Directed Last-Passage Site Percolation with Bernoulli Site Distribution

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Abstract

In this document we investigate directed last-passage site percolation in the case that the underlying site distribution is Bernoulli. We also provide relevant preliminary material, experimental findings and background/reference information.

1 Preliminaries

1.1 Longest Increasing Subsequence

Given a permutation $\pi \in S_n$, where S_n is the symmetric group, one can ask for the length of its longest increasing subsequence (denoted by $l_n(\pi)$), i.e what is the longest subsequence $(a_1, ..., a_k)$ of $(1, 2, ..., n)$ s.t $\pi(a_1) < ... < \pi(a_k)$, viewing π as a function from $1, 2, ..., n$ to itself. First we give an algorithm, with an accompanying example, for computing $l_n(\pi)$.

Step 1: Start an array of "piles" with $\pi(1)$.
Step 2: For $i = 2$ to n, starting from the left, place $\pi(i)$ on top of the first pile whose top value is greater than $\pi(i)$. If no such pile exists, start a new pile to the right of the existing piles.

Here is an example for $n = 6$, $\pi = 325641$.

										1		
	2	2		2			2	4		2	4	
3	3	3	5	3	5	6	3	5	6	3	5	6

Theorem 1.1. *The number of piles resulting from the preceding algorithm is equal to $l_n(\pi)$.*

Proof. If $\pi(a_1) < ... < \pi(a_k)$ is an increasing subsequence, then each $\pi(a_i)$ is placed to the right of $\pi(a_{i-1})$ since $\pi(a_i)$ is greater than $\pi(a_{i-1})$ and the piles can only decrease with height. This shows that there are at least $l_n(\pi)$ piles at the end of the algorithm. Conversely, for each new entry not in the first pile, place a pointer from that entry to the top entry of the pile directly to the left (which it is greater than). One can then create an increasing subsequence of π by following one of the sequences of pointers through all of the piles. This shows that the number of piles is less than or equal to $l_n(\pi)$. $\square$

This algorithm, called *patience sorting* in [1] for example, is a stripped down version of the "bumping" algorithm of the Robinson-Schensted correspondence, which will be discussed in more detail at the end of this section. Also see [1] for everything discussed here about longest increasing subsequences.

Given S_n with the uniform distribution, one can also ask what is $\mathbb{E}(l_n)$, the expected value of the length of the longest increasing subsequence of a random permutation. Here is one of the first results towards that end.

Theorem 1.2 (Erdös-Szekeres). *Given $\pi \in S_n, n = mq + 1$ either π has an increasing subsequence of length $m + 1$ or a decreasing subsequence of length $q + 1$.*

Proof. Using patience sorting and the pigeonhole principle, there are at least $m + 1$ piles after completion of the algorithm or one of the piles has at least $q + 1$ entries. This implies that we have an increasing subsequence of length $m + 1$ or, since the entries in the piles are decreasing and follow each other in the permutation, there is a pile that is itself a decreasing subsequence of length $q + 1$. □

1.2 Asymptotics

Using the last result of the previous subsection, one can derive a lower bound for $\mathbb{E}(l_n)$ and $\liminf_{n\to\infty} \frac{\mathbb{E}(l_n)}{\sqrt{n}}$.

Theorem 1.3. *For $n = N^2 + 1$, $\mathbb{E}(l_n) \geq \frac{N+1}{2}$. Furthermore, $\liminf_{n\to\infty} \frac{\mathbb{E}(l_n)}{\sqrt{n}} \geq \frac{1}{2}$.*

Proof. Let $n = N^2 + 1$ and $\pi \in S_n$. Let l_n and l'_n denote the length of the longest increasing and decreasing subsequence of π respectively. From the theorem of Erdös-Szekeres above, either $l_n \geq N + 1$ or $l'_n \geq N + 1$. However, there is a bijection between $\{\pi \mid l_n \geq N+1\}$ and $\{\pi \mid l'_n \geq N+1\}$ so that $\mathbb{P}(l_n \geq N+1) \geq \frac{1}{2}$. We can now derive the first result

$$\mathbb{E}(l_n) = \sum_{k=0}^{n} k \cdot \mathbb{P}(l_n = k) = \sum_{k=0}^{n} k \cdot (\mathbb{P}(l_n \geq k) - \mathbb{P}(l_n \geq k+1))$$

$$= \sum_{k=0}^{n} \mathbb{P}(l_n \geq k) \geq \frac{N+1}{2}$$

As for the second assertion, given n, $\exists$ M s.t. $M^2 + 1 \leq n < (M+1)^2 + 1$ and so, by the fact that $\mathbb{E}(l_{k+1}) \geq \mathbb{E}(l_k) \forall$ k (which can easily be checked),we have $\mathbb{E}(l_n) \geq \mathbb{E}(l_{M^2+1}) \geq \frac{M+1}{2}$. As $n \to \infty$, $M \to \infty$ and so

$$\liminf_{n\to\infty} \frac{\mathbb{E}(l_n)}{\sqrt{n}} \geq \liminf_{M\to\infty} \frac{\frac{1}{2}(M+1)}{\sqrt{M^2+1}} = \frac{1}{2}$$

□

As it turns out, $\lim_{N\to\infty} \frac{\mathbb{E}(l_N)}{\sqrt{N}}$ exists ([8]) and is equal to 2 ([11]). In [3] it is also shown that in the limit, l_N, appropriately scaled and centered, converges to F_{TW}, the Tracy-Widom distribution for the largest eigenvalue of a GUE matrix as the matrix dimensions go to infinity. This distribution will be defined and discussed in a subsequent section.

1.3 Robinson-Schensted Correspondence

A *partition* λ of a positive integer k (denoted $\lambda \vdash k$) is a sequence of positive integers, $(\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r)$ such that $\sum_{i=1}^{r} \lambda_i = k$. A *Young diagram* or *Ferrer diagram* of a partition λ is an array with λ_1 cells in the first row, λ_2 cells in the second row, etc. A (standard) *Young tableau* is a Young diagram with the cells filled with the numbers $1, ..., k$ such that the numbers in the cells increase along the rows (left to right) and columns (top to bottom). An important result is the following.

Theorem 1.4 (Robinson-Schensted). *There is a bijection between S_n and pairs (P, Q) of Young tableaux of the same shape $\lambda \vdash n$.*

We present an algorithm for creating the P and Q tableaux; the details of the proof are left to the reader or can be found in [14] for example. Let $\pi = (a_1 = \pi(1), ..., a_n = \pi(n) \in S_n$. To create the P tableau:
Step 1: Start a Young tableau with a_1.

For $i = 2$ to n:

Step 2: Starting from the top-left, replace by a_i the first entry larger than a_i, say, a_j. If a_i is larger than all entries in the first row, append it to the end of the row.
Step 3: Repeat Step 2 with the displaced or "bumped" entry a_j along the second row, third row, etc. until the current displaced entry is appended to the end of a row or is bumped into an empty row.

To create the Q tableau, place the entry i into the the new position created by running the above algorithm with a_i, i.e. place i in the tableaux into the place where the cascade of "bumps" ends.

Here is an example for $n = 6$, $\pi = 325641$.

$$P: \quad \begin{array}{l}3\end{array} \quad \begin{array}{l}2\\3\end{array} \quad \begin{array}{ll}2&5\\3&\end{array} \quad \begin{array}{lll}2&5&6\\3&&\end{array} \quad \begin{array}{lll}2&4&6\\3&5&\end{array} \quad \begin{array}{lll}1&4&6\\2&5&\\3&&\end{array}$$

$$Q: \quad \begin{array}{l}1\end{array} \quad \begin{array}{l}1\\2\end{array} \quad \begin{array}{ll}1&3\\2&\end{array} \quad \begin{array}{lll}1&3&4\\2&&\end{array} \quad \begin{array}{lll}1&3&4\\2&5&\end{array} \quad \begin{array}{lll}1&3&4\\2&5&\\6&&\end{array}$$

Note that the length of the first row, λ_1, is $l_6(\pi = 325641)$. This is always the case as shown in the next theorem.

Theorem 1.5. *Given $\pi \in S_n$ and the corresponding pair of tableaux (P, Q) (of shape $\lambda = (\lambda_1 \geq ... \geq \lambda_k)$) under the Robinson-Schensted correspondence, $l_n(\pi) = \lambda_1$.*

Proof. We need only concern ourselves with the consequences of the bumping algorithm on the first row of P. We see that the current number bumps an entry if and only if it would be placed on top of the corresponding pile in the patience sorting algorithm (so that the number on top of a pile is the same as the current entry in the P tableau) and the current number is placed at the end of the first row if and only if it would create a new pile in patience sorting. Since these are the only two operations on the first row of the tableau (and correspondingly in patience sorting) we see that number of piles and λ_1 are the same and equal to $l_n(\pi)$. □

1.4 The Tracy-Widom Distribution

The Tracy-Widom distribution (see [15]) can be defined as

$$F_{TW}(x) := exp\left\{-\int_x^\infty (s-x)q^2(s)ds\right\}$$

where $q(x)$ solves the Painlevé II equation,

$$q'' = 2q^3 + xq$$

subject to the condition that $q(x) \sim -Ai(x)$ as $x \to +\infty$ where $Ai(x)$ denotes the Airy function. This function is the distribution of the largest eigenvalue of a GUE matrix as the dimensions go to infinity, and is also the distribution function of the length of the longest increasing subsequence of a random permutation properly scaled and centered, i.e. (see [3])

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{l_N - 2\sqrt{N}}{N^{\frac{1}{6}}} \leq x\right) = F_{TW}(x)$$

Hence, it is also the distribution function for the length of the first row of a random Young tableau under the Plancherel measure $\frac{f_P^2}{n!}$ (the measure imposed on the Young tableaux by the uniform distribution on S_n and the Robinson-Schensted correspondence) where $P \vdash n$ and f_P is the number of young tableaux of shape P. Moreover, it turns out that the distribution functions related to F_{TW}, those for the second, third,... largest eigenvalues of a GUE matrix correspond to the distributions of the second, third,... rows of a random Young tableau under the Plancherel measure ([4], [13], [7] and [10]).

1.5 Generalized Permutations

Formally, a *generalized permutation* of length n is a $2 \times n$ array with the first row a weakly increasing sequence with entries from $\{1, ..., M\}$; the entries of the second row are from $\{1, ..., N\}$ and are weakly increasing within each block where the entries in the first row repeat. There is a natural bijection between these objects and non-negative integer matrices. Let $A = (a_{i,j})$ be an $M \times N$ non-negative integer matrix with $\sum_{(i,j)} a_{i,j} = n$. Construct the associated generalized permutation π of length n as follows:

Start with π empty.
For $j = 1$ to N
For $i = 1$ to M
Append $\begin{pmatrix} i \\ j \end{pmatrix}$ to π, $a_{i,j}$ times.

For example,

$$\begin{pmatrix} 1 & 0 & 3 & 1 \\ 2 & 0 & 2 & 0 \\ 1 & 2 & 0 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\ 1 & 3 & 3 & 3 & 4 & 1 & 1 & 3 & 3 & 1 & 2 & 2 & 4 & 4 & 4 \end{pmatrix}$$

The name generalized permutation comes from the fact that if A were a permutation matrix, π would be a permutation in the usual sense. The following theorem can be found in [12]. To state it, we need the definition of a *semi-standard Young tableau*. A semi-standard (or column strict) Young tableau of shape $\lambda \vdash k$, is a Young diagram with non-negative integer entries such that the entries in the rows are weakly increasing and those of the columns are strictly increasing. We present the theorem without proof, but it is just a generalization of the Robinson-Schensted correspondence presented earlier.

