Properties of Steady State Solutions for Critical Two-Layer Shallow Water Flows

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Abstract

The properties of the steady state solutions for two-layer shallow water flows are divided into two schemes for study: bounded and unbounded water surface. Analysis of the bounded water surface, as presented in this paper, provides admissible energy and individual layer height ranges. Further, useful conditions of the topography and channel geometry that are necessary for the existence of a critical point within the flow region are described. Analysis of the unbounded water surface scheme is the analysis of the unrestricted steady state solutions to the shallow water equations. The results appear in a much more limited form, in contrast to the bounded water surface scheme. They however, still provide significant insight into the admissibility ranges at the critical point of relevant flow parameters such as the channel geometry, topography, and layer energies.

1 Introduction

In this paper, some relevant properties exhibited at the critical point by the two-layer quasi one dimensional shallow water model for channels with non-uniform rectangular cross sections are presented. The equations for these types of flows are easily derived from the two-layer shallow water equations in [2] with the additional assumption that the flow cross sections are rectangular. They may also be derived through the vertical averaging of the Euler equations of gas dynamics on each individual layer for an arbitrary rectangular cross section. The equations obtained are perfectly compatible with those presented in [1, 4]. As shown in Figure 1, two-layer shallow water flows are assumed to be constituted by two homogeneous layers that are differentiated by their respective densities ρ_1 and ρ_2, where $\rho_2 \leq \rho_1$ by convention.

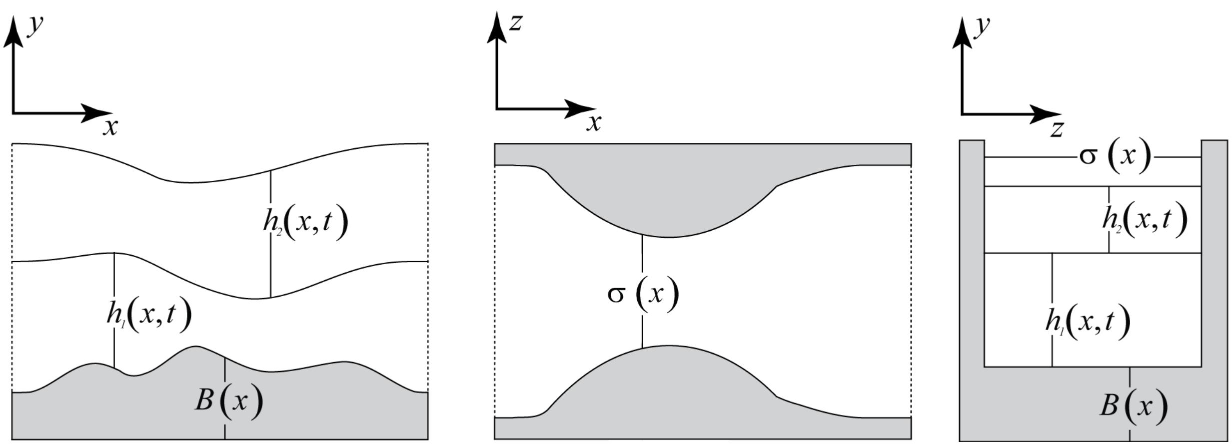

Figure 1: Sideview, Topview, and Backview of the flow channel. The x coordinate marks positive flow direction.

The equations that describe this type of flow are

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\sigma h_1\right)+\frac{\partial}{\partial x}\left(\sigma h_1 u_1\right)=0, \\
&\frac{\partial}{\partial t}\left(\sigma h_1 u_1\right)+\frac{\partial}{\partial x}\left(\sigma h_1 u_1^2+\frac{1}{2} g \sigma h_1^2+r g \sigma h_1 h_2\right)=\left(r g h_1 h_2+\frac{1}{2} g h_1^2\right) \frac{\partial \sigma}{\partial x}+r g \sigma h_2 \frac{\partial h_1}{\partial x}-g \sigma h_1 \frac{\partial B}{\partial x}, \\
&\frac{\partial}{\partial t}\left(\sigma h_2\right)+\frac{\partial}{\partial x}\left(\sigma h_2 u_2\right)=0, \\
&\frac{\partial}{\partial t}\left(\sigma h_2 u_2\right)+\frac{\partial}{\partial}\left(\sigma h_2 u_2^2+\frac{1}{2} g \sigma h_2^2\right)=\frac{1}{2} g h_2^2 \frac{\partial \sigma}{\partial x}-g \sigma h_2 \frac{\partial h_1}{\partial x}-g \sigma h_2 \frac{\partial B}{\partial x},
\end{aligned} \tag{1.1}$$

