

NEW RESULTS IN EXTREMAL NUMBER THEORY

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RANDOLPH F. PISTOR

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109-1109

1. INTRODUCTION

Let $[1, N] := \{1, 2, \dots, N\}$. In recent years there have been advancements in extremal number theory which have investigated the size and structure of the smallest subsets of $[1, N]$ which hit every set in some specified family of subsets of $[1, N]$. A number of these results can be found in the work of Chung, Erdős, and Graham [1], as well as that of Erdős and Spencer [2]. Similar work has been done investigating the size and structure of the *largest* subsets of $[1, N]$ that do not contain any sets in some specified family of subsets of $[1, N]$. A number of these results can be found in a paper by Graham, Spencer, and Witsenhausen [3]. There is in fact a large amount of overlap between these two related problems. In my research I explored this overlap, and in this paper I present my results.

2. PRELIMINARIES

A number of definitions are needed before we begin. Some are taken directly from [1] and [3], with some slight modifications, while others are my own creation.

Definition 2.1. For a fixed $m \times n$ integer matrix $A = (a_{ij})$, $1 \leq i \leq m$ and $1 \leq j \leq n$, we call a subset $S = S_A(N) \subseteq [1, N]$ *A-hitting* if for every non-trivial vector $\bar{x} = (x_1, x_2, \dots, x_n)^T$ with $x_i \in [1, N]$ for all i , and satisfying $A\bar{x} = \bar{0}$, we always have $x_k \in S$ for some k . Here, non-trivial means all x_i are distinct, and y^T denotes the transpose of y .

Definition 2.2. We define $s(N) = s_A(N)$ to be the minimal possible size of an *A-hitting* set $S_A(N)$.

For use in the next two definitions, let

$$\mathfrak{L} = \left\{ L_i(x_1, x_2, \dots, x_r) \equiv \sum_{j=1}^r b_{ij}x_j : 1 \leq i \leq n \right\}$$

be a set of linear forms in the variables x_j with integer coefficients b_{ij} .

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Definition 2.3. We say that a set $R \subseteq [1, N]$ is \mathfrak{L} -free if for any choice of positive integers t_1, \dots, t_r , at least one of the values $L_i(t_1, \dots, t_r)$ does not belong to R . If R is not \mathfrak{L} -free we say that \mathfrak{L} hits R .

Definition 2.4. We define the function $s_{\mathfrak{L}}(N)$ by

$$s_{\mathfrak{L}}(N) = \max_R |R|,$$

where the max is taken over all $R \subseteq [1, N]$ that are \mathfrak{L} -free and $|R|$ denotes the cardinality of R . In other words, $s_{\mathfrak{L}}(N)$ is the maximal possible size of an \mathfrak{L} -free set $R \subseteq [1, N]$.

Later in the paper we will generally be interested in the asymptotic behavior of $s_A(N)$ and $s_{\mathfrak{L}}(N)$, which leads to this next definition.

Definition 2.5. For a given matrix A and a set of linear forms \mathfrak{L} we define

$$\sigma(A) := \liminf_{N \rightarrow \infty} \frac{s_A(N)}{N} \quad \text{and} \quad \delta(\mathfrak{L}) := \liminf_{N \rightarrow \infty} \frac{s_{\mathfrak{L}}(N)}{N}.$$

It is natural to wonder what kind of relationship may exist between a matrix A and a set of linear forms \mathfrak{L} . To illustrate some of the many possible relationships, consider the following two examples.

Example 2.1. Suppose

$$\mathfrak{L} = \{L_1(x), L_2(x), L_3(x)\} = \{x, 2x, 3x\},$$

which we obtain by setting $r = 1$, $n = 3$, and $b_{ij} = i$ for $1 \leq i \leq 3$. In addition, let

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}.$$

Then if $\bar{x} = (x, 2x, 3x)^T$ we see that

$$A\bar{x} = \begin{bmatrix} 2x - 2x + 0 \\ 3x + 0 - 3x \end{bmatrix} = \bar{0}.$$

Thus, the solution sets to $A\bar{x} = \bar{0}$ are sets of the form $\{x, 2x, 3x\} \subseteq [1, N]$ for some integer x , which is exactly the set \mathfrak{L} . This shows that an A -hitting set $S_A(N)$ is one which intersects \mathfrak{L} for every choice of x such that $\{x, 2x, 3x\} \subseteq [1, N]$.