Theorem 1.6 (Robinson-Schensted-Knuth). *There is a bijection from non-negative integer matrices whose entries sum to n (equivalently, generalized permutations of length n) and pairs of semi-standard Young tableaux (P, Q) of the same shape $\lambda = (\lambda_1 \leq ... \leq \lambda_k) \vdash n$. Moreover, the length of the longest non-decreasing subsequence of the associated generalized permutation is λ_1.*

2 Last Passage Percolation Model

Let $\mathbb{Z}_+^2 = \{(x, y) \in \mathbb{Z}^2 | x, y > 0\}$ and denote by Π_γ the set of all up-right paths from $(1, 1)$ to $(N, \gamma N)$ i.e., letting $M = \gamma N + N - 1$,

$$\Pi_\gamma = \{(z_1, z_2, ..., z_M) \in (\mathbb{Z}_+^2)^M | z_1 = (1, 1), z_M = (N, \gamma N)$$
$$\text{and } z_i - z_{i-1} = (1, 0) \text{ or } (0, 1)\}$$

Let $R_{(i,j)}$ be i.i.d random variables $\forall\ (i, j) \in \mathbb{Z}_+^2$. One can think of $R_{(i,j)}$ as the passage time or weight of the site (i, j). Denote by

$$L(N, \gamma N) = \max_{\pi \in \Pi_\gamma} \{ \sum_{(i,j) \in \pi} R_{(i,j)} \} \text{ for } \gamma \in [1, \infty)$$

the last passage time or heaviest weight of a path from $(1, 1)$ to $(N, \gamma N)$. We only consider $\gamma \in [1, \infty)$ since, in the limit, there is an obvious symmetry between $L(N, \gamma N)$ and $L(N, \frac{N}{\gamma})$. Limiting distributions for the cases where the $R_{(i,j)}$ are geometrically or exponentially distributed random variables have been studied in [9]; in particular, appropriately scaled and centered, these converge to the Tracy-Widom distribution. Universality results for paths asymptotically close to the axis (i.e. $L(N^a, N)$ for a suitably small) are discussed in [6] and [5] where the limiting distribution is also the Tracy-Widom distribution. It seems likely that similar results should hold more generally for a large class of distributions on the $R_{(i,j)}$ (cf. section 3.3).

A naive approach to simulating this model might involve going over each path individually and taking the maximum over all of the paths for $L(N, \gamma N)$. However, there are $\binom{(\gamma+1)N-2}{N-1}$ paths from (1,1) to $(N, \gamma N)$

and calculations with this method become unreasonable. Instead, when dealing with discrete $R_{(i,j)}$ one can use the following method based on the RSK algorithm. Given a realization of the model, we can view $\{(i,j) \in \mathbb{Z}_+^2 | 1 \le i \le N, 1 \le j \le \gamma N\}$ as a non-negative integer matrix and compute the associated generalized permutation. Then calculating $L(N, \gamma N)$ is the same as calculating the length of the longest non-decreasing subsequence of the associated permutation, which can be done with the RSK algorithm presented earlier. With this approach we go over the γN^2 sites once to create the generalized permutation and go through the permutation once using the RSK algorithm to find $\lambda_1 = L(N, \gamma N)$. There is an obvious reduction in computational complexity by using this approach which allows better numerical exploration and, in the cases that have been more thoroughly explored, a theoretical advantage with this perspective.

In [9], this approach is used to find and prove the limiting distribution and other information for geometric $R_{(i,j)}$; every non-negative integer valued matrix is a realization of the model in the geometric case and the problem in some sense reduces to counting the number of pairs of semi-standard Young tableaux of the same shape with first rows of specified length. The exponential case is solved by taking the limit as $q \to 1$ in the geometric case and the case of the longest increasing subsequence of a random permutation can be likened to taking the limit as $q \to 0$ in the geometric case.

2.1 Bernoulli Site Distribution

In this paper, the ideas developed in previous sections will be used to study this model under the assumption that the $R_{(i,j)}$ are Bernoulli random variables, i.e. $R_{(i,j)} = 1$ with probability p and $R_{(i,j)} = 0$ with probability $1 - p$ for some $p \in (0, 1)$. For the rest of the paper, unless otherwise stated, we will be working under the assumption that the $R_{(i,j)}$ are Bernoulli.

In particular, one object of our study is the behavior of $\lim_{N\to\infty} \frac{\mathbb{E}(L(N,\gamma N))}{N}$, so the first step is showing the existence of this limit. For that, we will need the following well-known theorem.

Theorem 2.1 (Subadditivity). *Given a sequence $(a_i)_{i=1}^{\infty}$ such that $a_{m+n} \le a_n + a_m$ then $\exists$ c, $-\infty \le c < \infty$ so that $\lim_{n\to\infty} \frac{a_n}{n} = c$.*

Proof. First note that $a_{2n} \le 2a_n, a_{3n} \le 3a_n$, etc. Fix a positive integer q so that for any n, $\exists p, r$ such that $p \ge 0, 0 \le r < q$ and $n = pq + r$. We now have $a_n = a_{pq+r} \le pa_q + a_r$ and $\frac{a_n}{n} \le \frac{a_q}{q+\frac{r}{p}} + \frac{a_r}{n}$. Taking the limit as $n \to \infty$, we have $p \to \infty$, r bounded and q fixed so that $\limsup_{n\to\infty} \frac{a_n}{n} \le \frac{a_q}{q}$. So the left hand side is less than $+\infty$ (but possibly equal to $-\infty$) and we take the limit as $q \to \infty$ to find that $\limsup_{n\to\infty} \frac{a_n}{n} \le \liminf_{q\to\infty} \frac{a_q}{q}$ which shows that $\limsup = \liminf$ as desired. ☐

Here is a simple lower bound for $\mathbb{E}(L(N, \gamma N))$ with Bernoulli $R_{(i,j)}$ with parameter p.

Theorem 2.2. *For Bernoulli $R_{(i,j)}$ with parameter p,*

$$\mathbb{E}(L(N, \gamma N)) \ge p((\gamma + 1)N - 1) \text{ and } \lim_{N\to\infty} \frac{\mathbb{E}(L(N, \gamma N))}{N} \ge p(\gamma + 1)$$

Proof. The expected value of the maximum over Π_γ will be greater than the expected value of a single path from Π_γ which is $\sum_{(i,j)\in\pi} p = p((\gamma + 1)N - 1)$. Dividing by N and taking the limit as $N \to \infty$ gives the result. ☐

We also have a trivial upper bound given by maximal paths, i.e. paths having $R_{(i,j)} = 1$ at each site along the path.

Theorem 2.3. *For Bernoulli $R_{(i,j)}$ with parameter p,*

$$\mathbb{E}(L(N, \gamma N)) \le \max(L(N, \gamma N)) \le (\gamma + 1)N - 1 \text{ and } \lim_{N\to\infty} \frac{\mathbb{E}(L(N, \gamma N))}{N} \le \gamma + 1$$

Proof. The length of a maximal (all 1's) path is $(\gamma + 1)N - 1$. Dividing by N and take the limit as $N \to \infty$ gives the result. ☐

With these bounds, we can now prove the existence of $\lim_{N\to\infty} \frac{\mathbb{E}(L(N,\gamma N))}{N}$ with the subadditivity theorem.

Theorem 2.4. *For Bernoulli site distribution with parameter p, $\lim_{N\to\infty} \frac{\mathbb{E}(L(N,\gamma N))}{N} = c(p,\gamma)$ for some constant $c(p,\gamma)$ depending on γ, p.*

Proof. Given a path π_n from (1,1) to $(n,\gamma n)$ and a path π_m from (1,1) to $(m,\gamma m)$ we can construct a path π_{n+m} from (1,1) to $(n+m,\gamma(n+m))$ by conjoining π_n and π_m (due to the independence of each site distribution). So we have $L(n,\gamma n)+L(m,\gamma m) \le L(n+m,\gamma(n+m))$ and hence $\mathbb{E}(L(n,\gamma n))+\mathbb{E}(L(m,\gamma m)) \le \mathbb{E}(L(n+m,\gamma(n+m)))$. The sequence $(-\mathbb{E}(L(k,\gamma k)))_{k=1}^{\infty}$ satisfies the hypotheses of the subadditive theorem above so $\exists\ c, -\infty < c \le \infty$ such that $\lim_{k\to\infty} \frac{\mathbb{E}(L(k,\gamma k))}{k} = c(p,\gamma)$. With the bounds presented above, we have $(\gamma+1)p \le c \le \gamma+1$. □

Figures 1 and 2 in Appendix E are images constructed from numerical estimates for $\lim_{N\to\infty} \frac{\mathbb{E}(L(N,\gamma N))}{N}$ with $N = 1000$, $\gamma = 1, 5$ to use as typical examples. The main features are the monotone increase from $p = 0$ until $p = p_c$, the *critical probability*, which depends on γ, where the graph levels out at $\gamma + 1$. Although we have no proof of the existence of such a critical probability, it is an obvious feature of the numerical experiments and has been proven to exist in a few other percolation models. Here p_c is defined by the smallest p such that $\mathbb{P}(\exists$ a maximal path from $(1,1)$ to $(N,\gamma N)) > 0$, a maximal path being one in which $R_{(i,j)} = 1$ for all (i,j) on the path. In our numerical study, we take p_c to be the smallest p such that a maximal path $(L(N,\gamma N) = (\gamma+1)N - 1)$ appears in the data. An idea of how p_c changes with γ can be seen in figures 3 and 4, the most extensive of the data sets we created. The critical probability is marked by the "choppy" line running through all of the other curves. For $N = 100, \gamma = 1, 1.1, 1.2, ..., 30.1, 30.2$ a table for p_c is in Appendix C along with figures 5 and 6 (in Appendix E) which show $\mathbb{P}(\exists$ a maximal path) as a function of p for $N = 1000$ and $\gamma = 1$ (figure 5), $\gamma = 5$ (figure 6). As one can see from this data, there is a large jump from 0 towards 1. It would be expected that as $N \to \infty$ this becomes a 0-1 correspondence, i.e there are maximal paths with probability 0 below p_c and maximal paths with probability 1 above p_c.

2.2 Relation to First-Passage Percolation and Point-Line Percolation

As a side note, not discussed further than the following statement, there is a simple transformation from last-passage percolation to first-passage percolation, where $F(N,\gamma N)$ is defined almost exactly as $L(N,\gamma N)$ except we take the *minimum* over all paths from $(1,1)$ to $(N,\gamma N)$ as opposed to the *maximum.*

Theorem 2.5. $\mathbb{E}(F(N,\gamma N)) = (\gamma+1)N - (1 + \mathbb{E}(L(N,\gamma N)))$ *where the left-hand side is over Bernoulli $R_{(i,j)}$ with parameter p and the right-hand side is over Bernoulli $R_{(i,j)}$ with parameter $1-p$.*

A result known to hold in the geometric and exponential cases, and that our numerical evidence indicates also holds in the Bernoulli case, relates last-passage percolation to a similar model, point-line percolation. Consider the following set of paths in $\mathbb{Z}_2^+$, the set of all up-right paths from $(1,1)$ to the line $y = 2N - x$. Letting $M = 2N - 1$,

$$\Pi_N^L = \{(z_1, z_2, ..., z_M) \in (\mathbb{Z}_+^2)^M | z_1 = (1,1), z_M \in \{(x,y)|x+y = 2N\}$$

$$\text{and } z_i - z_{i-1} = (1,0) \text{ or } (0,1)\}$$

For the geometric and exponential cases it is known that $\lim_{N\to\infty} \frac{\mathbb{E}(L(N,N))}{N}$ and the analogous limit for the length of the longest up-right path in point-line percolation are the same. Our numerical evidence seems to show that this is also true in the Bernoulli case as shown in figure 7 (Appendix E) for $N = 500$ (unscaled). Somewhat more thorough numerical evidence is provided in tables in Appendix B.