where g is the gravitational acceleration, $r=\rho_2/\rho_1$ is the ratio of the densities of the layers, $B(x)$ represents the channel's topography, $\sigma(x)$ represents the channel's width, $h_i(x,t)$ is the height of the flow for layer i, and $u_i(x,t)$ is the vertically averaged velocity of the flow for layer i in correspondence with Figure 1.

The steady state solution of this system is given by the equations

$$\begin{aligned}
Q_1 &= \sigma h_1 u_1 \\
E_1 &= \frac{1}{2} u_1^2 + g\left(h_1+B\right) + r g h_2 \\
Q_2 &= \sigma h_2 u_2 \\
E_2 &= \frac{1}{2} u_2^2 + g\left(h_1+h_2+B\right)
\end{aligned} \tag{1.2}$$

where Q_i and E_i are conserved quantities within the layers. They are further defined as: Q_i the *discharge* of layer i and E_i the *energy* of layer i. These parameters are obtained through the boundary conditions of the channel and fully determine the system.

The paper is structured as follows. §2 is a brief introduction to the relevant equations and nondimensionalizations of the bounded water surface scheme, *rigid-lid* approximation. In this section, several properties that must be satisfied at the critical point by the channel topography, geometry, and flow energy are presented and discussed. Lastly, a discussion on the admissible values for the internal Froude number of the layers at the critical point is given. In complete symmetry to the analysis in §2, §3 expands the methods presented in §2 to the full two-layer steady state shallow water problem, exact solution. The solutions include those mentioned for §2.

2 Rigid-Lid Approximation

The *rigid-lid* approximation refers to the assumption that the solution of the two-layer steady state system (1.2) can be accurately modeled by assuming that the water's total elevation remains constant. This assumption holds very well for most practical cases, and is an acceptable model given certain the permissibility parameters as discussed in [3]. The rigid-lid approximation is given as

$$h_1(x)+h_2(x)+B(x)=H_0$$

in dimensional form, where H_0 is the constant total water height of the channel. This expression can be written in non-dimensional form using the following parameters that are constantly used throughout the paper:

$$\tilde{h}_i=\frac{h_i}{H_0}, \quad \tilde{B}=\frac{B}{H_0}, \quad \tilde{\sigma}=\frac{\sigma}{\sigma_0}, \quad \tilde{Q}_i^2=\frac{Q_i^2}{g' \sigma_0^2 H_0^3},$$

where H_0 is the total water height, $g'=(1-r)g$ is known as the reduced gravity, and σ_0 is the point of greatest width in the channel such that $0<\tilde{\sigma}\le 1$. Another useful non-dimensional parameter is the layer *internal Froude number* as given by

$$F_i^2=\frac{u_i^2}{g' h_i}=\frac{\tilde{Q}_i^2}{\tilde{\sigma}^2 \tilde{h}_i^3}.$$

With these variables, the rigid-lid assumption is written non-dimensionally as

$$\tilde{h}_1 + \tilde{h}_2 + \tilde{B} = 1 \quad \text{or} \quad F_1^{-2/3} + Q_r^{2/3} F_2^{-2/3} = \left(1 - \tilde{B}\right) \left(\frac{\tilde{Q}_1}{\tilde{\sigma}}\right)^{-2/3}, \tag{2.1}$$

where $Q_r = Q_2/Q_1$.