Example 2.2. Suppose $\mathfrak{L} = \{x, a_1x, a_2x, \dots, a_sx\}$ for some positive integers $1 < a_1 < \dots < a_s$. Let A be the $s \times (s+1)$ matrix

$$A = \begin{bmatrix} a_1 & -1 & 0 & 0 & \dots & 0 \\ a_2 & 0 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_s & 0 & 0 & 0 & \dots & -1 \end{bmatrix}.$$

If $\bar{x} = (x, a_1x, a_2x, \dots, a_sx)^T$ we see that

$$A\bar{x} = \begin{bmatrix} a_1 & -1 & 0 & 0 & \dots & 0 \\ a_2 & 0 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_s & 0 & 0 & 0 & \dots & -1 \end{bmatrix} \begin{bmatrix} x \\ a_1x \\ a_2x \\ \vdots \\ a_sx \end{bmatrix} = \begin{bmatrix} a_1x - a_1x \\ a_2x - a_2x \\ \vdots \\ a_sx - a_sx \end{bmatrix} = \bar{0}.$$

Thus, the solution sets to $A\bar{x} = \bar{0}$ are sets of the form $\{x, a_1x, a_2x, \dots, a_sx\} \subseteq [1, N]$ for some integer x . These examples lead to the following useful definition.

Definition 2.6. We say that a matrix A and a set of linear forms \mathfrak{L} are *compatible* and write $\mathfrak{L} \sim A$ if the column vector

$$\bar{x} = \begin{bmatrix} L_1(x_1, \dots, x_r) \\ L_2(x_1, \dots, x_r) \\ \vdots \\ L_n(x_1, \dots, x_r) \end{bmatrix}$$

derived from the linear forms in \mathfrak{L} satisfies the equation $A\bar{x} = \bar{0}$. It should be clear that an A -hitting set $S_A(N)$ is one which contains one of the values $L_i(t_1, \dots, t_r)$ for every choice of t_1, \dots, t_r such that $\{L_i(t_1, \dots, t_r) : 1 \leq i \leq n\} \subseteq [1, N]$.

3. INTEGER SOLUTIONS TO $A\bar{x} = \bar{0}$

Much of [1] was devoted to determining (or estimating) $\sigma(A)$ for various choices of A , while much of [3] was devoted to doing the same for $\delta(\mathfrak{L})$ for various choices of \mathfrak{L} . Before we can establish the relationship between these two asymptotic functions, we need the following lemma.

Lemma 3.1. *Let A be an $m \times n$ matrix, let $\mathfrak{L} = \{L_i(x_1, \dots, x_r) : 1 \leq i \leq n\}$ be a set of linear forms, and suppose $\mathfrak{L} \sim A$. If $S \subseteq [1, N]$ is an A -hitting set, then the complement $R = S^C$ of S defined by*

$$R = [1, N] \setminus S$$

is \mathfrak{L} -free.

Proof. Suppose $R = S^C$ is not \mathfrak{L} -free. By definition there must exist positive integers t_1, \dots, t_r such that $\{L_i(t_1, \dots, t_r) : 1 \leq i \leq n\} \subseteq R$. Since $\mathfrak{L} \sim A$ this implies that there must exist a vector $\bar{x} = (x_1, x_2, \dots, x_n)^T$ such that $A\bar{x} = \bar{0}$, with the set of vector components $\{x_1, x_2, \dots, x_n\} \subseteq R$. Since $R = S^C$ it follows that $S \cap \{x_1, x_2, \dots, x_n\} = \emptyset$, hence S is not A -hitting. This yields a contradiction, and we see that R must be \mathfrak{L} -free when S is an A -hitting set. \square

We can now relate the functions $\sigma(A)$ and $\delta(\mathfrak{L})$.

Theorem 3.2. *Let A be an $m \times n$ matrix, let $\mathfrak{L} = \{L_i(x_1, \dots, x_r) : 1 \leq i \leq n\}$ be a set of linear forms, and suppose $\mathfrak{L} \sim A$. Then $\delta(\mathfrak{L}) = 1 - \sigma(A)$.*

Proof. To begin, recall that $s_A(N)$ is defined as the minimal possible size of an A -hitting set $S_A(N) \subseteq [1, N]$. Let S be any A -hitting set. By definition we must have

$$|S| = s_A(N) + \epsilon \leq N$$

for some $\epsilon \geq 0$. It follows that the complement $R = S^C$ of S has cardinality

$$|R| = N - s_A(N) - \epsilon.$$

Now, as $\epsilon \rightarrow 0$, the set S becomes smaller while still retaining its property as an A -hitting set. By Lemma (4.1) it follows that as $\epsilon \rightarrow 0$, the set R becomes larger while still remaining \mathfrak{L} -free. Thus, as $\epsilon \rightarrow 0$ we see that $|S| \rightarrow s_A(N)$ and $|R| \rightarrow N - s_A(N)$, with S still A -hitting and R still \mathfrak{L} -free. This implies that

$$(1) \quad s_{\mathfrak{L}}(N) = N - s_A(N).$$

Dividing both sides of (1) by N and taking a limit we have

$$\liminf_{N \rightarrow \infty} \frac{s_{\mathcal{L}}(N)}{N} = \liminf_{N \rightarrow \infty} \left(1 - \frac{s_A(N)}{N} \right).$$

From Definition (2.5) we conclude that $\delta(\mathcal{L}) = 1 - \sigma(A)$. \square