The following is an heuristic argument for a lower bound of point-line (and hence last-passage for $\gamma = 1$). Given a point $(i,j) \in \mathbb{Z}_+^2$ with Bernoulli site distribution (parameter p), what is the expected value of the length of a contiguous path of 1's starting from (i,j)? The probability that one can move either up, right, or both is $1-(1-p)^2 = p(2-p)$. The probability that one can do this exactly n times (randomly choosing whether to go up or right when both are available) is $(p(2-p))^n(1-p)^2$. We have

$$\mathbb{E}(n) = \sum_{n=0}^{\infty} n(p(2-p))^n(1-p)^2 = (1-p)^2 p(2-p)\sum_{n=0}^{\infty} n(p(2-p))^{n-1}$$

$$= \frac{(1-p)^2 p(2-p)}{2(1-p)}\sum_{n=0}^{\infty} n(p(2-p))^{n-1}2(1-p) = \frac{(1-p)^2 p(2-p)}{2(1-p)}\sum_{n=0}^{\infty}\frac{d}{dp}(p(2-p))^n$$

$$= \frac{(1-p)^2 p(2-p)}{2(1-p)}\frac{d}{dp}\left(\sum_{n=0}^{\infty}(p(2-p))^n\right) = \frac{1-(1-p)^2}{(1-p)^2}$$

Now consider a path where one takes $\mathbb{E}(n)$ steps, hits a dead end and starts again, taking another $\mathbb{E}(n)$ steps. So the ratio of 1's to total steps is $\frac{\mathbb{E}(n)}{1+\mathbb{E}(n)} = p(2-p)$. Multiplying by the number of total steps $(2N-1)$, dividing by N and taking the limit as $N \to \infty$ gives us a lower bound for $\lim_{N\to\infty}\frac{\mathbb{E}(L(N,N))}{N}$ of $2p(1-p)$ shown in Figure 8 for $N = 1000$.

2.3 Bernoulli and Tracy-Widom

It is clear that the distribution of $L(N,\gamma N)$ cannot be related to the Tracy-Widom distribution for values of $p \geq p_c$. What can be said for values of $p \leq p_c$? Our numerical evidence seems to show that the distribution of $L(N,\gamma N)$ is related to F_{TW} for these values of p (see figures 9, 10 and 11 in Appendix E). These figures show that $\frac{L(N,\gamma N)-a}{b}$ approaches the Tracy-Widom distribution where $a = \overline{L} + \mu_{TW}\frac{S}{\sigma_{TW}}$ and $b = \frac{S}{\sigma_{TW}}$ with $\overline{L}$ and S^2 the sample mean and sample variance respectively (see section 3 below), and $\mu_{TW} = -1.77109...$ and $\sigma_{TW}^2 = .81324...$ the Tracy-Widom mean and variance respectively. For contrast, figures 10 ($\gamma = 1$) and 11 ($\gamma = 5$) show the scaled distributions for $p = .1, .2, ..., .8, .9$. As one can see, for $p > p_c$ the distributions deviate from the Tracy-Widom and $p < p_c$ distributions.

3 Numerical Observations

Here we discuss some of the details of the numerical methods employed to gather the data in the appendices. Let $L_i(N,\gamma N)$ for $i = 1, 2, ..., t$ be t independent trials of the last passage percolation time. Define

$$\overline{L}(N,\gamma N) := \frac{1}{t}\sum_{i=1}^{t} L_i(N,\gamma N)$$

to be the sample mean of $L(N,\gamma N)$. The sample variance is defined as

$$S^2 := \frac{1}{t-1}\sum_{i=1}^{t}(L_i(N,\gamma N) - \overline{L}(N,\gamma N))^2.$$

Where we have looked at the values of $\overline{L}(N,\gamma N)$ over a range of values of p we have used in most cases 100 trials, and when looking at the distribution for a single value of p (in order to compare it with F_{TW}) we have used 1000 trials.

As for values of N, we have often used 500 or 1000 as a “large enough” value to compare with the limiting values. However, for the largest data set in the appendix, the table of critical values as a function of γ, we used $N = 100$ in order to make the running time of the program reasonable. From just looking at figure 12 ($N = 100$ is slightly the odd one out), it does not seem as though this overly compromises the data; although the larger numbers are preferable. Below is a table of critical values for $N = 100, 500, 1000$, $\gamma = 1, 5$ (using 100 trials) so as to “justify” the use of $N = 100$.

n	γ	p_c	n	γ	p_c
100	1	0.69	100	5	0.84
500	1	0.70	500	5	0.85
1000	1	0.70	1000	5	0.85

The critical values in Appendix C are recorded as the first value of p such that the probability of there being a maximal path (i.e. $R_{(i,j)} = 1$ for all (i,j) on the path) was non-zero (or, more precisely, the value of p such that some trials of $L(N, \gamma N)$ were maximal). Figures 5 and 6 contain graphs of the ratio of maximal paths to total number of trials.

As a further note, we did similar simulations using geometric $R_{(i,j)}$ (where there are known results) with comparable values n. Figure 13 shows $\frac{\overline{L}(100,100)}{100}$ for geometric $R_{(i,j)}$ with $p = .01, .02, ..., .49, .5$ and 100 trials along with the known limiting values (see [9]). As one can see in this figure, they are close but not exact. Basically, $n = 100$ isn't quite as large as we would like, but using $n = 100$ allowed more data to be collected in a shorter amount of time and still gives a good estimate for p_c.

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A Excerpts from Simulations of $L(N, \gamma N)$, Means and Variances

The following tables are some examples of the output of programs written by the author in C++ (parts of the code are included in appendix X). These tables contain sample means $\overline{L}(N, \gamma N)$ and variances S^2 for $N = 500, 1000$ and $\gamma = 1, 5$ as we let $p = .01, .02, ..., .99$. The 3rd and 7th columns are scaled sample means $\frac{\overline{L}(N,\gamma N)}{N}$ and the 5th and 9th columns are scaled sample variances $\frac{S^2}{N^{\frac{2}{3}}}$. Observe that $|\frac{\overline{L}(1000,\gamma 1000)}{1000} - \frac{\overline{L}(500,\gamma 500)}{500}| \approx .01$. However, $\frac{S^2}{N^{\frac{2}{3}}}$ for $N = 500, 1000$ are quite different, possibly because the sample size of 100 was inadequate. Graphs of $\frac{\overline{L}(N,\gamma N)}{N}$ for $N = 1000, \gamma = 1, 5$ are figures 1 and 2 in Appendix E respectively.

p	$\overline{L}(500, 500)$	$\frac{\overline{L}(500,500)}{500}$	S^2	$\frac{S^2}{500^{\frac{2}{3}}}$	$\overline{L}(1000, 1000)$	$\frac{\overline{L}(1000,1000)}{1000}$	S^2	$\frac{S^2}{1000^{\frac{2}{3}}}$
0.01	102.33	0.20466	13.3951	2.84823	209.54	0.20954	16.2105	2.6278
0.02	150.84	0.30168	16.257	4.19533	308.02	0.30802	28.5248	8.13667
0.03	189.97	0.37994	20.2718	6.52337	386.88	0.38688	26.3087	6.92147
0.04	223.72	0.44744	22.2036	7.82591	454.23	0.45423	33.5122	11.2307
0.05	254.15	0.5083	30.5328	14.7986	517.28	0.51728	39.5976	15.6797
0.06	282.11	0.56422	23.1898	8.53651	571.78	0.57178	47.9713	23.0125
0.07	309.2	0.6184	33.2525	17.5524	624.56	0.62456	42.5115	18.0723
0.08	333.36	0.66672	39.4651	24.7236	675.58	0.67558	54.8521	30.0876
0.09	356.03	0.71206	40.2516	25.719	720.95	0.72095	53.8258	28.9721
0.1	378.06	0.75612	32.4812	16.7475	765.43	0.76543	44.49	19.7936
0.11	399.81	0.79962	35.6504	20.1751	808.96	0.80896	51.6752	26.7032
0.12	420.69	0.84138	37.6504	22.5022	849.02	0.84902	45.1309	20.368
0.13	440.93	0.88186	39.6415	24.9452	889.03	0.88903	53.4637	28.5837
0.14	459.61	0.91922	33.21	17.5075	926.55	0.92655	57.6237	33.205
0.15	477.89	0.95578	28.2201	12.6417	965.26	0.96526	49.2853	24.2904
0.16	495.45	0.9909	28.7146	13.0886	998.84	0.99884	53.8125	28.9579
0.17	513.24	1.02648	38.5277	23.5631	1033.91	1.03391	47.1938	22.2726
0.18	529.19	1.05838	26.1555	10.8595	1068.33	1.06833	64.6274	41.767
0.19	546.06	1.09212	34.2186	18.5871	1102.88	1.10288	70.7733	50.0886
0.2	562.89	1.12578	38.3817	23.3849	1133.16	1.13316	68.1358	46.4248
0.21	577.33	1.15466	32.1425	16.4001	1164.24	1.16424	51.6186	26.6448
0.22	592.32	1.18464	32.9673	17.2525	1195.61	1.19561	45.4524	20.6592
0.23	608.24	1.21648	42.1842	28.248	1224.06	1.22406	62.2792	38.787
0.24	622.59	1.24518	34.6484	19.0569	1254.98	1.25498	56.7875	32.2482
0.25	636.52	1.27304	35.5855	20.1017	1282.46	1.28246	52.8368	27.9172
0.26	648.93	1.29786	32.9344	17.2182	1310.6	1.3106	53.0303	28.1221
0.27	663.74	1.32748	28.5984	12.9828	1335.75	1.33575	51.7854	26.8172
0.28	676.82	1.35364	38.9168	24.0414	1362.5	1.3625	61.5455	37.8784
0.29	689.98	1.37996	35.9592	20.5261	1387.13	1.38713	55.6496	30.9688
0.3	701.5	1.403	36.3535	20.9788	1412.21	1.41221	53.9858	29.1446
0.31	714.73	1.42946	30.5829	14.8472	1438.34	1.43834	47.4792	22.5427
0.32	726.25	1.4525	36.5328	21.1862	1461.88	1.46188	43.0562	18.5383
0.33	738.52	1.47704	37.444	22.2563	1484.9	1.4849	47.3636	22.4331

