Changes in Betti numbers of Hessenberg varieties on restricted tableaux

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Abstract

It has been observed in all known cases that moving to a less restrictive Hessenberg function on a fixed tableau design will never cause a decrease in the Betti numbers b_i. In this paper we prove this observation on certain restricted tableaux.

1 Introduction

In this paper we seek to find a solution to a question posed by Tymoczko regarding the way in which Betti numbers of Hessenberg varieties, a subvariety of the full flag variety, change with a change in the Hessenberg function. The Hessenberg function is a nondecreasing function $h : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ such that $h(i) \geq i$ for all i. We seek to prove the observation that on a fixed Young Diagram changing the function h to g where $g(i) \geq h(i)$ for each i results in Betti numbers b'_i such that $b'_i \geq b_i$.

We will achieve this goal on certain restricted tableau which lend themselves to rearrangement of their boxes. In this paper I will follow some notation conventions established by Iveson in [1].

Before we can discuss the Betti numbers we need to take care of some preliminaries. A cell is a particular filling of a Young tableau. Given a Hessenberg function h cell is valid or h-allowed if whenever $\boxed{a \mid b}$ occurs in a cell $a \leq h(b)$ [1].

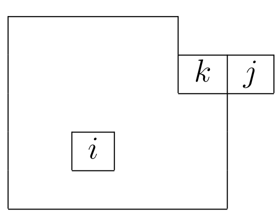


Figure 1: An Inversion Diagram

Definition 1 *The* **number of inversions on a box** *occupied by k is the number of i such that:*

- *i is anywhere below or to the left of k,*
- *$k > i$, and*

- *if there is a box immediately to the right of k filled by j then $i < h(j)$.*

Rephrasing Tymoczko's result in [2] we find

Theorem 2 *The sum over all boxes in a cell of the number of inversions on each box is the dimension of the cell.*

The Betti numbers b_i are the number of valid fillings of a tableau of dimension i. For the purposes of this paper h is a Hessenberg function on a Young Diagram, h' is another Hessenberg function such that $h'(j) = h(j) + 1$ for exactly one j and $h'(i) = h(i)$ for all $i \neq j$.

Lemma 1.1 *Given a fixed tableau design λ and Hessenberg functions h and h'. When we move from h to h' the dimension of each cell is either unchanged or increases by exactly 1.*

Proof. First we show that the dimension of a cell can never decrease. Each inversion corresponds to a unique triple i, j, k such that i is below or to the left of k, $k < i$, and if there is a box immediately to the right of k filled by j, then $i \leq h(j)$. Since $h'(j) \geq h(j)$ for all j we know $i \leq h(j)$ implies $i \leq h'(j)$. Therefore all triples which correspond to an inversion in the case of h correspond to an inversion in the case of h', and the dimension of a cell cannot decrease.

Now suppose the dimension of the cell increases. The dimension increase must occur as an increase in the number of inversions on some box filled by $i < h(j) + 1$ where j is immediately to the right of i. Furthermore we know $h(j) + 1$ is below or somewhere to the left of i. This defines a unique triple in a given cell, because both j and $h(j) + 1$ can occur only once in the filling, and i is then specified because it must be the number immediately to the left of j. This triple then increases the dimension by exactly one. □

Definition 3 *A* **bumped up cell** *is a cell whose dimension increases by one with the change in Hessenberg functions from h to h'.*

2 First restricted tableaux

a_1	β
a_2	
...	
a_{N-2}	
a_{N-1}	

Figure 2: An example of a very restrictive tableau design

Lemma 2.1 *Given Hessenberg functions h and h' such that $h'(\beta) = h(\beta) + 1$ for some β and $h'(j) = h(j)$ for all $j \neq \beta$, a valid cell λ of design similar to Figure 2 (having 2 boxes in the first row, and any number of additional rows with a single box) will have a dimension increase if and only if β is in the second column of the first row and $a_1 < h'(\beta)$.*

Proof. Suppose we have β in the second column and $a_1 < h'(\beta)$. Note that for h' to be a valid Hessenberg function on a cell with N boxes, we have $\beta < h'(\beta) \leq N$. Therefore $h'(\beta)$ exists somewhere in the sequence $a_1, a_2, ..., a_{N-3}, a_{N-2}$. We know $a_1 \leq h(\beta)$ and hence $a_1 \neq h'(\beta)$. The triple a_1, $h'(\beta)$, β gives us an inversion with the new function h' but not with the old h since the third condition for an inversion was not previously met. By Lemma 1.1 this type of function increase from h to h' leaves all previous inversions intact. Therefore the dimension of our cell has increased by 1.