Corollary 3.3. *Let $\mathcal{L} = \{x, 2x, 3x\}$. Then*

$$0.8003194838 < \delta(\mathcal{L}) < 0.8009611.$$

Proof. This follows immediately from Theorem 2 and Theorem 3 of [1], as well as Theorem (3.2) above. \square

Throughout the rest of this section we will only consider the set of linear forms $\mathcal{L} = \{x, a_1x, \dots, a_sx\}$ and the matrix A from Example (2.2). We denote $\sigma(A)$ in this case by $\sigma(1, a_1, \dots, a_s)$ and define the set $A = \{a_1 < \dots < a_s\}$ which corresponds to the matrix $A = (a_{ij})$. I do this to follow the conventions of [1].

In [2] Erdős and Spencer proved the result

$$\sigma(1, 2, \dots, s) = O\left(\frac{1}{s \log s}\right)$$

for the set of linear forms $\mathcal{L} = \{x, 2x, \dots, sx\}$. In their paper they initially demonstrated that

$$\sigma(1, 2, \dots, s) > s^{-1} \prod_{q \leq s} \left(1 - \frac{1}{q}\right) \sim \frac{1}{s \log s},$$

where q above runs over primes only. To help the reader understand this result, which will be used in later results, I walk through their argument in some detail.

For an integer t define $B_t = \{t, 2t, \dots, st\}$. Let S be an arbitrary A -hitting set. Then S must hit every set B_t for which $1 \leq t \leq \frac{N}{s}$. Let $M(t)$ denote the smallest prime dividing t , with $M(1) = \infty$. We want to look at the density of integers with $M(t) > p$, where p denotes a fixed prime. This is equivalent to determining the density of integers which are coprime to all primes q such that $q \leq p$.

As an example of what this might look like, consider the primes 2 and 3. It is clear that the density of integers which are coprime to 2 is $\frac{1}{2}$, since every integer is either even or odd. Similarly, the density of integers which are coprime to 3 is $\frac{2}{3}$, so that the density of integers which are coprime to both 2 and 3 is $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$.

Based on this example, we conclude that density of integers with $M(t) > p$ is given by

$$\begin{aligned} \prod_{q \leq p} \left(1 - \frac{1}{q}\right) &= \frac{e^{-\gamma}}{\log p} \left(1 + O\left(\frac{1}{\log p}\right)\right) \\ &\sim \frac{1}{\log p}, \end{aligned}$$

where q runs over primes only and γ is the Euler-Mascheroni constant. This result is a well-known fact of analytic number theory (see Theorem (8.8e) of [4]). Now, the integers t for which $M(t) > s$ give disjoint sets B_t . From the equation above we know that there are asymptotically $\frac{1}{\log s}$ sets B_t with this property. Since there are $\frac{N}{s}$ sets B_t that lie in $[1, N]$, it follows that

$$\frac{|S|}{N} > \frac{N}{s} \cdot \frac{1}{\log s},$$

where S is defined above as an arbitrary A -hitting set. Dividing by N and taking a limit as $N \rightarrow \infty$ we conclude that

$$\sigma(1, 2, \dots, s) > \frac{1}{s \log s}.$$

This argument is worth noting because it can be extended in order to establish a lower bound for $\sigma(1, a_1, \dots, a_s)$. To see how, we begin with some preliminary definitions taken from [1]. Given a set of linear forms $\mathfrak{L} = \{x, a_1x, \dots, a_sx\}$ with $a_1 < \dots < a_s$, let $\pi(A)$ be the set of primes q dividing $\prod_{i=1}^s a_i$. Let $D_A = \{d_1 < d_2 < \dots\}$ be the set of numbers with only the primes q as prime factors, and for $k \in \mathbf{N}$ let $D_A(k) = \{d_1 < d_2 < \dots < d_k\}$. Next, let $f(k)$ denote the size of the smallest set hitting all $\{x, a_1x, \dots, a_sx\} \subseteq D_A(k)$, where x is an integer. Finally, define $K(A) = \{k : f(k) > f(k-1)\}$. In [3] Graham, Spencer, and Witsenhausen proved the following theorem.

Theorem 3.4.

$$\sigma(A) = \sigma(1, a_1, \dots, a_s) = \prod_{q \in \pi(A)} \left(1 - \frac{1}{q}\right) \sum_{k \in K(A)} \frac{1}{d_k}.$$

While this gives an exact formula for $\sigma(A)$, the summation involved is rather difficult to work with because of the unpredictable nature of the function $f(k)$. In the next two propositions I offer estimates for $\sigma(A)$ that depend only on a_s and the primes in $\pi(A)$.

Proposition 3.5.

$$\frac{1}{a_s} \geq \sigma(A) > \frac{e^{-\gamma}}{a_s \log(a_s)} (1 + o(1)),$$

where γ is the Euler–Mascheroni constant.