p	$\overline{L}(500,500)$	$\frac{\overline{L}(500,500)}{500}$	S^2	$\frac{S^2}{500^{\frac{2}{3}}}$	$\overline{L}(1000,1000)$	$\frac{\overline{L}(1000,1000)}{1000}$	S^2	$\frac{S^2}{1000^{\frac{2}{3}}}$
0.34	750.62	1.50124	44.44	31.3498	1507.24	1.50724	53.962	29.119
0.35	760.84	1.52168	28.0347	12.4761	1529.6	1.5296	49.697	24.6979
0.36	773.18	1.54636	27.5228	12.0247	1553.5	1.5535	51.202	26.2165
0.37	781.83	1.56366	34.4456	18.8345	1573.07	1.57307	48.4698	23.4932
0.38	793.31	1.58662	30.5393	14.8049	1594.47	1.59447	53.2011	28.3036
0.39	803.58	1.60716	28.9329	13.2884	1614.91	1.61491	37.8807	14.3495
0.4	813.59	1.62718	26.4666	11.1194	1634.41	1.63441	42.2241	17.8288
0.41	823.29	1.64658	19.1575	5.8259	1654.22	1.65422	52.6178	27.6863
0.42	832.67	1.66534	33.6779	18.0043	1673.69	1.67369	40.2565	16.2058
0.43	842.39	1.68478	27.513	12.0161	1690.94	1.69094	41.4711	17.1985
0.44	850.77	1.70154	24.3607	9.42034	1709.56	1.70956	51.2792	26.2956
0.45	859.66	1.71932	23.1156	8.48194	1727.24	1.72724	29.0933	8.46422
0.46	867.46	1.73492	21.564	7.38154	1744.73	1.74473	37.7142	14.2236
0.47	876.34	1.75268	25.1358	10.0293	1760.83	1.76083	30.5062	9.30626
0.48	885.07	1.77014	22.288	7.88548	1777	1.777	31.0101	9.61626
0.49	892.09	1.78418	15.8605	3.9932	1793.16	1.79316	44.5802	19.8739
0.5	901.14	1.80228	28.5055	12.8986	1807.49	1.80749	27.6868	7.66557
0.51	908.19	1.81638	17.9534	5.1166	1822.06	1.82206	27.5923	7.61336
0.52	915.13	1.83026	17.69	4.96755	1837.37	1.83737	27.2254	7.4122
0.53	921.91	1.84382	15.214	3.67431	1850.83	1.85083	25.193	6.34689
0.54	928.12	1.85624	12.3693	2.42871	1863.48	1.86348	22.7168	5.16052
0.55	934.89	1.86978	10.4625	1.73764	1876.71	1.87671	21.8241	4.76293
0.56	940.64	1.88128	11.4044	2.0646	1888.74	1.88874	18.2549	3.33243
0.57	946.91	1.89382	10.5474	1.76594	1901.61	1.90161	19.9777	3.99108
0.58	952.56	1.90512	15.764	3.94477	1911.65	1.91165	15.5631	2.42211
0.59	957.65	1.9153	10.9167	1.89176	1921.98	1.92198	19.7572	3.90346
0.6	962.52	1.92504	11.04	1.93475	1932.18	1.93218	11.6844	1.36526
0.61	967.97	1.93594	10.0294	1.59675	1941.32	1.94132	15.9168	2.53344
0.62	972.44	1.94488	7.78424	0.961877	1950.64	1.95064	12.2933	1.51126
0.63	976.86	1.95372	7.79838	0.965375	1958.46	1.95846	10.8368	1.17436
0.64	980.38	1.96076	7.28848	0.843259	1966.97	1.96697	9.20111	0.846604
0.65	984.28	1.96856	6.12283	0.595101	1973.22	1.97322	7.91071	0.625793
0.66	987.25	1.9745	6.43182	0.656681	1979.67	1.97967	8.16273	0.666301

p	$\overline{L}(500,500)$	$\frac{\overline{L}(500,500)}{500}$	S^2	$\frac{S^2}{500^{\frac{2}{3}}}$	$\overline{L}(1000,1000)$	$\frac{\overline{L}(1000,1000)}{1000}$	S^2	$\frac{S^2}{1000^{\frac{2}{3}}}$
0.67	990.06	1.98012	3.30949	0.173864	1984.66	1.98466	8.91354	0.794511
0.68	992.53	1.98506	3.60515	0.206316	1989.13	1.98913	4.01323	0.16106
0.69	994.53	1.98906	2.35263	0.0878603	1992.83	1.99283	3.11222	0.0968593
0.7	996.26	1.99252	2.67919	0.113945	1995.39	1.99539	2.46253	0.0606403
0.71	996.9	1.9938	1.44444	0.0331198	1997	1.997	1.29293	0.0167167
0.72	997.4	1.9948	1.41414	0.0317448	1997.61	1.99761	1.31101	0.0171875
0.73	997.87	1.99574	0.942525	0.0141017	1997.78	1.99778	0.900606	0.00811091
0.74	997.8	1.9956	1.07071	0.0181982	1997.94	1.99794	0.905455	0.00819848
0.75	998.15	1.9963	0.674242	0.00721637	1998.13	1.99813	0.740505	0.00548348
0.76	998.1	1.9962	0.616162	0.00602665	1998.11	1.99811	0.886768	0.00786357
0.77	998.26	1.99652	0.598384	0.0056839	1998.25	1.99825	0.593434	0.00352164
0.78	998.36	1.99672	0.596364	0.00564559	1998.4	1.9984	0.505051	0.00255076
0.79	998.37	1.99674	0.518283	0.00426403	1998.32	1.99832	0.603636	0.00364377
0.8	998.44	1.99688	0.450909	0.00322749	1998.44	1.99844	0.390303	0.00152336
0.81	998.43	1.99686	0.449596	0.00320872	1998.53	1.99853	0.332424	0.00110506
0.82	998.47	1.99694	0.39303	0.0024521	1998.51	1.99851	0.373636	0.00139604
0.83	998.48	1.99696	0.433939	0.00298913	1998.62	1.99862	0.278384	0.000774976
0.84	998.62	1.99724	0.318788	0.00161321	1998.56	1.99856	0.349899	0.00122429
0.85	998.57	1.99714	0.38899	0.00240195	1998.56	1.99856	0.309495	0.000957871
0.86	998.63	1.99726	0.376869	0.00225459	1998.66	1.99866	0.267071	0.000713268
0.87	998.61	1.99722	0.260505	0.00107726	1998.67	1.99867	0.263737	0.000695574
0.88	998.78	1.99756	0.193535	0.000594576	1998.68	1.99868	0.260202	0.000677051
0.89	998.64	1.99728	0.313535	0.00156049	1998.75	1.99875	0.229798	0.000528071
0.9	998.81	1.99762	0.175657	0.000489796	1998.86	1.99886	0.141818	0.000201124
0.91	998.68	1.99736	0.24	0.000914343	1998.81	1.99881	0.175657	0.000308552
0.92	998.89	1.99778	0.0988889	0.000155232	1998.75	1.99875	0.209596	0.000439305
0.93	998.88	1.99776	0.106667	0.000180611	1998.84	1.99884	0.15596	0.000243234
0.94	998.91	1.99782	0.0827273	0.000108639	1998.84	1.99884	0.15596	0.000243234
0.95	998.88	1.99776	0.126869	0.000255503	1998.87	1.99887	0.154646	0.000239155
0.96	998.9	1.9978	0.131313	0.000273718	1998.93	1.99893	0.0657576	4.32406e-05
0.97	998.95	1.9979	0.0479798	3.65429e-05	1998.97	1.99897	0.0293939	8.64004e-06
0.98	998.94	1.99788	0.0569697	5.15198e-05	1998.93	1.99893	0.0657576	4.32406e-05
0.99	998.98	1.99796	0.04	2.53984e-05	1998.97	1.99897	0.0293939	8.64004e-06

p	$\overline{L}(500, 2500)$	$\frac{\overline{L}(500,2500)}{500}$	S^2	$\frac{S^2}{500^{\frac{2}{3}}}$	$\overline{L}(1000, 5000)$	$\frac{\overline{L}(1000,5000)}{1000}$	S^2	$\frac{S^2}{1000^{\frac{2}{3}}}$
0.01	242.52	0.48504	22.2925	7.8887	491.41	0.49141	41.4363	17.1696
0.02	360.13	0.72026	30.0334	14.3185	727.31	0.72731	58.5191	34.2448
0.03	455.34	0.91068	36.2065	20.8094	920.07	0.92007	68.1062	46.3845
0.04	538.21	1.07642	40.7736	26.3904	1088.18	1.08818	86.5531	74.9144
0.05	616.93	1.23386	57.3587	52.2258	1241.96	1.24196	62.261	38.7643
0.06	687.12	1.37424	56.3289	50.3673	1383.76	1.38376	116.043	134.659
0.07	753.7	1.5074	58.0505	53.4932	1517.92	1.51792	92.2158	85.0375
0.08	815.73	1.63146	57.7142	52.8753	1646.25	1.64625	84.3914	71.2191
0.09	876.74	1.75348	49.8913	39.5127	1764.97	1.76497	114.312	130.673
0.1	936.39	1.87278	89.311	126.618	1884.39	1.88439	105.21	110.691
0.11	990.21	1.98042	69.5413	76.7666	1992.26	1.99226	125.891	158.486
0.12	1043.86	2.08772	84.3034	112.818	2101.66	2.10166	115.479	133.354
0.13	1097.75	2.1955	79.6439	100.691	2204.84	2.20484	102.52	105.103
0.14	1148.64	2.29728	77.5459	95.4561	2308.78	2.30878	109.486	119.873
0.15	1197.68	2.39536	89.3107	126.617	2408.85	2.40885	93.1187	86.7109
0.16	1245.92	2.49184	75.0238	89.3481	2506.06	2.50606	118.825	141.193
0.17	1293.72	2.58744	93.1733	137.807	2599.56	2.59956	129.501	167.706
0.18	1338.41	2.67682	92.507	135.842	2691.56	2.69156	128.37	164.789
0.19	1383.36	2.76672	79.5863	100.546	2781.64	2.78164	118.758	141.035
0.2	1428.13	2.85626	77.8516	96.2104	2868.61	2.86861	129.271	167.109
0.21	1471.42	2.94284	96.7511	148.593	2956.5	2.9565	154.798	239.624
0.22	1513.76	3.02752	104.184	172.302	3040.09	3.04009	152.022	231.107
0.23	1553.8	3.1076	87.4747	121.465	3121.67	3.12167	157.86	249.197
0.24	1595.56	3.19112	75.2388	89.8608	3202.29	3.20229	119.562	142.95
0.25	1634.97	3.26994	91.8678	133.972	3283.37	3.28337	144.417	208.563
0.26	1673.7	3.3474	80.4747	102.803	3361.58	3.36158	200.468	401.875
0.27	1713.05	3.4261	69.7449	77.2169	3440.82	3.44082	123.119	151.582
0.28	1749.78	3.49956	102.981	168.347	3516.74	3.51674	151.528	229.606
0.29	1788.2	3.5764	74.7677	88.739	3587.54	3.58754	119.382	142.521
0.3	1824.52	3.64904	82.4541	107.922	3663.4	3.6634	193.939	376.125
0.31	1859.78	3.71956	74.739	88.6709	3733.42	3.73342	114.145	130.291
0.32	1894.51	3.78902	86.0504	117.542	3806.25	3.80625	111.018	123.249
0.33	1929.14	3.85828	80.061	101.749	3877.2	3.8772	131.838	173.814