Now suppose we have a cell whose dimension has changed. By Lemma 1.1 we know that β must be somewhere to the right of $h'(\beta)$, which means β must be in the only box in the second column. We also know a_1 is less than $h'(\beta)$ since $a_1 \leq h(\beta)$ is a condition for a valid cell. □

Proposition 4 *Uniquely mapping each bumped up cell to another cell of dimension reduced by one, which is either also bumped up, or was not previously valid, is enough to show that if h gives us k cells of some dimension d and h' gives us j cells of dimension d, $j \geq k$.*

Proof. This should be very clear, we will be forming chains of cells which take each others place in their former dimension, these chains will then be terminated by cells which were not previously valid. Thus if a cell λ of dimension d moves to dimension $d+1$ and we can always find a cell of dimension d which takes it's place we will always be left with at least as many cells of dimension d as we had previously. □

This proposition motivates the remainder of our work. We seek to find methods to establish these chains of cells which move up in dimension, terminating with a new cell. This will give us the results that we are searching for on the Betti numbers.

Theorem 5 *Suppose we have a bumped up cell with 2 boxes in the first row and any number of additional rows with a single box as before. Exchanging a_1 with a_l the smallest a_j greater than a_1 will give us a unique cell with dimension reduced by one.*

Proof. It should be clear that a_l exists since $h'(\beta)$ is somewhere in the a_j, and $h'(beta)$ is necessarily greater than a_1 since a condition for validity of our cell is $a_1 \leq h(beta) < h'(beta)$.

Suppose $\beta \neq a_1+1$. Then a_1+1 is in the a_j and $a_l = a_1+1$. The condition for our cell to be valid with the function h is $a_1 \leq h(\beta)$. This tells us $a_1+1 \leq h(\beta)+1 = h'(\beta)$, so our new cell is valid. Now we look at the inversions in this cell. The number of inversions on the box occupied by a_l before we exchange is the number of a_j such that $a_j > a_l$ and $j > l$. After we exchange we see that the number of inversions on this box is constant since the sequence $a_{l+1}, ..., a_{N-2}$ of (nonrepeating) integers cannot contain any a_j such that $a_1 < a_j < a_l$. The inversions on boxes occupied by a_j such that $j > l$ must be constant since no changes were made to boxes below these numbers. Since we have no repetition if any a_j of $a_1, ..., a_{l-1} < a_l$ then $a_j < a_1$ and thus the number of inversions on the boxes occupied by $a_1, ..., a_{l-1}$ is unchanged. The number of inversions on the box occupied by β is unchanged because every number in the sequence $a_1, ..., a_{N-2}$ remains to the left of β after the permutation. Finally we come to the box previously occupied by a_1. There is exactly one integer which is greater than a_1 but not greater than a_l, it is a_l. Since we have switched these two numbers we get a reduction of one in the number of inversions on this box. This is exactly what we were looking for.

Now suppose $\beta = a_1 + 1$ so now $a_l = a_1 + 2$. We know $a_l \leq h'(\beta)$ since $h(\beta) \geq \beta$ and $h'(\beta) = h(\beta) + 1$ so exchanging a_1 and a_l gives us a valid cell. The previous arguments for the changes in the number of inversions on a cell hold if we add the fact that $a_j \neq a_1 + 1$ for all a_j.

To show uniqueness we can simply show that this process is reversible. We only need to reverse a cell if it has been bumped up, and it could come from some other cell. Thus we first eliminate all cells where β is not in the upper right, and all cells where a_1 is the smallest of the a_j, as these cells could not be generated by our previous algorithm. The rule for recovering our original cell is then exchange a_1 and a_m where a_m is the maximum a_j such that $a_j < a_1$

It is obvious that we get a h-allowed filling here since $a_m < a_1 \leq h'(\beta)$ gives us $a_m \leq h(\beta)$. We know that a_m exists in the first column since we have already eliminated all cells were a_1 is the smallest a_j. Since our algorithm is reversible we have one-to-one correspondence between a cell whose dimension increases by one, and another cell with the dimension of the original cell.