The inclusion of the rigid-lid assumption to the system (1.2) leads to an over-determined system. For this reason, in complete congruence with the procedure in [4], the preferred system of equations for the solution of the rigid-lid problem consists of the two discharge equations, the rigid-lid equation, and the difference between the energy equations. The energy difference equation can be written as

$$\frac{r}{2} \frac{\tilde{Q}_2^2}{\tilde{\sigma}^2 \tilde{h}_2^2} - \frac{1}{2} \frac{\tilde{Q}_1^2}{\tilde{\sigma}^2 \tilde{h}_1^2} + \tilde{h}_2 = \frac{rE_2 - E_1}{g' H_0} + 1 = \Delta E \tag{2.2}$$

in terms of the non-dimensional layer height $\tilde{h}_i$. Similarly, in terms of the Froude number the equation becomes

$$\left(\frac{\tilde{Q}_1}{\tilde{\sigma}}\right)^{2/3} \left(\frac{r}{2} Q_r^{2/3} F_2^{4/3} - \frac{1}{2} F_1^{4/3} + Q_r^{2/3} F_2^{-2/3}\right) = \Delta E. \tag{2.3}$$

In conclusion, the full solution of the rigid-lid approximation is given by (2.1) and (2.2) or (2.3) as needed.

2.1 Properties at the Critical Point

A location within the channel of particular interest in shallow water flows is the critical point. For the rigid-lid case the critical point is defined in [4] as the a location where $G^2 = F_1^2 + rF_2^2 = 1$, where G is referred to as the composite Froude number of the system. Properties of the system at the critical point can be obtained from the previous equations (2.1) and (2.3) upon differentiation with respect to x. This produces the following system:

$$\begin{bmatrix} F_1^2 & 1 - rF_2^2 & -\Delta E & 0 \\ 1 & 1 & \tilde{B} - 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{2}{3} F_1^{-5/3} F_{1x} \\ -\frac{2}{3} Q_r^{2/3} F_2^{-5/3} F_{2x} \\ \frac{2}{3} \tilde{Q}_1^{-2/3} \tilde{\sigma}^{-1/3} \tilde{\sigma}_x \\ \tilde{Q}_1^{-2/3} \tilde{\sigma}^{2/3} \tilde{B}_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.4}$$

In turn, this system can be row reduced and evaluated at the critical point $G^2 = 1$ to obtain the following more interesting set of equations

$$\begin{aligned} 2\left[(1 - \tilde{B}) F_1^2 - \Delta E\right] \tilde{\sigma}_x - 3F_1^2 \tilde{\sigma} \tilde{B}_x = 0, \\ 2\left[(1 - \tilde{B})\left(1 - rF_2^2\right) - \Delta E\right] \tilde{\sigma}_x - 3\left(1 - rF_2^2\right) \tilde{\sigma} \tilde{B}_x = 0. \end{aligned} \tag{2.5}$$

It is important to note, that unlike single-layer shallow water flows with area contraction, the occurrence of a peak in the topography, $\tilde{B}_x = 0$, at the critical point does not uniquely imply the existence of a peak in the channel's width, $\tilde{\sigma}_x = 0$.

The top most equation of the system is simple to analyze and characteristic of the properties at the critical point for the rigid-lid system. A simple analysis of this equation can be obtained directly and is summarized below.

Properties 2.1. The following properties must hold at the critical point

- If $\tilde{B}_x = 0$, then $\tilde{\sigma}_x = 0$ or $F_1^2 = \frac{\Delta E}{1-\tilde{B}}$.
- For $\tilde{B}_x > 0$.
  - If $\tilde{\sigma}_x > 0$, then $F_1^2 > \frac{\Delta E}{1-\tilde{B}}$.
  - If $\tilde{\sigma}_x < 0$, then $F_1^2 < \frac{\Delta E}{1-\tilde{B}}$.
- For $\tilde{B}_x < 0$.
  - If $\tilde{\sigma}_x > 0$, then $F_1^2 < \frac{\Delta E}{1-\tilde{B}}$.
  - If $\tilde{\sigma}_x < 0$, then $F_1^2 > \frac{\Delta E}{1-\tilde{B}}$.