p	$\overline{L}(500, 2500)$	$\frac{\overline{L}(500,2500)}{500}$	S^2	$\frac{S^2}{500^{\frac{2}{3}}}$	$\overline{L}(1000, 5000)$	$\frac{\overline{L}(1000,5000)}{1000}$	S^2	$\frac{S^2}{1000^{\frac{2}{3}}}$
0.34	1965.39	3.93078	61.6342	60.3019	3942.82	3.94282	130.008	169.02
0.35	1999.28	3.99856	93.1329	137.687	4010.68	4.01068	157.533	248.166
0.36	2032.22	4.06444	100.638	160.772	4074.14	4.07414	106.808	114.081
0.37	2065.38	4.13076	85.9147	117.172	4140.54	4.14054	129.241	167.032
0.38	2095.72	4.19144	62.3855	61.7808	4204.45	4.20445	96.9369	93.9676
0.39	2128.1	4.2562	71.3434	80.7969	4269.23	4.26923	109.876	120.727
0.4	2159.91	4.31982	78.0221	96.6323	4331.83	4.33183	101.395	102.81
0.41	2191.33	4.38266	70.1021	78.0098	4391.13	4.39113	118.094	139.462
0.42	2220.12	4.44024	91.9451	134.197	4451.15	4.45115	101.866	103.767
0.43	2249.23	4.49846	90.5021	130.018	4511.57	4.51157	113.662	129.19
0.44	2277.54	4.55508	69.4428	76.5493	4569.18	4.56918	105.725	111.777
0.45	2306.01	4.61202	56.3938	50.4836	4625.83	4.62583	92.6072	85.7609
0.46	2334.4	4.6688	69.4949	76.6643	4682.03	4.68203	121.726	148.173
0.47	2362.53	4.72506	52.2314	43.3062	4739.5	4.7395	123.545	152.635
0.48	2388.39	4.77678	62.2201	61.4537	4791.2	4.7912	106.505	113.433
0.49	2416.37	4.83274	66.8617	70.9646	4843.52	4.84352	88.1915	77.7774
0.5	2442.85	4.8857	59.5631	56.3173	4897.06	4.89706	81.8347	66.9693
0.51	2468.63	4.93726	70.8011	79.5732	4946.48	4.94648	70.4339	49.6094
0.52	2493.34	4.98668	72.9741	84.5327	4997.52	4.99752	96.1107	92.3727
0.53	2517.05	5.0341	40.7753	26.3925	5046.81	5.04681	83.1454	69.1315
0.54	2541.81	5.08362	56.7615	51.144	5096.37	5.09637	84.0536	70.6501
0.55	2564.66	5.12932	58.4085	54.155	5142.73	5.14273	84.1587	70.8268
0.56	2588.32	5.17664	46.7248	34.6563	5189.08	5.18908	72.2158	52.1512
0.57	2613.22	5.22644	50.84	41.0296	5233.61	5.23361	90.1191	81.2145
0.58	2634.47	5.26894	65.6052	68.3223	5279.2	5.2792	79.7778	63.6449
0.59	2655.93	5.31186	47.4597	35.755	5322.04	5.32204	82.5034	68.0682
0.6	2676.62	5.35324	50.4602	40.4189	5366.14	5.36614	82.7681	68.5056
0.61	2697.5	5.395	48.5758	37.4564	5405.3	5.4053	68.6162	47.0818
0.62	2717.7	5.4354	47.3232	35.5497	5445.3	5.4453	54.8384	30.0725
0.63	2736.94	5.47388	41.0267	26.7189	5484.43	5.48443	50.7324	25.7378
0.64	2756.12	5.51224	46.8137	34.7883	5523.05	5.52305	61.1187	37.3549
0.65	2774.35	5.5487	36.5328	21.1862	5558.47	5.55847	48.9183	23.93
0.66	2791.75	5.5835	28.8561	13.2178	5595.34	5.59534	66.9539	44.8283

p	$\overline{L}(500,2500)$	$\frac{\overline{L}(500,2500)}{500}$	S^2	$\frac{S^2}{500^{\frac{2}{3}}}$	$\overline{L}(1000,5000)$	$\frac{\overline{L}(1000,5000)}{1000}$	S^2	$\frac{S^2}{1000^{\frac{2}{3}}}$
0.67	2809.26	5.61852	39.4469	24.7008	5627.99	5.62799	54.9999	30.2499
0.68	2826.71	5.65342	36.7534	21.4429	5663.16	5.66316	37.6307	14.1607
0.69	2843.42	5.68684	26.0844	10.8006	5693.63	5.69363	37.0435	13.7222
0.7	2858.44	5.71688	24.7943	9.7587	5723.83	5.72383	35.0516	12.2862
0.71	2872.49	5.74498	30.6363	14.899	5752.46	5.75246	34.1701	11.676
0.72	2887.03	5.77406	29.1001	13.4424	5781.49	5.78149	30.5959	9.36107
0.73	2899.84	5.79968	16.2772	4.20576	5806.88	5.80688	30.9552	9.58221
0.74	2913.69	5.82738	17.7312	4.99072	5832.86	5.83286	34.6873	12.0321
0.75	2925.57	5.85114	15.5405	3.83369	5856.49	5.85649	25.0201	6.26005
0.76	2935.98	5.87196	12.444	2.45816	5879.3	5.8793	20.9596	4.39305
0.77	2947.26	5.89452	11.8711	2.23702	5899.8	5.8998	20.4444	4.17975
0.78	2956.01	5.91202	10.6969	1.81635	5918.35	5.91835	16.1288	2.60138
0.79	2965.39	5.93078	12.6847	2.55417	5935.41	5.93541	17.0726	2.91475
0.8	2973.05	5.9461	9.17929	1.33753	5951.15	5.95115	13.1591	1.73162
0.81	2979.58	5.95916	6.06424	0.583767	5964.11	5.96411	8.56354	0.733341
0.82	2985.59	5.97118	5.51707	0.483174	5975.91	5.97591	6.87061	0.472052
0.83	2990.44	5.98088	3.64283	0.210651	5984.8	5.9848	6.26263	0.392205
0.84	2994.3	5.9886	1.74747	0.048474	5991.68	5.99168	3.29051	0.108274
0.85	2996.89	5.99378	1.1898	0.0224716	5995.83	5.99583	1.94051	0.0376556
0.86	2998.5	5.997	0.414141	0.0027226	5998.29	5.99829	0.61202	0.00374569
0.87	2998.69	5.99738	0.276667	0.00121507	5998.67	5.99867	0.364747	0.00133041
0.88	2998.78	5.99756	0.193535	0.000594576	5998.69	5.99869	0.276667	0.000765444
0.89	2998.79	5.99758	0.187778	0.000559725	5998.73	5.99873	0.239495	0.000573578
0.9	2998.73	5.99746	0.199091	0.000629201	5998.83	5.99883	0.142525	0.000203134
0.91	2998.87	5.99774	0.134444	0.000286928	5998.8	5.9988	0.20202	0.000408122
0.92	2998.81	5.99762	0.155455	0.000383613	5998.88	5.99888	0.106667	0.000113778
0.93	2998.85	5.9977	0.14899	0.000352371	5998.9	5.9989	0.0909091	8.26446e-05
0.94	2998.82	5.99764	0.169293	0.000454951	5998.92	5.99892	0.0743434	5.52695e-05
0.95	2998.86	5.99772	0.141818	0.000319264	5998.86	5.99886	0.121616	0.000147905
0.96	2998.96	5.99792	0.0387879	2.38824e-05	5998.92	5.99892	0.0743434	5.52695e-05
0.97	2998.9	5.9978	0.111111	0.000195975	5998.96	5.99896	0.0387879	1.5045e-05
0.98	2998.94	5.99788	0.0569697	5.15198e-05	5998.92	5.99892	0.0743434	5.52695e-05
0.99	2999	5.998	0	0	5998.99	5.99899	0.01	1e-06

B Numerical Results for Last-Passage with $\gamma = 1$ and Point-Line Percolation Equivalence

In these tables we compare point-line and last-passage ($\gamma = 1$) percolation for $N = 500, 1000$ and $p = .01, .02, ..., .99$ with 50 trials for each p. The graphs of $\overline{L}(500,500)$ and the unscaled sample mean for point-line percolation are in figure 7. Also observe in these tables (and in figure 7) that the point-line average is greater than the last-passage average as one would expect since there are more paths from $(1,1)$ to the line $x + y = 2N$ than from $(1,1)$ to (N,N), the second being a subset of the first.

p	$\overline{L}(500)$	$\frac{\overline{L}}{500}$	ptln mean 500	$\frac{\text{ptln mean}}{500}$	$\overline{L}$ (1000)	$\frac{\overline{L}}{1000}$	ptln mean 1000	$\frac{\text{ptln mean}}{1000}$
0.01	102.66	0.20532	105.88	0.21176	209.54	0.20954	213.96	0.21396
0.02	151.74	0.30348	154.38	0.30876	308.02	0.30802	313.4	0.3134
0.03	189.74	0.37948	195.06	0.39012	386.88	0.38688	392.08	0.39208
0.04	223.66	0.44732	229.04	0.45808	454.23	0.45423	460.83	0.46083
0.05	253.8	0.5076	259.3	0.5186	517.28	0.51728	522.74	0.52274
0.06	283.2	0.5664	288	0.576	571.78	0.57178	579.88	0.57988
0.07	308.32	0.61664	313.48	0.62696	624.56	0.62456	631.14	0.63114
0.08	334.06	0.66812	338.72	0.67744	675.58	0.67558	682.27	0.68227
0.09	357.5	0.715	362.24	0.72448	720.95	0.72095	728.12	0.72812
0.1	378.66	0.75732	384.42	0.76884	765.43	0.76543	774.22	0.77422
0.11	399.9	0.7998	406.84	0.81368	808.96	0.80896	816.51	0.81651
0.12	421.42	0.84284	425.66	0.85132	849.02	0.84902	857.86	0.85786
0.13	439.82	0.87964	446.88	0.89376	889.03	0.88903	896.68	0.89668
0.14	460.5	0.921	465.02	0.93004	926.55	0.92655	936.12	0.93612
0.15	477.3	0.9546	483.3	0.9666	965.26	0.96526	973.07	0.97307
0.16	496.38	0.99276	501.52	1.00304	998.84	0.99884	1007.43	1.00743
0.17	513.24	1.02648	519.94	1.03988	1033.91	1.03391	1043.43	1.04343
0.18	529.48	1.05896	536.1	1.0722	1068.33	1.06833	1076.91	1.07691
0.19	545.82	1.09164	553.6	1.1072	1102.88	1.10288	1108.6	1.1086
0.2	561.7	1.1234	568.48	1.13696	1133.16	1.13316	1139.45	1.13945
0.21	577.3	1.1546	584.32	1.16864	1164.24	1.16424	1172.93	1.17293
0.22	593.2	1.1864	597.86	1.19572	1195.61	1.19561	1203.96	1.20396
0.23	608.64	1.21728	613.66	1.22732	1224.06	1.22406	1232.75	1.23275
0.24	621.2	1.2424	627.46	1.25492	1254.98	1.25498	1262.34	1.26234
0.25	636.68	1.27336	641.86	1.28372	1282.46	1.28246	1289.24	1.28924
0.26	649.7	1.2994	656.88	1.31376	1310.6	1.3106	1316.91	1.31691
0.27	664.16	1.32832	669.38	1.33876	1335.75	1.33575	1344.63	1.34463
0.28	677.14	1.35428	683.64	1.36728	1362.5	1.3625	1369.72	1.36972
0.29	689.92	1.37984	695.08	1.39016	1387.13	1.38713	1394.73	1.39473
0.3	700.9	1.4018	708.82	1.41764	1412.21	1.41221	1421.28	1.42128
0.31	714.24	1.42848	719.18	1.43836	1438.34	1.43834	1444.39	1.44439
0.32	726.26	1.45252	733.18	1.46636	1461.88	1.46188	1468.73	1.46873
0.33	737.28	1.47456	742.96	1.48592	1484.9	1.4849	1492.21	1.49221