Proof. There are clearly $\lfloor \frac{N}{a_s} \rfloor$ sets $\{x, a_1x, \dots, a_sx\} \subseteq [1, N]$. From this it immediately follows that $\sigma(A) \leq \frac{1}{a_s}$, since we need only take one element from each of these sets.

To establish the lower bound, let $P = \{\text{primes } p : p \leq a_s\}$. I make the claim that $|P| \geq |\pi(A)|$, a fact which is easily verified. Using Theorem (8.8e) of [4] once more we have

$$\begin{aligned} \prod_{p \in P} \left(1 - \frac{1}{p}\right) &= \frac{e^{-\gamma}}{\log(a_s)} \left(1 + O\left(\frac{1}{\log(a_s)}\right)\right) \\ &\sim \frac{e^{-\gamma}}{\log(a_s)}. \end{aligned}$$

To complete the proof it is now enough to show that

$$(1) \quad \frac{1}{a_s} < \sum_{k \in K(A)} \frac{1}{d_k},$$

for this would imply that

$$\sigma(A) > \frac{1}{a_s} \prod_{p \in P} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{a_s \log(a_s)} (1 + o(1))$$

by Theorem (3.4) above. This comes from the fact that

$$\prod_{p \in P} \left(1 - \frac{1}{p}\right) \leq \prod_{q \in \pi(A)} \left(1 - \frac{1}{q}\right).$$

Proving the inequality in (1) is quite simple. We begin by listing the primes in $\pi(A)$ in order so that $\pi(A) = \{q_1 < q_2 < \cdots < q_r\}$ for some integer r . Observe that $D_A(1) = \{1\}$, $D_A(2) = \{1, q_1\}$, and $D_A(3) = \{1, q_1, q_2\}$. Also, if $q_1 q_2 > q_3$ then $D_A(4) = \{1, q_1, q_2, q_3\}$; otherwise, $D_A(4) = \{1, q_1, q_2, q_1 q_2\}$. Ultimately, we see that $f(k) = 0$ for all k such that $d_k < a_s$, since there cannot be any sets $\{x, a_1 x, \dots, a_s x\} \subseteq D_A(k)$ for any integer x . Now, by definition we know that $a_s \in D_A$, and there exists a value of k , say $k = i$, such that $a_s \notin D_A(i-1)$ and $a_s \in D_A(i)$. In other words, a_s is the i th element of D_A . It follows that $f(i-1) = 0$ and $f(i) = 1$, thus the summation $\sum_{k \in K(A)} d_k^{-1}$ must include the term a_s^{-1} . This proves that

$$\frac{1}{a_s} < \sum_{k \in K(A)} \frac{1}{d_k}.$$

The inequality here is strict because the summation is really infinite and must include more terms. \square

Corollary 3.6.

$$\sigma(1, a_1, \dots, a_s) \leq \frac{1}{3}$$

for any choice of integers $a_1 < \cdots < a_s$.

Proof. This follows immediately from the fact that $\sigma(1, 2) = \frac{1}{3}$ (see [1]) and

$$\sigma(1, a_1, \dots, a_s) \leq \frac{1}{a_s}$$

by Proposition (3.5). \square

While not terrible, this estimate for $\sigma(A)$ is lacking in some cases because it uses the set P instead of the set $\pi(A)$. For example, if $a_i = (i+1)^2$ for $1 \leq i \leq s$, then P contains far more primes than $\pi(A)$. It seems rather unnecessary to consider all primes $\leq (s+1)^2$. It would be better if we instead considered all primes $\leq s+1$. In doing so, we arrive at this next proposition.

Proposition 3.7. *Let $M = \prod_{q \in \pi(A)} q$. Then*

$$\sigma(A) > \frac{\phi(M)}{M a_s} (1 + o(1)),$$

where ϕ is the Euler totient function.

Proof. Let R be any A -hitting set and let

$$B = \left\{ b \leq \frac{N}{a_s} : b \in \mathbf{N} \text{ and } (b, M) = 1 \right\}.$$

Observe that the density of integers coprime to M is just $\prod_{q \in \pi(A)} (1 - q^{-1})$, so that

$$|B| \sim \frac{N}{a_s} \prod_{q \in \pi(A)} \left(1 - \frac{1}{q}\right).$$

Recall from basic number theory that for any positive integer n we have

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

where the product above is over primes p . Using this fact we conclude that

$$|B| \sim \frac{N\phi(M)}{Ma_s} = \frac{N\phi(M)}{Ma_s} (1 + o(1)).$$

Note that for all $b \in B$ we have $(b, a_i) = 1$ for $1 \leq i \leq s$, hence each element of the $(s+1)$ -tuple (b, a_1b, \dots, a_sb) is distinct. Furthermore, for $b_1, b_2 \in B$ with $b_1 \neq b_2$ we have $\{b_1, a_1b_1, \dots, a_sb_1\} \cap \{b_2, a_1b_2, \dots, a_sb_2\} = \emptyset$. To see why, suppose that $b_1a_j = b_2a_k$ for $a_j, a_k \in A$. Since we need not have $(a_j, a_k) = 1$, we remove all common factors between a_j and a_k until $b_1l_1 = b_2l_2$ for $(l_1, l_2) = 1$ and $l_1, l_2 \leq a_s$. Since $(l_1, l_2) = 1$ it follows that $l_2|b_1$. However, $l_2|M$ and $(b, M) = 1$, hence we must have $l_2 = 1$. By this same argument we conclude that $l_1 = 1$, yielding $b_1 = b_2$ as desired.