From the system (2.5), one may further solve for the internal Froude numbers F_1 and F_2 to obtain the system

$$\begin{aligned} F_1^2 &= \frac{2\Delta E\tilde{\sigma}_x}{2\left(1-\tilde{B}\right)\tilde{\sigma}_x - 3\tilde{\sigma}\tilde{B}_x}, \\ F_2^2 &= \frac{1}{r}\left(1 - \frac{2\Delta E\tilde{\sigma}_x}{2\left(1-\tilde{B}\right)\tilde{\sigma}_x - 3\tilde{\sigma}\tilde{B}_x}\right), \end{aligned} \tag{2.6}$$

of internal Froude number valid at the critical point. The significance of the system above is that these values of the internal Froude number must be satisfied at the critical point unless the system (2.5) is satisfied trivially, $\tilde{B}_x = 0$ and $\tilde{\sigma}_x = 0$. Hence, the values above may be substituted in the rigid-lid equation (2.1) or energy difference equation (2.3) to provide a root finding scheme to determine the actual location of the critical point given the discharge Q_i, layer energy difference ΔE, total water height H_0, and the known channel geometry and topography.

Several properties of the flow at the critical point can be obtained from the system (2.6) using the fact that $F_1^2 > 0$ and $F_2^2 > 0$ by definition. These properties are summarized below.

Properties 2.2. The following properties must hold at the critical point.

- For $\Delta E > 0$.
  - If $\tilde{\sigma}_x > 0$, then $\tilde{B}_x < \frac{2}{3}\left(1-\tilde{B}\right)\frac{\tilde{\sigma}_x}{\tilde{\sigma}}$.
  - If $\tilde{\sigma}_x < 0$, then $\tilde{B}_x > \frac{2}{3}\left(1-\tilde{B}\right)\frac{\tilde{\sigma}_x}{\tilde{\sigma}}$.
- For $\Delta E < 0$.
  - If $\tilde{\sigma}_x > 0$, then $\tilde{B}_x > \frac{2}{3}\left(1-\tilde{B}\right)\frac{\tilde{\sigma}_x}{\tilde{\sigma}}$.
  - If $\tilde{\sigma}_x < 0$, then $\tilde{B}_x < \frac{2}{3}\left(1-\tilde{B}\right)\frac{\tilde{\sigma}_x}{\tilde{\sigma}}$.
- For $\Delta E = 0$.
  - If $\tilde{\sigma}_x > 0$, then $\tilde{B}_x > 0$.
  - If $\tilde{\sigma}_x < 0$, then $\tilde{B}_x < 0$.

An equivalent system of equations to (2.5) can be obtained by differentiating equations (2.1) and (2.2). This system is of the form

$$\begin{bmatrix} F_1^2 & 1-rF_2^2 & F_1^2\tilde{h}_1 - rF_2^2\tilde{h}_2 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\sigma}\tilde{h}_{1x} \\ \tilde{\sigma}\tilde{h}_{2x} \\ \tilde{\sigma}_x \\ \tilde{\sigma}\tilde{B}_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.7}$$

Now, row reducing and evaluating at the critical point renders two equations

$$\begin{aligned} \left[\left(F_1^2-1\right)\left(1-\tilde{B}\right)+\tilde{h}_1\right]\tilde{\sigma}_x - F_1^2\tilde{\sigma}\tilde{B}_x &= 0, \\ \left[\left(1-rF_2^2\right)\left(1-\tilde{B}\right)-\tilde{h}_2\right]\tilde{\sigma}_x - \left(1-rF_2^2\right)\tilde{\sigma}\tilde{B}_x &= 0. \end{aligned} \tag{2.8}$$