Note that if the cell formed by this rule was valid with the function h it is of the form given by Lemma 2.1 and this formula can be applied again to find a cell which has its dimension reduced by one. These chains are terminated when we place $h'(\beta)$ in front of β which gives us a cell which was not valid with the function h. $\square$

This case motivates the three lemmas that follow. First we look at the possibility of reducing big cells to smaller ones by examining which parts of a cell are important when we exchange the contents of two boxes. Then we will establish methods for changing the number of inversions on a single box by 1, either up or down, through exchanging the contents of that box.

Lemma 2.2 *Assume we exchange two elements in any cell diagram such that the result is a valid cell. The number of inversions on boxes either directly above or anywhere to the right of each exchanged box is unchanged.*

Proof. Let us exchange i_1 and i_2 and let k represent a box which is either directly above or to the right of i_1 and i_2. Since the position criteria are met on both locations an inversion on k by either i_1 or i_2 remains in the final cell. $\square$

Lemma 2.3 *Let $a_{l(i)}$ be the $max(L(a_i))$ where $L(a_i) = \{a_j | j > i \text{ and } a_j > a_i\}$. Thus $a_{l(i)}$ is the least a_j simultaneously greater than and below a_i. Assuming a valid cell is created swapping a_i with $a_{l(i)}$ will result in a reduction by one of the number of inversions on the i-th box in the first column.*

This swap will also increase by one the number of inversions on any other box occupied by a_n if and only if the following conditions are met

- $i < n < l$
- $n \leq k$ *(the n-th row contains 2 boxes)*

- $a_n < a_i < a_{l(i)}$, *and*
- $a_i \leq h(b_n) < a_{l(i)}$.

The number of inversions on all other boxes is left alone.

Proof. Without loss of generality we assume a_i is a_1. Since boxes anywhere above a_i do not effect our result by Lemma 2.2. The permutations are occurring on boxes which are to the left of $b_1, ..., b_k$. Therefore the number of inversions on these boxes is also unchanged by Lemma 2.2.

The number of inversions on the first box in the in first column is the number of a_j such that $a_1 < a_j \leq h(b_1)$. There is no repetition in the a_j. Therefore, for all $j \neq l$ if $a_j > a_1$ then $a_j > a_{l(i)}$, and similarly $a_j < a_1$ implies $a_j < a_{l(i)}$. However the inversion caused by $a_{l(i)}$ on the first box has been removed, so the number of inversions on this box has been reduced by one.

The number of inversions on the l-th box in the first column has not decreased since $a_j > a_{l(i)}$ implies $a_j > a_1$. It has also not increased because $a_{l(i)} > a_j > a_1$ contradicts our conditions on $a_{l(i)}$. The number of inversions on boxes occupied by $a_{l+1}, ..., a_{N-k}$ remains the same because no permutations occur below these boxes.

Now we will look a_n where $1 < n < l$. If $n > k$ (that is to say if the n-th row has only one box) the number of inversions cannot change on a_n. This is because we know either $a_1 < a_{l(i)} < a_n$ or $a_n < a_1 < a_{l(i)}$, and without a b_n these conditions are sufficient to maintain the previous inversion or lack of inversion by $a_{l(i)}$ on a_n. Therefore we know $n > k$. If $a_n > a_1$ then $a_n > a_{l(i)}$ by our conditions on $a_{l(i)}$. In this case there was no inversion on a_n by $a_{l(i)}$ and there cannot be an inversion on a_n by a_1. Now suppose $a_n < a_1$. If $a_{l(i)} \leq h(b_n)$ then $a_1 < h(b_n)$ and the inversion on a_n by $a_{l(i)}$ is maintained by a_1. Similarly if $a_1 \geq h(b_n)$ then $a_{l(i)} > h(b_n)$ and no inversion by $a_{l(i)}$ on a_n remains no inversion after the swap. However if $a_1 \leq h(b_n) < a_{l(i)}$ no inversion by $a_{l(i)}$ on a_n becomes an inversion by a_1 on a_n. In this case the number of inversions on the box occupied by a_n increases by one.