Now, for all $b \in B$ at least one element of the set $\{b, a_1b, \dots, a_sb\}$ is in R , since R is A -hitting. We must have $|R| > |B|$, hence

$$\frac{|R|}{N} > \frac{\phi(M)}{Ma_s} (1 + o(1)).$$

Since R is arbitrary, if we take a limit as $N \rightarrow \infty$ we conclude that

$$\sigma(A) > \frac{\phi(M)}{Ma_s} (1 + o(1)).$$

□

In order to determine the error of this estimate we define the function

$$\Delta(1, a_1, \dots, a_s) := \sigma(1, a_1, \dots, a_s) - \frac{\phi(M)}{Ma_s}.$$

For simplicity we denote $\Delta(1, a_1, \dots, a_s)$ by $\Delta(A)$. We have the following result.

Proposition 3.8. *Let $M = \prod_{q \in \pi(A)} q$. Then*

1. $\Delta(A) \leq \frac{1}{Ma_s}$ if M is prime;
2. $\Delta(A) \leq \frac{2}{3a_s}$ if $M = 6$;
3. $\Delta(A) \leq \frac{M - \sqrt{M}}{Ma_s}$ if M is composite and $M \neq 6$.

Proof. From Proposition (3.5) we know that $\sigma(A) \leq \frac{1}{a_s}$ and from Proposition (3.7) we know that $\sigma(A) > \frac{\phi(M)}{Ma_s} (1 + o(1))$, hence it follows that

$$\Delta(A) \leq \frac{1}{a_s} - \frac{\phi(M)}{Ma_s}.$$

Using this inequality we consider the three cases individually.

1. If M is prime then $\phi(M) = M - 1$ and

$$\Delta(A) \leq \frac{1}{a_s} - \frac{M - 1}{Ma_s} = \frac{1}{Ma_s}.$$

2. If $M = 6$ then $\phi(M) = 2$ and

$$\Delta(A) \leq \frac{1}{a_s} - \frac{2}{6a_s} = \frac{2}{3a_s}.$$

3. Suppose M is composite and $M \neq 6$. Then in this case we know that $\phi(M) \geq \sqrt{M}$, hence

$$\Delta(A) \leq \frac{1}{a_s} - \frac{\phi(M)}{Ma_s} \leq \frac{1}{a_s} - \frac{\sqrt{M}}{Ma_s} = \frac{M - \sqrt{M}}{Ma_s}.$$

□

4. REAL SOLUTIONS TO $A\bar{x} = \bar{0}$

In the previous section we were interested in integer solutions to the matrix equation $A\bar{x} = \bar{0}$. In this section, we ask similar questions for *real* solutions \bar{x} to $A\bar{x} = \bar{0}$: How small can the measure of a set $S \subseteq [0, 1]$ be which hits every non-trivial solution \bar{x} to $A\bar{x} = \bar{0}$? (As before, we say that such a set is *A-hitting*.) For a given set of linear forms $\mathfrak{L} = \{L_i(x_1, \dots, x_r) : 1 \leq i \leq n\}$ with $\mathfrak{L} \sim A$, how large can the measure of a set $R \subseteq [0, 1]$ be so that for every choice of positive real numbers t_1, \dots, t_r at least one of the values $L_i(t_1, \dots, t_r)$, $1 \leq i \leq n$, is not in R ? To explore these questions, we require two new definitions, the real analogues of $s_A(N)$ and $s_{\mathfrak{L}}(N)$.

Definition 4.1. Let A be an $m \times n$ matrix and let $S \subseteq [0, 1]$ be a set which hits every solution \bar{x} to $A\bar{x} = \bar{0}$. We define the function $\rho(A)$ by

$$\rho(A) = \inf\{\mu(S) : S \subseteq [0, 1] \text{ hits every solution } \bar{x} \text{ to } A\bar{x} = \bar{0}\},$$

where μ denotes Lebesgue measure. Thus, $\rho(A)$ is the smallest measure that a set $S \subseteq [0, 1]$ can have and hit every solution \bar{x} to $A\bar{x} = \bar{0}$.