Note that the system obtained is equivalent to (2.5). Hence, one may freely subtract the systems (2.5) and (2.8) to obtain the conditions

$$\begin{aligned} \tilde{h}_1 &= -\frac{2}{3}\Delta E + \left(1-\tilde{B}\right)\left(1-\frac{1}{3}F_1^2\right), \\ \tilde{h}_2 &= \frac{2}{3}\Delta E + \frac{1}{3}\left(1-\tilde{B}\right)F_1^2, \end{aligned} \tag{2.9}$$

upon solving for the non-dimensional height $\tilde{h}_i$. Note that because at the critical point $0 < F_1^2 < 1$, the heights must be bounded as follows

$$\frac{2}{3}\left(1-\Delta E-\tilde{B}\right) < \tilde{h}_1 < 1-\frac{2}{3}\Delta E - \tilde{B} \quad \text{and} \quad \frac{2}{3}\Delta E < \tilde{h}_2 < \frac{1}{3}\left(1+2\Delta E-\tilde{B}\right).$$

Equivalently, since $\tilde{h}_1 > 0$ and $\tilde{h}_2 > 0$, ΔE is bounded at the critical point as follows

$$-\frac{1}{2}\left(1-\tilde{B}\right)F_1^2 < \Delta E < \frac{1}{2}\left(1-\tilde{B}\right)\left(3-F_1^2\right) \quad \text{or more generally} \quad -\frac{1}{2}\left(1-\tilde{B}\right) < \Delta E < \frac{3}{2}\left(1-\tilde{B}\right).$$

2.2 Internal Froude Number at the Critical Point

It is important to note that (2.8) can be solved for ΔE and written entirely in terms of the internal Froude number. The significance of this fact is that it can in turn be replaced in (2.3). This new equation may be used to determine the value of F_1 or F_2 at the critical point, the remaining Froude number can be obtained by using the criticality condition $F_1^2 + rF_2^2 = 1$, respectively. The substitution renders the equation

$$\frac{r}{2}Q_r^{2/3}F_2^{4/3} - \frac{1}{2}F_1^{4/3} + Q_r^{2/3}F_2^{-2/3} = \frac{3}{2}F_1^{-2/3} - \frac{3}{2}\left(1-\tilde{B}\right)\left(1-\frac{1}{3}F_1^2\right)\left(\frac{\tilde{Q}_1}{\tilde{\sigma}}\right)^{-2/3}.$$

The following more helpful form of this equation is obtained after some simplification

$$\varphi = rF_2^{8/3} - F_2^{2/3} + \frac{3}{1-\tilde{B}}\left(\frac{\tilde{Q}_2}{\tilde{\sigma}}\right)^{2/3} = 0. \tag{2.10}$$

It is important to note that φ implies that the value of the internal Froude numbers at the critical point does not depend on the energy directly. It does however, depend on the discharge and the location of the critical point. The location of the critical point in turn does depend on the energy. Thus, if we are to pick an arbitrary point as the critical point for a given discharge, the energy of the flow is determined.

The usefulness of φ lies in the fact that this equation can be treated as a quartic polynomial of $y = F_2^{2/3}$. That is,

$$\varphi = ry^4 - y + \frac{3}{1-\tilde{B}}\left(\frac{\tilde{Q}_2}{\tilde{\sigma}}\right)^{2/3} = 0.$$

One can easily examine that the equation will only have roots if the critical point (calculus) of φ is below the x-axis, since $\lim_{y\to\infty}\varphi=\infty$ and $\lim_{y\to-\infty}\varphi=\infty$. Note that the critical point is $y=(1/4r)^{1/3}$. Thus, the statement of the existence of roots is reduced to

$$\frac{1}{r}\left(\frac{1}{4r}\right)^{4/3}-\left(\frac{1}{4r}\right)^{1/3}+\frac{3}{1-\tilde{B}}\left(\frac{\tilde{Q}_2}{\tilde{\sigma}}\right)^{2/3}\leq 0.$$

Simplifying

$$\frac{1}{1-\tilde{B}}\left(\frac{\tilde{Q}_2}{\tilde{\sigma}}\right)^{2/3}\leq\frac{1}{4}\left(\frac{1}{4r}\right)^{1/3}.$$

Hence, φ will only have roots for F_2 at the location under the above condition. It is important to note that φ will have more than one root for most cases.