p	$\overline{L}$ (500)	$\frac{\overline{L}}{500}$	ptln mean 500	$\frac{\text{ptln mean}}{500}$	$\overline{L}$ (1000)	$\frac{\overline{L}}{1000}$	ptln mean 1000	$\frac{\text{ptln mean}}{1000}$
0.34	748.3	1.4966	755.86	1.51172	1507.24	1.50724	1515.46	1.51546
0.35	760.6	1.5212	768	1.536	1529.6	1.5296	1538.52	1.53852
0.36	770.58	1.54116	777.84	1.55568	1553.5	1.5535	1559.3	1.5593
0.37	784.1	1.5682	787.96	1.57592	1573.07	1.57307	1580.32	1.58032
0.38	791.54	1.58308	799.06	1.59812	1594.47	1.59447	1602.48	1.60248
0.39	804.52	1.60904	808.46	1.61692	1614.91	1.61491	1621.4	1.6214
0.4	813.2	1.6264	820.2	1.6404	1634.41	1.63441	1641.4	1.6414
0.41	823.54	1.64708	829.34	1.65868	1654.22	1.65422	1661.19	1.66119
0.42	832.2	1.6644	837.3	1.6746	1673.69	1.67369	1679.94	1.67994
0.43	841.9	1.6838	847.26	1.69452	1690.94	1.69094	1698.1	1.6981
0.44	851.3	1.7026	856.34	1.71268	1709.56	1.70956	1716.22	1.71622
0.45	860	1.72	864.44	1.72888	1727.24	1.72724	1734.38	1.73438
0.46	868.24	1.73648	872.34	1.74468	1744.73	1.74473	1750.89	1.75089
0.47	875.78	1.75156	881.42	1.76284	1760.83	1.76083	1767.75	1.76775
0.48	885.1	1.7702	888.68	1.77736	1777	1.777	1783.07	1.78307
0.49	892.36	1.78472	898.14	1.79628	1793.16	1.79316	1798.3	1.7983
0.5	899.18	1.79836	905.82	1.81164	1807.49	1.80749	1813.44	1.81344
0.51	908.78	1.81756	912.24	1.82448	1822.06	1.82206	1827.81	1.82781
0.52	915.78	1.83156	919.2	1.8384	1837.37	1.83737	1842.37	1.84237
0.53	921.62	1.84324	925.7	1.8514	1850.83	1.85083	1855.86	1.85586
0.54	927.48	1.85496	932.76	1.86552	1863.48	1.86348	1868.59	1.86859
0.55	935.7	1.8714	938.88	1.87776	1876.71	1.87671	1881.85	1.88185
0.56	941	1.882	944.98	1.88996	1888.74	1.88874	1893.53	1.89353
0.57	947.1	1.8942	950.94	1.90188	1901.61	1.90161	1905.03	1.90503
0.58	952.84	1.90568	956.24	1.91248	1911.65	1.91165	1916.21	1.91621
0.59	958.18	1.91636	961.18	1.92236	1921.98	1.92198	1926	1.926
0.6	963.42	1.92684	966.74	1.93348	1932.18	1.93218	1936.71	1.93671
0.61	968.96	1.93792	970.74	1.94148	1941.32	1.94132	1945.27	1.94527
0.62	972.3	1.9446	975.54	1.95108	1950.64	1.95064	1954.43	1.95443
0.63	976.3	1.9526	979.56	1.95912	1958.46	1.95846	1962.11	1.96211
0.64	980.12	1.96024	983.04	1.96608	1966.97	1.96697	1969.91	1.96991
0.65	983.22	1.96644	987	1.974	1973.22	1.97322	1976.09	1.97609
0.66	987.56	1.97512	989.42	1.97884	1979.67	1.97967	1982.36	1.98236

p	$\overline{L}$ (500)	$\frac{\overline{L}}{500}$	ptln mean 500	$\frac{\text{ptln mean}}{500}$	$\overline{L}$ (1000)	$\frac{\overline{L}}{1000}$	ptln mean 1000	$\frac{\text{ptln mean}}{1000}$
0.67	990.32	1.98064	992.52	1.98504	1984.66	1.98466	1987.5	1.9875
0.68	992.66	1.98532	994.66	1.98932	1989.13	1.98913	1991.49	1.99149
0.69	994.28	1.98856	996.18	1.99236	1992.83	1.99283	1994.72	1.99472
0.7	995.88	1.99176	997.28	1.99456	1995.39	1.99539	1996.92	1.99692
0.71	996.82	1.99364	998.2	1.9964	1997	1.997	1997.89	1.99789
0.72	997.56	1.99512	998.14	1.99628	1997.61	1.99761	1998.2	1.9982
0.73	997.64	1.99528	998.32	1.99664	1997.78	1.99778	1998.34	1.99834
0.74	998.1	1.9962	998.36	1.99672	1997.94	1.99794	1998.5	1.9985
0.75	998.2	1.9964	998.42	1.99684	1998.13	1.99813	1998.47	1.99847
0.76	998.48	1.99696	998.64	1.99728	1998.11	1.99811	1998.47	1.99847
0.77	998.28	1.99656	998.62	1.99724	1998.25	1.99825	1998.64	1.99864
0.78	998.18	1.99636	998.56	1.99712	1998.4	1.9984	1998.62	1.99862
0.79	998.46	1.99692	998.54	1.99708	1998.32	1.99832	1998.59	1.99859
0.8	998.24	1.99648	998.78	1.99756	1998.44	1.99844	1998.71	1.99871
0.81	998.54	1.99708	998.74	1.99748	1998.53	1.99853	1998.73	1.99873
0.82	998.46	1.99692	998.74	1.99748	1998.51	1.99851	1998.8	1.9988
0.83	998.46	1.99692	998.86	1.99772	1998.62	1.99862	1998.73	1.99873
0.84	998.64	1.99728	998.84	1.99768	1998.56	1.99856	1998.82	1.99882
0.85	998.76	1.99752	998.82	1.99764	1998.56	1.99856	1998.9	1.9989
0.86	998.52	1.99704	998.8	1.9976	1998.66	1.99866	1998.75	1.99875
0.87	998.64	1.99728	998.82	1.99764	1998.67	1.99867	1998.86	1.99886
0.88	998.64	1.99728	998.84	1.99768	1998.68	1.99868	1998.86	1.99886
0.89	998.7	1.9974	998.84	1.99768	1998.75	1.99875	1998.89	1.99889
0.9	998.8	1.9976	998.94	1.99788	1998.86	1.99886	1998.96	1.99896
0.91	998.72	1.99744	998.84	1.99768	1998.81	1.99881	1998.83	1.99883
0.92	998.92	1.99784	998.96	1.99792	1998.75	1.99875	1998.95	1.99895
0.93	998.84	1.99768	998.92	1.99784	1998.84	1.99884	1998.89	1.99889
0.94	998.86	1.99772	998.84	1.99768	1998.84	1.99884	1998.93	1.99893
0.95	998.86	1.99772	998.98	1.99796	1998.87	1.99887	1998.96	1.99896
0.96	998.9	1.9978	998.96	1.99792	1998.93	1.99893	1998.94	1.99894
0.97	998.92	1.99784	998.98	1.99796	1998.97	1.99897	1998.94	1.99894
0.98	998.94	1.99788	998.98	1.99796	1998.93	1.99893	1998.95	1.99895
0.99	999	1.998	999	1.998	1998.97	1.99897	1998.99	1.99899

C Table of Critical Values

The following tables are our numerical estimates of p_c for $\gamma = 1, 1.1, ..., 30.1, 30.2$. For these we used $N = 100$ and $p = .01, .02, ..., .98, .99$ with 100 trials for each p. Figure 3 plots these values along with $\frac{\overline{L}(100,\gamma 100))}{100}$ for the integer values of γ. p_c has been taken to be the first value of p such that there has been a realization of a maximal path, see figures 5 and 6 for visualization of the critical probability as we have calculated it based on the ratio of maximal paths to total paths for each p.

γ	p_c	γ	p_c	γ	p_c	γ	p_c	γ	p_c	γ	p_c
1	0.69	5.5	0.85	10	0.91	14.5	0.93	19	0.95	23.5	0.95
1.1	0.67	5.6	0.85	10.1	0.91	14.6	0.93	19.1	0.95	23.6	0.96
1.2	0.67	5.7	0.85	10.2	0.9	14.7	0.93	19.2	0.95	23.7	0.96
1.3	0.69	5.8	0.86	10.3	0.91	14.8	0.93	19.3	0.95	23.8	0.96
1.4	0.68	5.9	0.85	10.4	0.9	14.9	0.93	19.4	0.95	23.9	0.96
1.5	0.7	6	0.84	10.5	0.91	15	0.94	19.5	0.95	24	0.96
1.6	0.69	6.1	0.86	10.6	0.91	15.1	0.93	19.6	0.95	24.1	0.95
1.7	0.7	6.2	0.86	10.7	0.92	15.2	0.93	19.7	0.95	24.2	0.96
1.8	0.7	6.3	0.85	10.8	0.91	15.3	0.94	19.8	0.95	24.3	0.96
1.9	0.72	6.4	0.85	10.9	0.91	15.4	0.94	19.9	0.95	24.4	0.96
2	0.72	6.5	0.86	11	0.92	15.5	0.94	20	0.95	24.5	0.96
2.1	0.72	6.6	0.86	11.1	0.91	15.6	0.94	20.1	0.95	24.6	0.96
2.2	0.73	6.7	0.86	11.2	0.91	15.7	0.93	20.2	0.95	24.7	0.96
2.3	0.75	6.8	0.87	11.3	0.91	15.8	0.94	20.3	0.94	24.8	0.96
2.4	0.75	6.9	0.87	11.4	0.92	15.9	0.94	20.4	0.95	24.9	0.96
2.5	0.75	7	0.86	11.5	0.91	16	0.93	20.5	0.95	25	0.96
2.6	0.75	7.1	0.88	11.6	0.91	16.1	0.94	20.6	0.95	25.1	0.96
2.7	0.75	7.2	0.87	11.7	0.92	16.2	0.94	20.7	0.95	25.2	0.96
2.8	0.75	7.3	0.88	11.8	0.91	16.3	0.94	20.8	0.95	25.3	0.96
2.9	0.77	7.4	0.88	11.9	0.92	16.4	0.93	20.9	0.95	25.4	0.96
3	0.78	7.5	0.88	12	0.92	16.5	0.94	21	0.95	25.5	0.96
3.1	0.78	7.6	0.88	12.1	0.92	16.6	0.94	21.1	0.95	25.6	0.96
3.2	0.78	7.7	0.89	12.2	0.92	16.7	0.94	21.2	0.95	25.7	0.96
3.3	0.78	7.8	0.88	12.3	0.92	16.8	0.94	21.3	0.95	25.8	0.96
3.4	0.79	7.9	0.88	12.4	0.93	16.9	0.94	21.4	0.95	25.9	0.96
3.5	0.79	8	0.89	12.5	0.92	17	0.94	21.5	0.95	26	0.96
3.6	0.79	8.1	0.88	12.6	0.92	17.1	0.94	21.6	0.95	26.1	0.96
3.7	0.78	8.2	0.89	12.7	0.92	17.2	0.94	21.7	0.96	26.2	0.96
3.8	0.81	8.3	0.89	12.8	0.92	17.3	0.94	21.8	0.95	26.3	0.96
3.9	0.81	8.4	0.89	12.9	0.92	17.4	0.94	21.9	0.95	26.4	0.96
4	0.82	8.5	0.89	13	0.92	17.5	0.94	22	0.95	26.5	0.96
4.1	0.81	8.6	0.89	13.1	0.93	17.6	0.95	22.1	0.95	26.6	0.96
4.2	0.81	8.7	0.88	13.2	0.93	17.7	0.94	22.2	0.95	26.7	0.96
4.3	0.82	8.8	0.89	13.3	0.93	17.8	0.94	22.3	0.95	26.8	0.96
4.4	0.83	8.9	0.89	13.4	0.92	17.9	0.94	22.4	0.95	26.9	0.96
4.5	0.82	9	0.9	13.5	0.92	18	0.95	22.5	0.96	27	0.96
4.6	0.83	9.1	0.89	13.6	0.93	18.1	0.94	22.6	0.95	27.1	0.96
4.7	0.82	9.2	0.89	13.7	0.93	18.2	0.94	22.7	0.95	27.2	0.96
4.8	0.84	9.3	0.9	13.8	0.92	18.3	0.94	22.8	0.96	27.3	0.96
4.9	0.83	9.4	0.9	13.9	0.93	18.4	0.94	22.9	0.95	27.4	0.96
5	0.84	9.5	0.9	14	0.92	18.5	0.94	23	0.96	27.5	0.97
5.1	0.83	9.6	0.9	14.1	0.93	18.6	0.95	23.1	0.96	27.6	0.96
5.2	0.82	9.7	0.9	14.2	0.93	18.7	0.94	23.2	0.96	27.7	0.96
5.3	0.84	9.8	0.9	14.3	0.93	18.8	0.94	23.3	0.96	27.8	0.96
5.4	0.84	9.9	0.91	14.4	0.93	18.9	0.95	23.4	0.96	27.9	0.96