Now if there is a box filled by a_n, $(n \neq 1)$ such that the number of inversions on a_n increases we know a_n is above $a_{l(i)}$. This is because if a_n is below $a_{l(i)}$ the permutations occur in boxes which cannot cause inversions on a_n. Furthermore we know $a_1 > a_n$, and $a_1 \leq h(b_n)$ because a_1 will cause the new inversion on a_n. If $a_{l(i)} \leq h(b_n)$ there was a preexisting inversion on a_n by $a_{l(i)}$ which is continued by a_1. Therefore in order to increase the number of inversions on a_n by one $a_{l(i)} > h(b_n)$. □

Example 2.1 As an example let us take the first cell below with the function $h(6) = 7$ and $h(i) = i$ for $i \neq 6$, and preform a swap on the first box in the first column.

4	6
3	5
2	
1	
7	

(a) Initial cell

7	6
3	5
2	
1	
4	

(b) After first swap

7	6
4	5
2	
1	
3	

(c) Final cell

Notice that we first move $a_1 = 4$, and this creates a change in the number of inversions on

3. We can then iterate the process on 3 to end with a cell whose total dimension is decreased by one.

Lemma 2.4 *Assume we have a valid cell and swap a_i with a_m such that a_m is the greatest a_j such that $j > i$ and $a_j < a_i$. Assuming this also gives us a valid cell (that is $a_1 \leq h(b_m)$ if b_m exists) will result in an increase by one of the number of inversions on the i-th box in the first column.*

This swap will also reduce by one the number of inversions on any box occupied by a_n if and only if

- $i < n < m$
- $n \leq k$ *(the n-th row contains 2 boxes)*
- $a_n < a_m$ *and*
- $a_m \leq h(b_n) < a_i$.

The number of inversions on all other boxes is left alone.

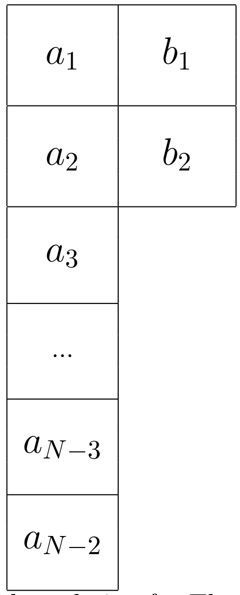
Proof. Similar to above all boxes in columns above the i-th are unchanged and we remove them without loss of generality. The number of inversions on the boxes in the second column remains the same by Lemma 2.2. The reasoning for the number of inversions being unchanged on boxes in the m-th column and below is equivalent to that of the l-th column and below in the proof of Lemma 2.3.

The number of inversions on the a_1 box is the number of a_j such that $a_1 < a_j \leq h(b_1)$. For all $j \neq m$ if $a_j > a_1$ then $a_j > a_m$. Also $a_j < a_1$ implies $a_j < a_m$ by our conditions on a_m. There is a new inversion caused by a_1 on the box now occupied by a_m since $a_1 > a_m$ and $a_1 \leq h(b_1)$ by our cell's previous validity. Thus the number of inversions on this box increases by one.

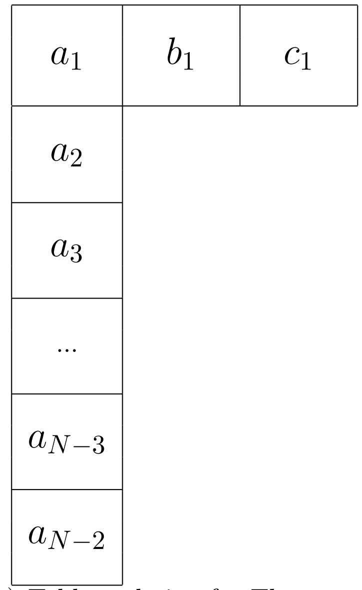
Again we examine a_n with $1 < n < m$. We know $a_n < a_m$ if $a_n < a_1$ by our conditions on a_m. This gives us no new inversions if $a_n > a_m$. If no b_n exists ($n > k$) the number of inversions on a_n remains the same by the reasons in the previous Lemma. Now suppose $a_1 < a_m$. As before if $a_m < a_1 \leq h(b_n)$ or $h(b_n) \leq a_m < a_1$ we get no change on the number of inversions on a_n. However if $a_m \leq h(b_n) < a_1$ an inversion by a_m on a_n has been removed, and the number of inversions on this box is reduced by one.

