, v_4; the set partition is written [6] = {1, 2, 5, 6} U {3, 4}.

An idea related to set partitions is that of refinement. If some of the blocks in set partition A are combined into one block in set partition B, then A refines B. For instance, the set partition {1,5} U {2, 6} U {3, 4} refines the set partition {1, 2, 5, 6} U {3, 4} from above.

Suppose we let G_1 and G_2 be two graphs created from dot diagrams D_1 and D_2 using the Left-Right Algorithm as described in section II. If the set partition corresponding to G_1 refines the set partition corresponding to G_2, it means that some blocks in the set partition of G_1 are combined into one block in the set partition of G_2. Since these blocks simply refer to connected components, or paths, of the graphs, this means that the paths of G_1 all exist within the paths of G_2. Recall that □(D) is the special subset of T_n associated with a dot diagram D, which in turn is associated by the Left-Right Algorithm with a graph G. Because all the paths of G_1 are contained in the paths of G_2, it means that the special subset □(D_1) is contained in the special subset □(D_2).

IV. Constructing Special Subsets

We can now return to the question of how to construct an entire special subset of T_n given only its Left-Right Pair. The smallest n such that T_n has more than one special subset containing more than two elements is T_4 so we will start with the fourth Temperley-Lieb algebra. It contains 35 special subsets, including 31 with at least two elements. We will focus on constructing these 31 subsets given their Left-Right Pairs.

We describe the elements as set partitions. Elements of the fourth Temperley-Lieb algebra have four components with two vertices each. Therefore, they will have a set partition of the form [8] = {a, b} U {c, d} U {e, f} U {g, h} with the eight vertices labeled a through h in any valid order.

In T_2, T_3, and T_4, whenever the two elements of the Left-Right Pair share a set of their set partitions, the rest of the elements of that special subset will also contain that same set. In T_4 the Left-Right elements share zero, one, or two sets.

Case 1: One shared set

One set of Left-Right Pair set partitions is the following:

{1, 2} U {3, 4} U {5, 6} U {7, 8} and
{1, 2} U {3, 8} U {4, 5} U {6, 7}

Notice that both partitions contain the set {1, 2}. As previously stated, whenever the Left-Right Pairs have a set in common, the rest of the elements of the special subset must also contain this set. Therefore, each element that we construct will be of the form {1, 2} U {a, b} U {c, d} U {e, f} with a through f representing vertices 3 through 8 in any valid order. Now we find all valid combinations that include exactly one set from each set partition (i.e. one of {3, 4}; {5, 6}; {7, 8} and one of {3, 8}; {4, 5}; {6, 7}.) A valid combination simply means that each vertex can only appear once in the set partition and that the set partitions are compatible with a dot diagram D.

This produces:

{1, 2} U {3, 4} U {6, 7}
{1, 2} U {5, 6} U {3, 8}
{1, 2} U {7, 8} U {4, 5}

To complete the set partitions we add the missing vertices as the fourth set.

The new set partitions are:

{1, 2} U {3, 4} U {6, 7} U {5, 8}
{1, 2} U {5, 6} U {3, 8} U {4, 7}
{1, 2} U {7, 8} U {4, 5} U {3, 6}

These are exactly the set partitions of elements belonging to the subset represented by the original Left-Right Pair. Using this method, we can generate all of the subsets given only the Left-Right Pair, assuming that the two elements that make up the Left-Right Pair share one set of their set partitions. This accounts for 8 of the 31 special subsets of T_4. However, a new method is needed for the special subsets where this is not the case.

Case 2: No shared sets

Another example of a Left-Right Pair of set partitions is:

{1, 8} U {2, 7} U {3, 4} U {5, 6} and
{1, 2} U {3, 6} U {4, 5} U {7, 8}

These set partitions obviously do not share any sets. However, we still need to use these sets to construct the rest of the special subset. We do this by finding all valid combinations of two sets from each set partition.

Suppose we start with {1, 8} from the first set partition. We cannot use any vertex more than once so from the second set partition we must choose the sets {3, 6} and {4, 5}. To obtain the fourth set, we choose from the first original set partition. We must choose {2, 7} in order to avoid repeating vertices, so this becomes the final set. Therefore, the new set partition is {1, 8} U {2, 7} U {3, 6} U {4, 5}.

To find another new set partition, we start with {3, 4} from the first set partition and use the same method to arrive at a second new set partition: {3, 4} U {5, 6} U {1, 2} U {7, 8}. This exhausts all possibilities of combining two sets from each set partition to form new set partitions, and thereby a complete special subset containing four elements has been obtained.

Case 3: Two sets shared

In cases where two sets are already shared by the Left-Right Pair, there is no way to recombine various sets of the set partitions, so the Left-Right Pair itself forms the entire special subset.

We now have a method for constructing an entire special subset based on its Left-Right Pair in T_4. However, this method is not general enough to apply to T_n except for a few special cases.

V. Posets of Special Subsets

Let P be the poset of special subsets of T_n, ordered by inclusion, meaning $S_1 <_P S_2$ if S_1 is a subset of S_2.

In order to provide more meaning to the poset P, it would be useful to find out some of its special properties. With a few definitions, we will be able to show that P is a join-semi-lattice, meaning there exists a unique upper bound $x \vee y$ for each two elements x and y.

We will use the idea of refinement of set partitions to find a formula for S_1 □ S_2. As with all lattices and semi-lattices, there exists a hierarchy of the elements, meaning that some elements are higher than others. In the case of the join-semi-lattice of special subsets of T_n, the special subset of all elements of T_n is the highest element, and the n identity elements are each the lowest.

To view this in terms of set partitions, the highest element of the semi-lattice (again, the special subset consisting of all elements of T_n) will have a set partition consisting of only one set: {1, 2, … , n}. The lowest elements (again, identity elements) will consist of n sets of two elements each. Therefore, the set partitions of lowest elements refine the set partition of the highest element. This pattern is consistent throughout the semi-lattice.

To find the unique upper bound S_1 □ S_2, start by finding the set partitions for both S_1 and S_2. Combine into one set any sets that contain the same vertex, continuing to combine as sets as long as there are two different sets with the same vertex. This yields a new set partition. The special subset represented by this set partition is then the unique upper bound.

For instance, suppose S_1 has the set partition {1, 8} U {2, 7} U {3, 6} U {4, 5} and S_2 has the set partition {2, 3} U {1, 4} U {5, 8} U {6, 7}. Here, {1, 8} and {1, 4} both contain vertex 1. Therefore, these sets should be combined into the set {1, 4, 8}. Now this new set {1, 4, 8} shares vertex 4 with set {4, 5} so these sets should be combined into {1, 4, 5, 8}. By continuing this way, we arrive at the new set partition, {1, 4, 5, 8} U {2, 3, 6, 7}. This set partition corresponds to a three element special subset in T_4, so we have successfully found the unique upper bound.

VI. Open Problems

There are a couple of open problems and opportunities for further study within the topics presented:

1. Currently the special subsets through the fourth Temperley-Lieb Algebra can be constructed using only the two elements of their Left-Right Pairs. In T_5 and beyond, this can only be done in special cases where the two elements of the Left-Right Pair share enough edges to reduce the problem to T_4 (that is, if T_5 had two shared edges, T_6 had three shared edges, etc.) However, these cases are relatively few and it would be interesting to come up with a generalized rule for creating the subsets.

2. We found a formula for $S_1 \square S_2$ in P and used this to show that P was a join-semilattice. An extension of this problem is given the graphs G_1 and G_2 constructed by the Left-Right Algorithm as described in section II, find a formula for $G_1 \square G_2$ in P.

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References

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