Definition 4.2. Let $\mathfrak{L} = \{L_i(x_1, \dots, x_r) : 1 \leq i \leq n\}$ be a set of linear forms. We define the function $\nu(\mathfrak{L})$ by

$$\nu(\mathfrak{L}) = \sup\{\mu(R) : R \subseteq [0, 1] \text{ and } R \text{ is } \mathfrak{L}\text{-free}\}.$$

Thus, $\nu(\mathfrak{L})$ is the largest measure an \mathfrak{L} -free set $R \subseteq [0, 1]$ can have.

These definitions lead to the following theorem relating $\rho(A)$ and $\nu(\mathfrak{L})$. As before, a lemma is needed first.

Lemma 4.1. *Let A be an $m \times n$ matrix, let $\mathfrak{L} = \{L_i(x_1, \dots, x_r) : 1 \leq i \leq n\}$ be a set of linear forms, and suppose $\mathfrak{L} \sim A$. If $S \subseteq [0, 1]$ is an A -hitting set, then the complement $R = S^C$ of S defined by*

$$R = [0, 1] \setminus S$$

is \mathfrak{L} -free.

Proof. With only some minor changes, the proof of Lemma (3.1) carries over almost verbatim to the present case. The only difference is that we choose positive real numbers t_1, \dots, t_r instead of positive integers and $S \subseteq [0, 1]$ instead of $S \subseteq [1, N]$. □

Theorem 4.2. *Let A be an $m \times n$ matrix, let $\mathfrak{L} = \{L_i(x_1, \dots, x_r) : 1 \leq i \leq n\}$ be a set of linear forms, and suppose $\mathfrak{L} \sim A$. Then $\nu(\mathfrak{L}) = 1 - \rho(A)$.*

Proof. To begin, recall that $\rho(A)$ is the smallest measure that a set $U \subseteq [0, 1]$ can have and hit every solution \bar{x} to $A\bar{x} = \bar{0}$. Let $S \in [0, 1]$ be an A -hitting set. By definition we must have

$$\mu(S) = \rho(A) + \epsilon \leq 1$$

for some $\epsilon \geq 0$. It follows that the complement $R = S^C$ of S has measure

$$\mu(R) = 1 - \rho(A) - \epsilon.$$

Now, as $\epsilon \rightarrow 0$, the measure of S becomes smaller with S still retaining its property as an A -hitting set. By Lemma (4.1) it follows that as $\epsilon \rightarrow 0$, the measure of R becomes larger with R still remaining \mathfrak{L} -free. Thus, as $\epsilon \rightarrow 0$ we see that $\mu(S) \rightarrow \rho(A)$ and $\mu(R) \rightarrow 1 - \rho(A)$, with S still A -hitting and R still \mathfrak{L} -free. This implies that $\mu(R)$ is as large as possible when $\mu(S)$ is as small as possible, from which we conclude that

$$\nu(\mathfrak{L}) = 1 - \rho(A).$$

□

Corollary 4.3. *Let $\mathfrak{L} = \{x, 2x, ax\}$ for $3 \leq a \leq 8$. Then*

$$\nu(\mathfrak{L}) = \begin{cases} 4/5 & \text{if } a = 3, \\ 1 - \frac{2}{a} + \frac{5}{4a-2} & \text{if } 4 \leq a \leq 8. \end{cases}$$

Proof. This follows immediately from Theorem 1 and Table 3 of [1], as well as Theorem (4.2) above. □

As in the previous section, we now turn our attention to the set of linear forms $\mathfrak{L} = \{x, a_1x, \dots, a_sx\}$ and the matrix A from Example (2.2). We denote $\rho(A)$ in this case by $\rho(1, a_1, \dots, a_s)$, following another convention of [1].

Proposition 4.4.

$$\rho(1, a_1, \dots, a_s) \geq \frac{\prod_{i=1}^{s-1} a_i}{\prod_{k=1}^{s-1} a_k + \prod_{l=1}^s a_l + \sum_{t=1}^{s-1} \prod_{\substack{j_t=1 \\ j_t \neq t}}^s a_{j_t}}.$$

Proof. Let S be an arbitrary A -hitting set and define the characteristic function $\chi_S(x)$ of S by

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

For $x \leq \frac{1}{a_s}$ it is clear that we have

$$1 \leq \chi_S(x) + \chi_S(a_1x) + \dots + \chi_S(a_sx),$$

since S is A -hitting. Integrating both sides of this inequality using Lebesgue integration we see that

$$\begin{aligned} \frac{1}{a_s} &\leq \int_0^{1/a_s} [\chi_S(x) + \chi_S(a_1x) + \cdots + \chi_S(a_sx)] dx \\ &\leq \int_0^1 \chi_S(x) dx + \int_0^{1/a_1} \chi_S(a_1x) dx + \cdots + \int_0^{1/a_s} \chi_S(a_sx) dx \\ &= \mu(S) + \frac{1}{a_1}\mu(S) + \cdots + \frac{1}{a_s}\mu(S) \\ &= \mu(S) \left(1 + \frac{1}{a_1} + \cdots + \frac{1}{a_s} \right). \end{aligned}$$