Suppose the number of inversions on a box filled by a_n, ($n \neq 1$) decreases. We know $1 < n < m$ since boxes below a_m are not affected by our swap. We also have $a_m > a_n$, and $a_m \leq h(b_n)$ since we need an inversion to take away in our swap. If $a_1 \leq h(b_n)$ we would still have this last inversion. Therefore in order to decrease the number of inversions on a_n by one $a_1 > h(b_n)$ □

Definition 6 *An* **Unexpected Inversion Change** *is the increase or decrease of the number of inversions on some box a_n as the result of an exchange of boxes a_i and a_j with $n \neq i, j$.*



(d) Tableau design for Theorem 7



(e) Tableau design for Theorem 8

Theorem 7 *Suppose we have a cell generated by a partition of the form $2, 2, 1^N - 2$, as in figure 2. Given h and h' ($h'(\beta) = h(\beta) + 1$) and a h allowed cell which is bumped up λ we can generate a cell with total dimension reduced by one with two rules (supposing β is b_i);*

1. *exchange a_i and $a_{l(i)}$ rename $a_{l(i)}$ to a'_i*
2. *if $i = 1$ and the dimension of a_2 changes from our previous move, exchange a_2 and $a_{l(2)}$, rename a_2 to a'_2*

Proof. First note that if $i = 2$ we will not make any permutations on a_1 and b_1. We can then strike the first row from our tableau and reduce to the case proved in Theorem 5. Therefore let $i = 1$. By Lemma 2.3 our first swap produces a reduction by 1 in the number of inversions on the box occupied by a'_1. If a_2 satisfies the conditions outlined in Lemma 2.3 for an unexpected inversion change it will receive an increase by one in the number inversions on that box. We then note that applying the second rule is similar to removing the first row from our diagram and reducing to the case of Theorem 5 since we know $a_2 < a_1 \leq h'(b_2)$ there exists a valid a_l for our swap, and taking the smallest one will reduce our total cell dimension to exactly one less than what we started with.

We can ensure uniqueness by reversing this process. Given some cell were if either β is in the left column or $h'(\beta)$ is in the right the cell was not bumped up and no changes are necessary. Else apply the rules

1. if $i = 1$, $a'_2 < a'_1$ and $a'_{m(1)} \leq h(b_2) < a'_1$ exchange a'_2 and $a'_{m(2)}$ remove primes
2. exchange a'_i and $a'_{m(i)}$ remove primes

The first step reverses the final step of our previous algorithm. The conditions are a check to ensure that the final step was run. In this step we know that $a'_{m(2)}$ exists since the conditions are sufficient to ensure that a_2 was exchanged with $a_{l(2)}$ before, and then by definition $a'_{m(2)} = a_2$. A valid cell is created here since $a'_{m(2)} < a'_2 \leq h(b_2)$.

The second step reverses the first step of the initial algorithm, since $a'_{m(i)} < a'_i$ and $h(i) = h'(i)-1$ we have a valid cell from this swap. Also note that if $i = 1$ we will not illegally move a'_1 into the a'_2 space, since an illegal move there would require that $a'_2 < a'_1$ and $a'_{m(1)} \leq h(b_2) < a'_1$, in which case we will have already moved a'_2 down, into a cell valid for this swap. These conditions cannot be met twice, that is if a first rule exchange is made we know $a_m(1) \neq a_2$, since $a_m(1) < a'_2 < a'_1$.

As was our goal we now have either another cell which meets the conditions for being bumped up, or a cell which was not previously valid. Again by Proposition 4 we are assured that in this cell design the Betti numbers b_i never decrease under an $h \to h'$ function change. □

Refer to Example 2.1 to see what this looks like in practice. Later we will develop this idea into a method for all tableaux with rows of length no greater than 2 but first lets look at another small addition to our tableaux design from Theorem 5 adding another box to the first row. For this proof we require a slight modification to the idea of $a_{l(i)}$. Let $a_{l(\alpha)}$ be exactly $a_{l(1)}$ if $a_1 = \alpha$, and if $\alpha = b_1$ take $L(\alpha)$ as the set of all $a_i > b_1$ and let $a_{l(\alpha)} = min(L(\alpha))$.