γ	p_c
28	0.96
28.1	0.96
28.2	0.96
28.3	0.96
28.4	0.96
28.5	0.97
28.6	0.96
28.7	0.97
28.8	0.96
28.9	0.97
29	0.97
29.1	0.96
29.2	0.96
29.3	0.97
29.4	0.96
29.5	0.97
29.6	0.97
29.7	0.97
29.8	0.97
29.9	0.97
30	0.97
30.1	0.97
30.2	0.97

D Some Source Code

The code below computes "numtrials" values of $L(M,N)$ for $p = \frac{1}{\text{numdiv}}, \frac{2}{\text{numdiv}}, \ldots, \frac{\text{numdiv}-1}{\text{numdiv}}$ and outputs them to "filename." Figure 14 is a graphical representation of the raw data for $N = 1000, \gamma = 1, \text{numdiv} = 100, \text{numtrials} = 100$ where the value of $L(N, \gamma N)$ has been plotted against its position in the output file, i.e. there are 100 data points for each value of p and 9900 data points total.

```
#include <iostream>
#include <fstream>
#include <cstdlib>
#include<ctime>
#include <cmath>

using namespace std;

int length(int s[]);

int main(){
    int N,c,k,i,j,z,b,numtrials,M;
    double x,y,numdiv;
    bool f;
    double p;
    typedef int* intptr;
    ofstream fout;
    char filename[16];
    intptr P,s;
    srand(time(NULL));

    cout<<"filename: ";
    cin>>filename;
    fout.open(filename);
    cout<<endl<<"number of partitions: ";
    cin>>numdiv;
    cout<<endl<<"M: ";
    cin>>M;
    cout<<endl<<"N: ";
    cin>>N;
    cout<<endl<<"# trials: ";
    cin>>numtrials;

    P=new int[N*M];
    s=new int[N*M];

    //begin loop for increments of p
    for(b=1;b<numdiv;b++){
    p=b/numdiv;
        //begin loop for incremets of numtrials
        for (c=1;c<=numtrials;c++){
            /*create empty arrays for generalized permuation
            and "patience sorted" result*/
            for(i=0;i<=M*N;i++){
                s[i]=0;
```

```
            P[i]=0;
        }
        k=0;
        //creates generalized permutation
        for(i=1;i<=N;i++){
            for(j=1;j<=M;j++){
                y=rand();
                x=y/RAND_MAX;
                if(x<=p){
                    P[k]=j;
                    k++;
                }
            }
        }
        //patience sorting of the generalized permutation
        s[0]=P[0];
        for(i=1;i<=k-1;i++){
            z=P[i];
            j=0;
            f=false;
            while(f==false){
                if( (z>=s[j])&&(s[j]!=0) ){
                    j++;
                }
                else{
                    s[j]=z;
                    f=true;
                }
            }
        }
        //outputs L(N,\gamma N) to output file
        fout<<" "<<length(s);
      }
    }
    delete [] P;
    delete [] s;
    fout.clear();
    fout.close();
    return 0;
}
/*measures the length of the patience sorted array (the number
of non-zero entries) and hence L(N,\gamma N)*/
int length(int s[])
{
  int i=0;
  while ( s[i] != 0 )
  {
    i++;
  }
  return i;
}
```

E Figures

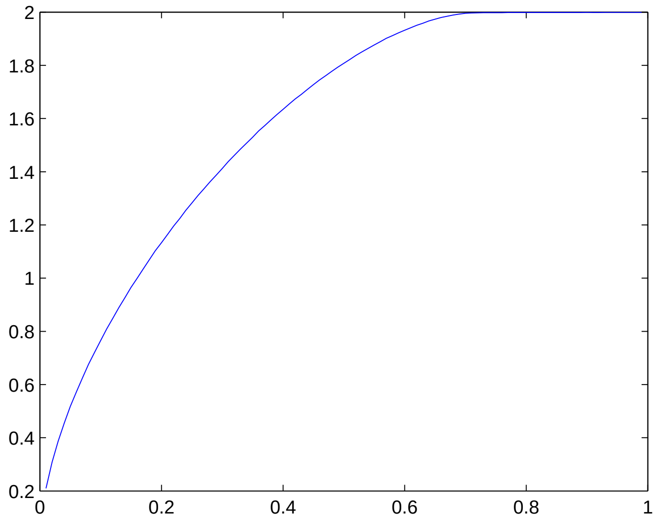

Figure 1: $\frac{\overline{L}(1000,1000)}{1000}$ for $p = .01, .02, ..., .98, .99$

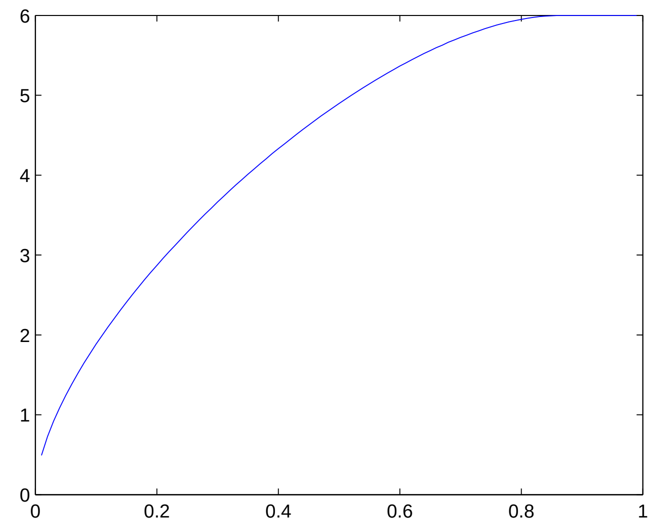

Figure 2: $\frac{\overline{L}(1000,5000)}{1000}$ for $p = .01, .02, ..., .98, .99$

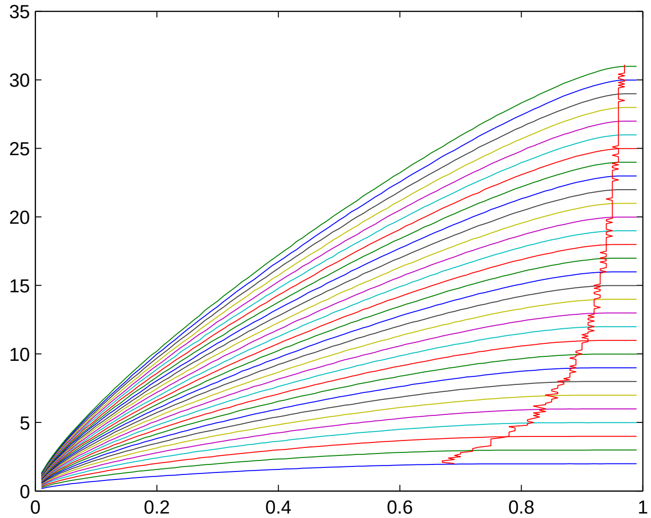

Figure 3: $\frac{\overline{L}(N,\gamma N)}{N}$ for $N = 100, \gamma = 1, 2, ..., 30$, $p = .01, .02, ..., .98, .99$ along with p_c (as a function of γ, $\gamma = 1, 1.1, ..., 30.1, 30.2$)

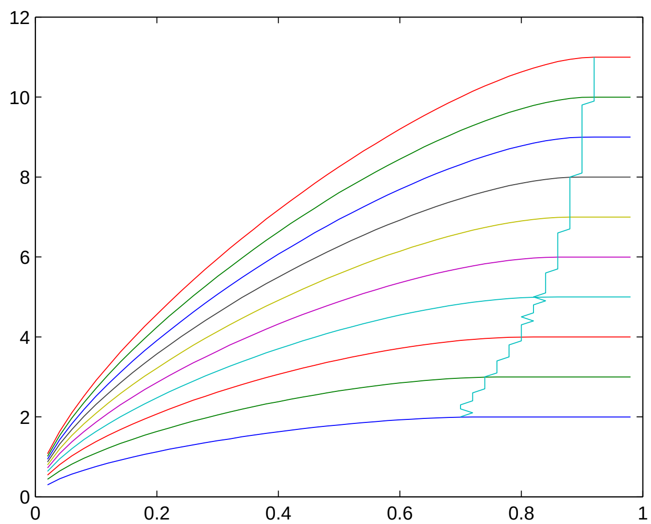

Figure 4: $\frac{\overline{L}(N,\gamma N)}{N}$ for $N = 500, \gamma = 1, 2, ..., 10$, $p = .02, .04, ..., .96, .98$ along with p_c (as a function of γ,$\gamma = 1, 1.1, ..., 9.9, 10$)

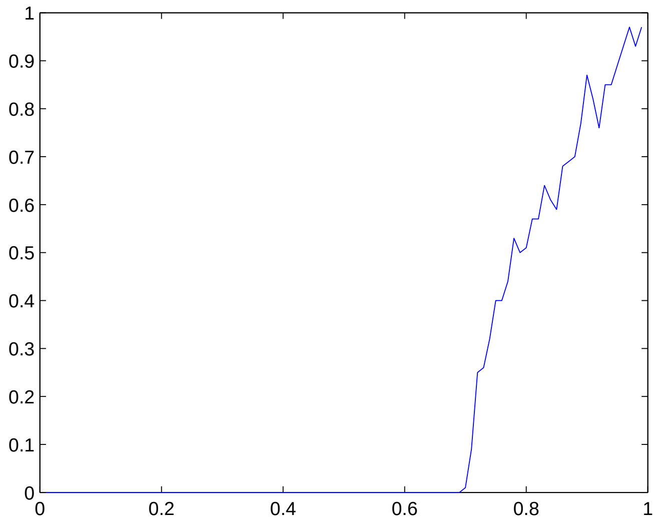

Figure 5: $\mathbb{P}(\exists$ a maximal path) as a function of p for $N = 1000, \gamma = 1$