Solving for $\mu(S)$ we conclude

$$\mu(S) \geq \frac{1}{1 + \frac{a_s}{a_1} + \cdots + \frac{a_s}{a_{s-1}} + a_s}.$$

Because $\rho(1, a_1, \dots, a_s)$ is defined as

$$\begin{aligned} \rho(1, a_1, \dots, a_s) &= \inf\{\mu(R) : R \subseteq [0, 1] \text{ hits every set } \{x, a_1x, \dots, a_sx\} \subseteq [0, 1]\} \end{aligned}$$

it must also follow that

$$\rho(1, a_1, \dots, a_s) \geq \frac{1}{1 + \frac{a_s}{a_1} + \cdots + \frac{a_s}{a_{s-1}} + a_s}.$$

To finish the proof, observe that

$$\begin{aligned} &\frac{1}{1 + \frac{a_s}{a_1} + \cdots + \frac{a_s}{a_{s-1}} + a_s} \cdot \frac{\prod_{i=1}^{s-1} a_i}{\prod_{i=1}^{s-1} a_i} \\ &= \frac{\prod_{i=1}^{s-1} a_i}{\prod_{k=1}^{s-1} a_k + \prod_{l=1}^s a_l + \frac{1}{a_1} \prod_{j_1=1}^s a_{j_1} + \cdots + \frac{1}{a_{s-1}} \prod_{j_{s-1}=1}^s a_{j_{s-1}}} \\ &= \frac{\prod_{i=1}^{s-1} a_i}{\prod_{k=1}^{s-1} a_k + \prod_{l=1}^s a_l + \sum_{t=1}^{s-1} \prod_{\substack{j_t=1 \\ j_i \neq t}}^s a_{j_t}}. \end{aligned}$$

The claim immediately follows. \square

This proposition confirms the fact that $\rho(1, a_1, \dots, a_s) > 0$, since the a_i 's are all integers.

5. EXTRANEOUS RESULTS

As I conducted my research this summer there were a number of small results that I proved either to understand a particular problem or because I thought they might lead to a bigger result, but ultimately did not. In this section I present two of these results. This first result came about in an effort to show that in dealing with real solutions to $A\bar{x} = \bar{0}$, a linear form can be made to be as small as desired.

Lemma 5.1. *Let $\mathfrak{L} = \{L_i(x_1, \dots, x_r) : 1 \leq i \leq n\}$ be a set of linear forms. Then for any $\epsilon > 0$ there exist positive real numbers t_1, \dots, t_r such that $L_i(t_1, \dots, t_r) < \epsilon$ for all i , $1 \leq i \leq n$.*

Proof. Fix $\epsilon > 0$. Recall that

$$L_i(x_1, \dots, x_r) = \sum_{j=1}^r b_{ij}x_j,$$

where the b_{ij} are integer coefficients. Since the b_{ij} may not be positive integers, it is clear that for any positive real numbers a_1, \dots, a_r we have

$$(1) \quad L_i(a_1, \dots, a_r) = \sum_{j=1}^r b_{ij}a_j \leq \sum_{j=1}^r |b_{ij}|a_j$$

for all i , $1 \leq i \leq n$. Let c denote the value

$$c = \max\{|b_{ij}| : 1 \leq i \leq n, 1 \leq j \leq r\}.$$

Let t_1, \dots, t_r denote distinct positive real numbers satisfying

$$(2) \quad t_j < \frac{\epsilon}{cr}$$

for all j , $1 \leq j \leq r$. Then according to (1) and (2), for every linear form in \mathfrak{L} we have

$$L_i(t_1, \dots, t_r) \leq \sum_{j=1}^r |b_{ij}|t_j < \underbrace{c \cdot \frac{\epsilon}{cr} + c \cdot \frac{\epsilon}{cr} + \dots + c \cdot \frac{\epsilon}{cr}}_r = \epsilon.$$

This is our desired result. \square

This next result came out of my investigation of measure theory, which was needed to prove Proposition (4.4).

Definition 5.1. Looking at the real interval $[0, 1]$, we define the *5-free set* X by

$$X = \{x \in [0, 1] : x \text{ does not contain a } 5 \text{ in its decimal expansion}\}.$$

We say that a real number x is *5-free* if $x \in X$. In many instances there are two equivalent ways to write the decimal expansion of a real number n , either by ending in an infinite string of 9's or an infinite string of 0's. For our purposes we choose to end n in an infinite string of 9's, so that numbers like the integer 1 are written

$$1 = 0.99999\dots = 0.\bar{9}$$

instead of

$$1 = 1.00000\dots$$

An alternate way to define the set X is to write every $x \in [0, 1]$ as an infinite geometric sum. If $x = 0.x_1x_2x_3x_4\dots$ then we can write

$$x = \sum_{n=1}^{\infty} \frac{x_n}{10^n}.$$

Using this sum we define the set X by

$$X = \left\{ x \in [0, 1] : x = \sum_{n=1}^{\infty} \frac{x_n}{10^n} \text{ and } x_n \neq 5 \text{ for all } n \right\}.$$