Theorem 8 *Suppose we have a cell generated by a partition of the form $3, 1^N - 2$, as in figure 2. Given h and h' ($h'(\beta) = h(\beta)+1$) and a h allowed cell which is bumped up λ we can generate a cell with total dimension reduced by one with two rules (let the box to the left of β be occupied by α);*

1. *exchange α and $a_{l(\alpha)}$*
2. *if we just moved b_1 and the dimension of a_1 changes from our previous move, exchange a_1 and $a_{l(1)}$, repeat until the number of inversions on the a_1 box is back to its original value*

Proof. Again if $\beta = b_1$ we may remove the last box in the first row and this reduces to Theorem 5. We need to show that we have corrected for the case of $\beta = c_1$ moving $a_{l(\alpha)}$ into the b_1 box. For simplicity call the new b_1 b'_1. First lets ensure that we have created a valid cell. Since $a_{l(\alpha)} > b_1$ we have $h'(a_{l(\alpha)}) > h'(b_1)$, furthermore either a_1 decreases if $l(\alpha) = 1$ or it stays the same if $l(\alpha) \neq 1$. Therefore we have an h-allowed filling.

There will now be 1 unexpected inversion change on a_1 for every a_j such that $h'(b_1) < a_j \leq h'(b'_1)$. Note that each of these a_j is in the set $L(a_1)$. And so we can preform Lemma 2.3 swaps on a_1 until the number of inversions on that box are back to its previous number.

To show this is unique we will show that no two distinct cells yield the same final cell. Suppose we have two cells λ and λ' which yield γ such that $\lambda \neq \lambda'$. We see immediately that β cannot be in the b_1 box because we already know this process is reversible. Furthermore $b_1 = b'_1$ since we preform exactly one switch on this box and $a_{l(b_1)} = a'_{l'(b'_1)}$ in order to receive the same final cell γ. Therefore $a_1 \neq a'_1$ else $\lambda = \lambda'$. Without loss of generality assume $a'_1 < a_1$. The number of unexpected inversion changes on a_1 is equal to the number of unexpected inversion changes on a'_1 since they are both equal to the number of a_j such that $h'(b_1) < a_j \leq h'(b'_1)$. Note there is no requirement on a_1 necessary since $a_j > h'(b_1)$ and $a_1 < h'(b_1)$, our valid cell condition, implies $a_j > a_1$. At each exchange we know $a'_{l(1)} \leq a_1 < a_{l(1)}$ because a_1 must exist in the cell λ' and since we have shown it is not in the first row it must be below a'_1 in the first column. Since both a_1 and a'_1 need to take the same number of steps we can be assured that they do not end on the same number. □

Example 2.2 Lets take an example with the function $h = 1, 2, 5, 5, 5, 6$ and $h' = 1, 2, 5, 5, 6, 6$

1	2	5
3		
4		
6		

(f) initial cell

1	3	5
2		
4		
6		

(g) swap 2 and 3

2	3	5
1		
4		
6		

(h) swap 1 and 2

4	3	5
1		
2		
6		

(i) swap 2 and 4

We can see that the initial swap produces a decrease by 1 in the number of inversions on the b_1 box, and also an increase by 2 on the a_1 box. Each swap after that reduces the number of inversions on the a_1 box by 1.

3 Generic two column tableaux

We will now look at general tableaux where all rows have no more than 2 boxes. These tableaux lend themselves to our methods because swaps on the a_i need only be checked for validity against the values of the Hessenberg function on the second column.

a_1	b_1
a_2	b_2
...	...
a_{k-1}	b_{k-1}
a_k	b_k
a_{k+1}	
...	
a_{N-k}	

Figure 3: A cell λ generated by the partition 2^k,1^{N-k}, where k is the number of two box rows.