3 Exact Solution

In the case of the exact solution, only the equations (1.2) are needed to determine the system. The procedure in this section, will try to mirror wherever possible the analysis done for the rigid-lid case. Thus, as in the rigid-lid case one can obtain the following two non-dimensional equations for the system of the energy difference between the layers and the energy of layer 2. The equations are of the form

$$\begin{aligned}\left(\frac{\tilde{Q}_1}{\tilde{\sigma}}\right)^{2/3}\left(\frac{r}{2}Q_r^{2/3}F_2^{4/3}-\frac{1}{2}F_1^{4/3}+F_1^{-2/3}\right)&=\Delta E-1+\tilde{B}=r\tilde{E}_2-\tilde{E}_1+\tilde{B},\\ \left(\frac{\tilde{Q}_1}{\tilde{\sigma}}\right)^{2/3}\left(\frac{1}{2}Q_r^{2/3}F_2^{4/3}+\frac{1}{1-r}F_1^{-2/3}+\frac{Q_r^{2/3}}{1-r}F_2^{-2/3}\right)&=\frac{E_2-gB}{g'H_0}=\tilde{E}_2-\frac{1}{1-r}\tilde{B},\end{aligned}\tag{3.1}$$

where one important distinction needs to be made, the value of H_0 no longer applies as a constant to the entire flow. It therefore must be determined given suitable boundary information. For the cases in this paper, the value H_0 is assumed to be the total water height directly at the boundary.

3.1 Properties at the Critical Point

As in the previous section, taking the derivative in terms of x of the system provides us with useful conditions at the critical point. For the exact solution, the critical point is defined as $G^2=F_1^2+F_2^2-(1-r)F_1^2F_2^2=1$ where G is the composite internal Froude number. For the system (3.1), taking derivatives

$$\begin{bmatrix}(1-r)\left(F_1^2-1\right) & -r(1-r)F_2^2 & -\Delta E+1-\tilde{B} & -(1-r)\\ 1 & 1-(1-r)F_2^2 & \tilde{E}_2-\frac{1}{1-r}\tilde{B} & 1\end{bmatrix}\begin{bmatrix}-\frac{2}{3}\frac{F_1^{-5/3}}{1-r}F_{1x}\\ -\frac{2}{3}\frac{\tilde{Q}_r^{2/3}F_2^{-5/3}}{1-r}F_{2x}\\ \frac{2}{3}\tilde{Q}_1^{-2/3}\tilde{\sigma}^{-1/3}\tilde{\sigma}_x\\ \tilde{Q}_1^{-2/3}\tilde{\sigma}^{2/3}\tilde{B}_x\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.\tag{3.2}$$

Again, this system can be row reduced and evaluated at the critical point. This renders the new system

$$\begin{aligned}2\left[\tilde{E}_2-\tilde{E}_1+\left(\tilde{B}-(1-r)\tilde{E}_2\right)F_1^2\right]\tilde{\sigma}_x+3F_1^2\tilde{\sigma}\tilde{B}_x=0,\\ 2\left[\Delta E+\tilde{B}-1+\left((1-r)\tilde{E}_1-\tilde{B}\right)F_2^2\right]\tilde{\sigma}_x+3\left(1-F_2^2\right)\tilde{\sigma}\tilde{B}_x=0.\end{aligned}\tag{3.3}$$

As before, the top most equation can be easily analyzed due to it's relative simplicity, revealing remarkable symmetry between the conditions for the rigid-lid case and the exact solution. They are summarized below.

Properties 3.1. The following conditions must hold at the critical point.