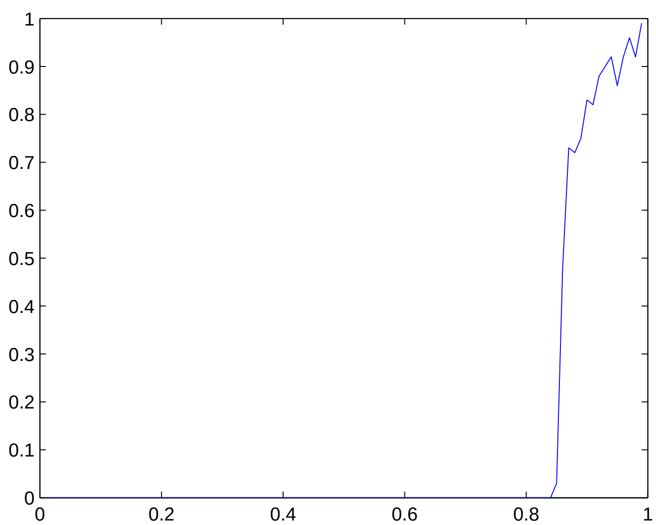

Figure 6: $\mathbb{P}(\exists$ a maximal path) as a function of p for $N = 1000, \gamma = 5$

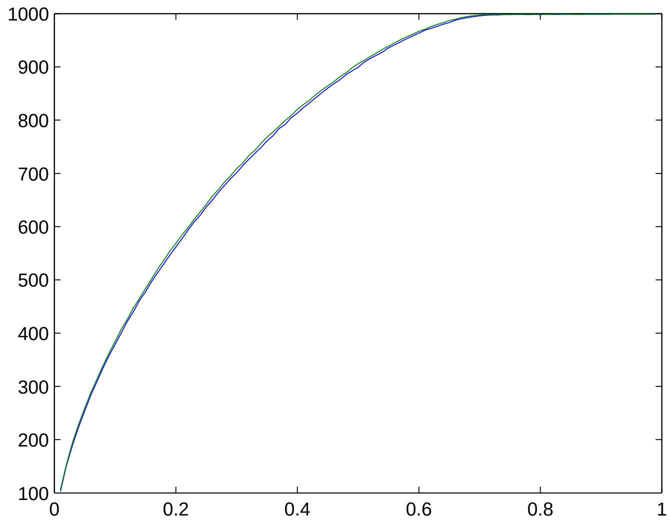

Figure 7: Comparison of (unscaled) Last-Passage and Point-Line Percolation for $N = 500$, the topmost (green) curve being the Point-Line data

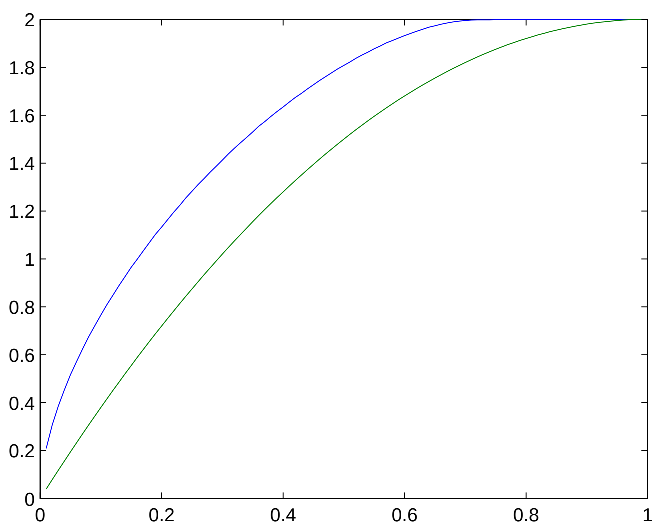

Figure 8: Heuristic Lower Bound for $N = 1000$ and $\frac{\overline{L}(1000,1000)}{1000}$, $\gamma = 1, p = .01, .02, ..., .98, .99$

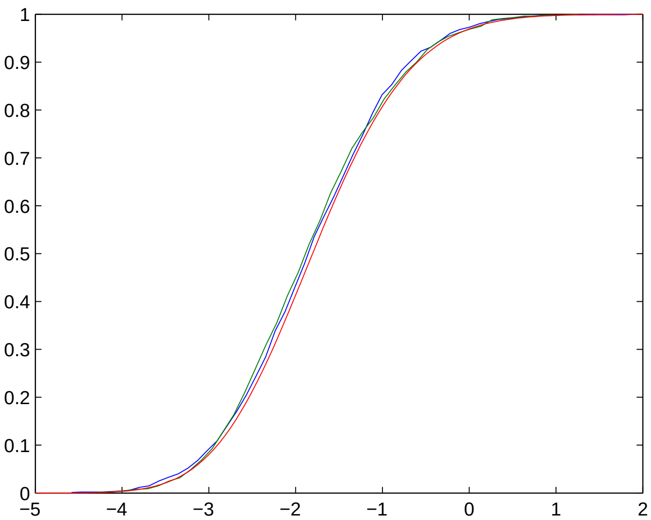

Figure 9: Tracy-Widom distribution (red) and scaled distributions of $L(1000, 1000)$ (green) and $L(500, 2500)$ (blue) for $p = .1$ with 1000 trials each

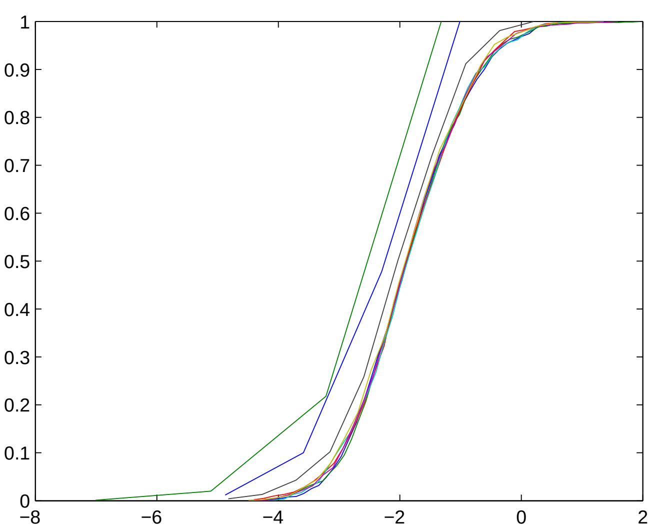

Figure 10: Scaled distribution of $L(1000, 1000)$ for $p = .1, .2, ..., .9$ with 1000 trials each

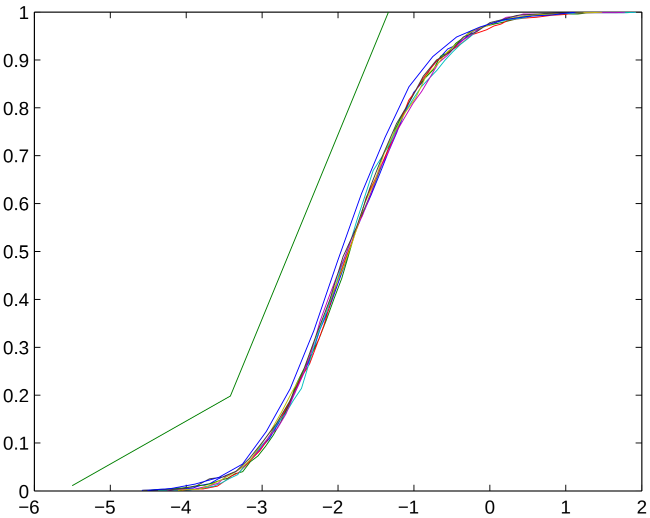

Figure 11: Scaled distribution of $L(500, 25000)$ for $p = .1, .2, ..., .9$ with 1000 trials each

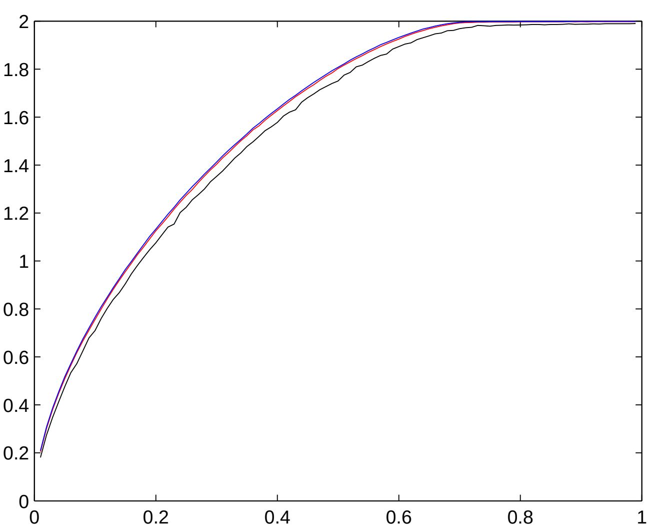

Figure 12: Comparison of $\frac{\overline{L}(N,N)}{N}$ for $N = 100, 500, 1000$

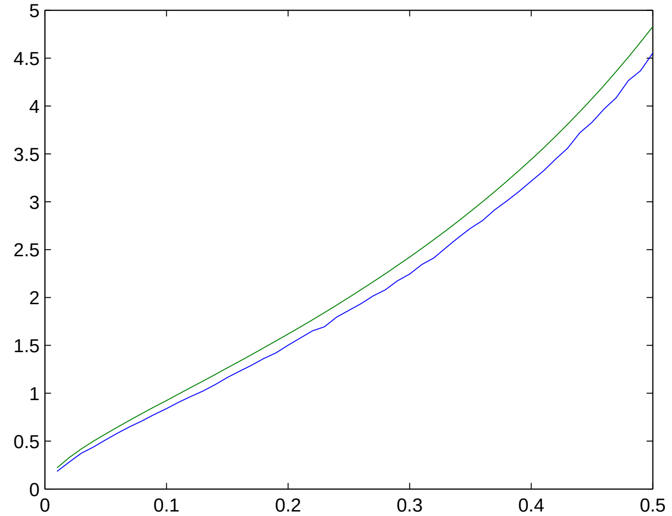

Figure 13: Comparison of $\frac{\overline{L}(100,100)}{100}$ for geometric $R_{(i,j)}$ with 100 trials for each p (lower or blue) and the theoretical values (see [9]) (topmost or green) for $p = .01, .02, ..., .49, .50$

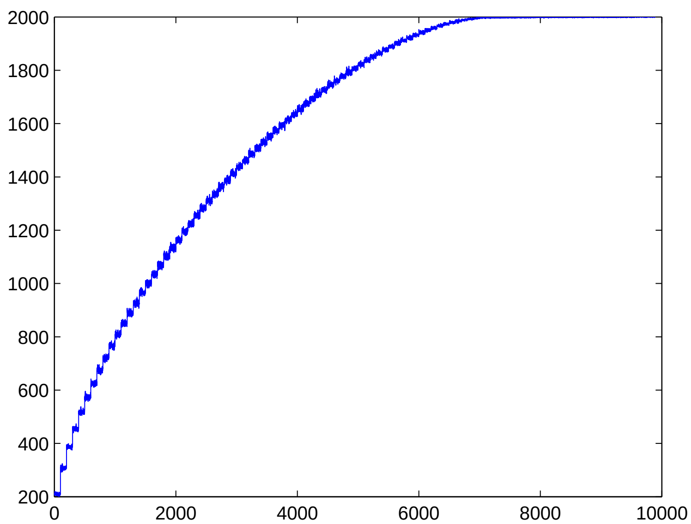

Figure 14: Graphical Representation of Code Output (see Appendix D)