Lemma 3.1 *Any box in the left column of a cell receives a maximum of a plus one unexpected inversion change from a_i and $a_{l(i)}$ (as in Lemma 2.3) swaps made in the left column.*

Proof. Suppose a we have a box a_n which has undergone an unexpected inversion change by exchange of a_i and $a_{l(i)}$. We then know that $a_i \leq h(b_n) < a_{l(i)}$. A further unexpected inversion change would require that there exists some a_k such that $a_k \leq h(b_n) < a_{l(k)}$. We will show that no such a_k exists. We know that $a_k \neq a_i$ because a_k must be above a_n and a_i below. If $a_k > a_i$ we have a contradiction on the definition of $a_{l(i)}$ because $a_k < a_{l(i)}$ and then $a_{l(i)}$ not the smallest a_j greater than and below a_i. If $a_k < a_i$ we have a contradiction on the definition of $a_{l(k)}$ because $a_i < a_{l(k)}$ and $a_{l(k)}$ not the smallest a_j greater than and below a_k. □

Theorem 9 *Given a bumped up cell λ generated by a partition of the form 2^k, 1^{N-k}, where $h'(b_i) = h(b_i) + 1$. The following rules will generate unique cell γ of total dimension reduced by one.*

1. *Exchange a_i and $a_{l(i)}$ rename all a_j to a'_j*
2. *Starting with the highest a'_n which has an unexpected inversion change (minimum n value). Swap a'_n and $a'_{l(n)}$ repeat until all unexpected inversion changes have been eliminated.*

Furthermore this is enough to show that if h gives us k cells of some dimension d and h' gives us j cells of dimension d, $j \geq k$.

Proof. **Part I: proof of reduction algorithm** First without loss of generality we remove all a_j and b_j where $j < i$. These boxes will be unaffected by permutations made on boxes below them by Lemma 2.2. Reindex the cell so that a_i is now a_1. Our first swap a_1 and $a_{l(i)}$ is valid because we know $h'(b_1)$ is below a_1 so there exists at least one a_j such that $a_1 < a_j \leq h(b_1)$, then $a_{l(i)}$ is the smallest such. By Lemma 2.3 we have reduced the number of inversions on a_1 by one.

Now we must account for all unexpected inversion changes. Our second rule starts with the highest, applying Lemma 2.3 again to remove each increase in dimension. Once again we know that each swap is valid because an unexpected inversion change on a_n requires moving some a_i below a_n such that

- $i < n < l$
- $n \leq k$ (the n-th row contains 2 boxes)
- $a_n < a_i$, and
- $a_i \leq h(b_n) < a_{l(i)}$.

The last condition tells us that there exists at least one valid swap for every unexpected inversion change. Furthermore since we have to make at most one swap on each box in the left column by Lemma 3.1 so this process terminates by the finite number of boxes in our cell.

We have now generated a cell from our previous cell which has its overall dimension reduced by one. Note that if $a_{l(i)} < h'(b_1)$ this is another bumped up cell, and if $a_{l(i)} = h'(b_1)$ this cell is a new cell, not valid under the function h.

Part II: proof of uniqueness Suppose two cells λ^1 and λ^2 with $\lambda^1 \neq \lambda^2$ give the same final cell γ. For all $i < k$ we know $b^1_i = b^2_i$ because we do not make any changes to the b_j. We also know $a^1_{l^1(1)} = a^2_{l^2(1)}$ in order to receive the same final cell. This tells us that $a^1_1 = a^2_1$, because this is required for $a^1_{l^1(1)} = a^2_{l^2(1)}$. We can see however that $l^1(1)$ is not necessarily equal to $l^2(1)$, because we have no restriction on the initial location of $a^1_{l^1(1)}$ or $a^2_{l^2(1)}$.

Take the set $D = \{d | a_d^1 \neq a_d^2\}$. As shown previously unexpected inversion changes must occur between two swapped numbers, therefore there exists at least one d such that, $1 < d \leq min(l^1(1), l^2(1))$. This is because if $l^1(1) \neq l^2(1))$ then $a_{min(l^1(1),l^2(1))} \neq a_{min(l^2(1),l^1(1))}$. Also if $l^1(1) = l^2(1)$ then one of λ^1 and λ^2 has an unexpected inversion change from swapping a_1 and $a_{l(1)}$. Otherwise, the algorithm terminates for both cells after the first swap. Since the same values in the same positions were exchanged, and the same cell γ was created, it follows that $\lambda^1 = \lambda^2$. This contradicts our hypothesis that $\lambda^1 \neq \lambda^2$. An unexpected inversion change can only occur between two swapped numbers, so some d exists with $1 < d \leq min(l(1), l(1)')$.

Let us take the $min(D)$ and call it δ. Furthermore without loss of generality let $a_\delta^1 > a_\delta^2$.