Proposition 5.2. *The set X has Lebesgue measure zero and is uncountable.*

Proof. Much like the Cantor set, the set X can be constructed through an infinite process of removing intervals from $[0, 1]$. Specifically, we create subintervals X_m of $[0, 1]$ such that

$$\lim_{m \rightarrow \infty} X_m = X.$$

We construct the subintervals X_m as follows. In the first iteration ($m = 1$) we remove the interval $[0.5, 0.6]$ from $[0, 1]$ so that

$$X_1 = [0, 0.5) \cup [0.6, 1] = \left\{ x \in [0, 1] : x = \sum_{k=1}^{\infty} \frac{x_k}{10^k} \text{ and } x_1 \neq 5 \right\}.$$

Clearly, $\mu(X_1) = \frac{9}{10}$, where μ denotes Lebesgue measure. In the second iteration ($m = 2$) we remove the nine intervals

$$[0.05, 0.06), \dots, [0.45, 0.46), [0.65, 0.66), \dots, [0.95, 0.96)$$

from X_1 so that

$$X_2 = \left\{ x \in [0, 1] : x = \sum_{k=1}^{\infty} \frac{x_k}{10^k} \text{ and } x_k \neq 5 \text{ for } k = 1, 2 \right\}.$$

Simple calculation shows that $\mu(X_2) = \frac{81}{100}$. In general, in the n th iteration ($m = n$) we have

$$X_n = \left\{ x \in [0, 1] : x = \sum_{k=1}^{\infty} \frac{x_k}{10^k} \text{ and } x_k \neq 5 \text{ for } k = 1, 2, \dots, n \right\},$$

with $\mu(X_n) = \frac{9^n}{10^n}$. At this point it becomes clear that we indeed have

$$\lim_{m \rightarrow \infty} X_m = X.$$

Because

$$\lim_{m \rightarrow \infty} \mu(X_m) = \lim_{m \rightarrow \infty} \frac{9^m}{10^m} = 0,$$

we conclude that $\mu(X) = 0$ as desired.

To show the second part of the theorem we use a proof by contradiction and suppose that X is countable. It follows that we may number the elements of X and write

$$X = \{a_1, a_2, a_3, a_4, \dots\},$$

where $a_i = a_{i1}a_{i2}a_{i3} \dots$ is the decimal expansion of each a_i . By supposition, the set $\{a_i : i \geq 1\}$ is the entire 5-free set X . We want to arrive at a contradiction by creating a new 5-free number x that does not lie in X . To do so, we make the decimal digits of

$$x = 0.x_1x_2x_3 \dots$$

such that $x_j \neq 5$ for $j \geq 1$; and $x_1 \neq a_{11}$, $x_2 \neq a_{22}$, and in general, $x_k \neq a_{kk}$. Thus, the new number x differs from every 5-free number in the set X by at least one digit. Since x is 5-free and is not included in X we arrive at a contradiction, hence X is uncountable. \square

6. DIRECTIONS FOR FURTHER RESEARCH

In my research there were a number of results which I confidently believed to be true, yet could not prove. I present these inferences below as conjectures. In each, I make use of the number $M = \prod_{q \in \pi(A)} q$.

Conjecture 6.1. *Define the sets*

$$B = \left\{ b \leq \frac{N}{a_s} : b \in \mathbf{N} \text{ and } (b, M) = 1 \right\}$$

and

$$C = \left\{ c \leq \frac{N}{a_s} : c \in \mathbf{N} \text{ and } (c, M) > 1 \right\}.$$

For every $c \in C$ and for N sufficiently large we have

$$|(c, a_1c, \dots, a_sc) \cap (y, a_1y, \dots, a_sy)| \geq 1$$

for some $y \in B$ or $y \in C$.

Conjecture 6.2.

$$\sigma(1, a_1, \dots, a_s) < \frac{M + \phi(M)}{2Ma_s} (1 + o(1)).$$

Conjecture 6.3. *There exists a constant k such that*

$$\sigma(1, a_1, \dots, a_s) < k \frac{\phi(M)}{Ma_s}.$$

This truth of Conjecture (6.3) would imply that

$$\sigma(1, a_1, \dots, a_s) = O\left(\frac{\phi(M)}{Ma_s}\right),$$

which is the natural extension of the main theorem in [2]. As it is, the only way I can see to prove this fact is to duplicate the argument used in [2] to establish an upper bound on $\sigma(1, a_1, \dots, a_s)$. Because the argument in [2] is rather complicated, it would be helpful to discover another way of finding a suitable upper bound.

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E-mail address: rpistor@umich.edu