- If $\tilde{B}_x = 0$, then $\tilde{\sigma}_x = 0$ or $F_1^2 = \frac{\tilde{E}_2 - \tilde{E}_1}{(1-r)\tilde{E}_2 - \tilde{B}}$.
- For $\tilde{B}_x > 0$
  - If $\tilde{\sigma}_x > 0$, then $F_1^2 > \frac{\tilde{E}_2 - \tilde{E}_1}{(1-r)\tilde{E}_2 - \tilde{B}}$.
  - If $\tilde{\sigma}_x < 0$, then $F_1^2 < \frac{\tilde{E}_2 - \tilde{E}_1}{(1-r)\tilde{E}_2 - \tilde{B}}$.
- For $\tilde{B}_x < 0$
  - If $\tilde{\sigma}_x > 0$, then $F_1^2 < \frac{\tilde{E}_2 - \tilde{E}_1}{(1-r)\tilde{E}_2 - \tilde{B}}$.
  - If $\tilde{\sigma}_x < 0$, then $F_1^2 > \frac{\tilde{E}_2 - \tilde{E}_1}{(1-r)\tilde{E}_2 - \tilde{B}}$.

The system (3.3) can be solved, as before, for the internal Froude numbers F_1 and F_2 respectively to obtain

$$\begin{aligned} F_1^2 &= \frac{2\left(\tilde{E}_2 - \tilde{E}_1\right)\tilde{\sigma}_x}{2\left[(1-r)\tilde{E}_2 - \tilde{B}\right]\tilde{\sigma}_x - 3\tilde{\sigma}\tilde{B}_x}, \\ F_2^2 &= 1 - \frac{2r\left(\tilde{E}_2 - \tilde{E}_1\right)\tilde{\sigma}_x}{2\left[(1-r)\tilde{E}_1 - \tilde{B}\right]\tilde{\sigma}_x - 3\tilde{\sigma}\tilde{B}_x}. \end{aligned} \tag{3.4}$$

Using the fact that $F_1 > 0$ and $F_2 > 0$, the properties on relationship between the two layer energies can be determined. This is summarized below.

Properties 3.2. The following properties must hold at the critical point

- For $\tilde{E}_2 > \tilde{E}_1$
  - If $\tilde{\sigma}_x > 0$, then $\tilde{B}_x < \frac{2}{3}\left[(1-r)\tilde{E}_2 - \tilde{B}\right]\frac{\tilde{\sigma}_x}{\tilde{\sigma}}$.
  - If $\tilde{\sigma}_x < 0$, then $\tilde{B}_x > \frac{2}{3}\left[(1-r)\tilde{E}_2 - \tilde{B}\right]\frac{\tilde{\sigma}_x}{\tilde{\sigma}}$.
- $\tilde{E}_2 < \tilde{E}_1$
  - If $\tilde{\sigma}_x > 0$, then $\tilde{B}_x > \frac{2}{3}\left[(1-r)\tilde{E}_2 - \tilde{B}\right]\frac{\tilde{\sigma}_x}{\tilde{\sigma}}$.
  - If $\tilde{\sigma}_x < 0$, then $\tilde{B}_x < \frac{2}{3}\left[(1-r)\tilde{E}_2 - \tilde{B}\right]\frac{\tilde{\sigma}_x}{\tilde{\sigma}}$.
- $\tilde{E}_2 = \tilde{E}_1$
  - If $\tilde{\sigma}_x > 0$, then $\tilde{B}_x > 0$ and $\tilde{E}_2 > \frac{1}{1-r}\tilde{B}$ or $\tilde{B}_x < 0$ and $\tilde{E}_2 < \frac{1}{1-r}\tilde{B}$.
  - If $\tilde{\sigma}_x < 0$, then $\tilde{B}_x < 0$ and $\tilde{E}_2 > \frac{1}{1-r}\tilde{B}$ or $\tilde{B}_x > 0$ and $\tilde{E}_2 < \frac{1}{1-r}\tilde{B}$.

Lastly, it is possible to continue this procedure as in the rigid-lid case to obtain a polynomial for the internal Froude number at the critical point. For the case of the exact solution this polynomial becomes increasingly complicated. This case can be treated more easily through root finding methods and computer software. The method for obtaining this polynomial should be similar as that outlined in the previous section.

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