It is possible that the a_δ are involved in previous algorithm swaps. Swapping some a_j with $j < \delta$ cannot give us $a_\delta^1 = a_\delta^2$ because then $a_j^1 \neq a_j^2$. Also a_δ^2 remains less than a_δ^1 since either $a_j < a_\delta^2 < a_\delta^1$ in which case a_j and a_δ^2 are swapped, or $a_\delta^2 < a_j < a_\delta^1$ in which case a_j and a_δ^1. By Lemma 3.1 each a_δ can receive only one swap. As some $a_j^2 = a_\delta$ and all $a_j^1 = a_j^2$ for $j < \delta$ by our definition of δ we know some $a_k^2 = a_\delta^1$ for $k > \delta$. Therefore $a_{l^2(\delta)}^2 \leq a_\delta^1$. Suppose both a_δ^1 and a_δ^2 receive unexpected inversion changes $a_{l^2(\delta)}^2 \neq a_{l^1(\delta)}^1$ as $a_{l^2(\delta)}^2 \leq a_\delta^1 < a_{l^1(\delta)}^1$ and the final cells cannot be equal. Then an unexpected inversion change must occur on a_δ^2 and not on a_δ^1. We have $b_\delta^1 = b_\delta^2$ which implies $h'(b_\delta^1) = h'(b_\delta^2)$, and since the unexpected inversion change must occur from some a_j, $a_{l(j)}$ exchange we know $a_\delta^2 < a_j < a_\delta^1$ and $a_{l(\delta)}^2$ cannot equal a_δ^1 (it must be no greater than a_j), a contradiction.

Again we see that we have retained the important element of our scheme, that is either $h'(b_1)$ is in front of b_1 which means that we have a cell which was not previously valid, or $h'(b_1)$ is below a_1 and we have another bumped up cell which we can then apply the algorithm to again to find a cell of further reduced dimension. This fits with Proposition 4 and ensures that $b_i \leq b_i'$ for all i. □

Example 3.1 An example of a chain with function $h = 2, 2, 5, 5, 5, 6, 7, 10, 10, 10, 11, 12$ and $h'(11) = 12$ is

1	11
2	10
4	8
6	9
5	3
7	
12	

2	11
1	10
4	8
6	9
5	3
7	
12	

4	11
1	10
2	8
6	9
5	3
7	
12	

5	11
1	10
2	8
6	9
4	3
7	
12	

6	11
1	10
2	8
5	9
4	3
7	
12	

7	11
1	10
2	8
5	9
4	3
6	
12	

12	11
2	10
4	8
6	9
5	3
1	
7	

Here each cell maps to the cell to its immediate right under our algorithm. The last mapping is (by design) where all the unexpected inversion changes occur and so we will do a step by step on this cell in the next example.

Example 3.2 We are going to do the final rearrangement in the last example in more detail. I will use +1 to the left of a box to indicate an increase in the number of inversions on that box by 1. We start with a single increase on the a_1 box because this is a bumped up cell.

+1	7	11
	1	10
	2	8
	5	9
	4	3
	6	
	12	

	12	11
+1	1	10
+1	2	8
+1	5	9
	4	3
	6	
	7	

	12	11
	2	10
+1	1	8
+1	5	9
	4	3
	6	
	7	

	12	11
	2	10
	4	8
+1	5	9
	1	3
	6	
	7	

	12	11
	2	10
	4	8
	6	9
+1	1	3
	5	
	7	

12	11
2	10
4	8
6	9
5	3
1	
7	

In this example each new cell represents a single swap from our algorithm.

4 Conclusions

Theorem 10 *For all the cell designs we have studied so far whenever we have two general Hessenberg functions h and g with $g(j) \geq h(j)$ for all j the number of cells of a particular dimension d under h is less than or equal to the number of cells of dimension d under the function g.*

Proof. We can step from h increasing each $h(i)$ individually until we reach g. While multiple paths may be legal, making our increases starting at the greatest i such that $h(i) < g(i)$ will always be legal. The result is then immediate since our previous theorems show that each increase results in no decrease in the number of cells of dimension d. □

This then is the result we were looking for since each of the Betti numbers b_i are the number of cells of dimension i we can say that moving from h to some other Hessenberg function g with each $g(i) \geq h(i)$ yields nondecreasing Betti numbers.

References

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