

REU Summer 2007: Two New Exceptional Quivers of Finite Mutation Type

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Abstract

In [1] the authors make a classification of all graphs that come from their geometric construction, and these graphs have finite-mutation type. It is noted there are more quivers of finite mutation type. In fact, it is noted that there are at least 9 exceptional finite mutation classes which do not come from their construction. Problem 12.7 asks whether there are only finitely many exceptional ones, and that perhaps the only exceptional ones are the 9. We have found 2 such quivers which do not conform to their classification and are not isomorphic to the 9 exceptional classes.

1 Introduction

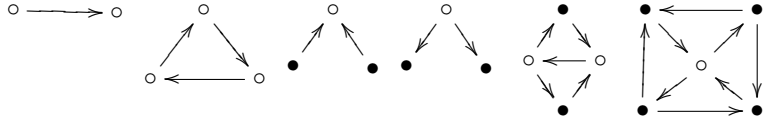
Using the notation introduced in [2] and [3], let Q be a quiver without loops and two-cycles, and let $b_{i,j}$ be the number of arrows from vertex i to vertex j minus the number of arrows from j to i . So we have $b_{i,j} = -b_{j,i}$ for all vertices i, j . The quiver Q is uniquely determined by the numbers $b_{i,j}$: If $b_{i,j} \geq 0$ then there are $b_{i,j}$ arrows from i to j and no arrows from j to i , and if $b_{i,j} \leq 0$ then there are $-b_{i,j}$ arrows from j to i and no arrows from i to j .

For a vertex k of Q , the mutation of Q at vertex k is defined as the quiver Q' associated to the integers $b'_{i,j}$ where $b'_{i,j}$ is defined as follows. $b'_{i,j} = -b_{i,j}$ if $i = k$ or $j = k$. If $k \neq i, j$ then $b'_{i,j} = b_{i,j} + |b_{i,k}b_{k,j}|$ if $b_{i,k}b_{k,j} > 0$, and $b'_{i,j} = b_{i,j}$ otherwise.

The mutation class of Q is the set of all isomorphism classes of graphs that can be obtained from Q by a sequence of mutations. We call Q of finite mutation type if the mutation class of Q is finite. We say that quiver (or a mutation class) is of geometric origin if it comes from a marked oriented Riemannian surface as described in [1].

We now give a combinatorial description of quivers of geometric origin, following [1], Section 13: The six basic blocks ...

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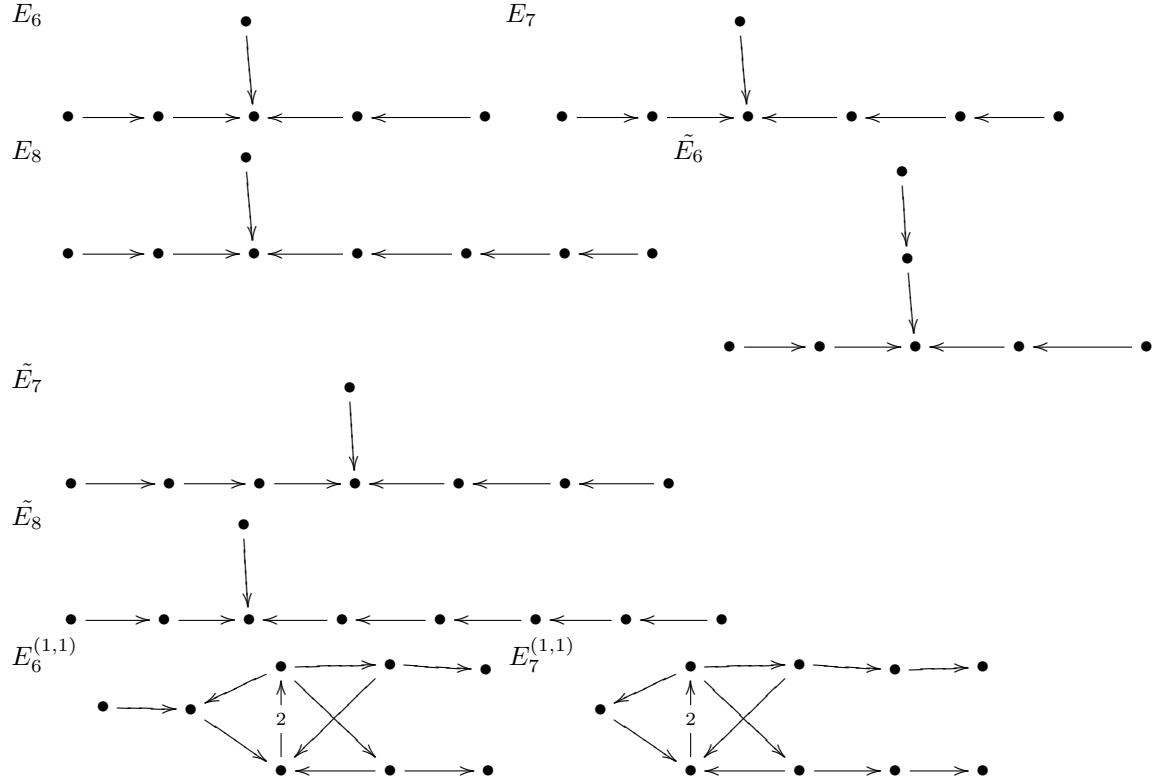


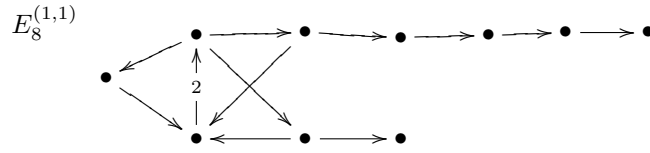
They are blocks of type I, II, IIIa, IIIb, IV, and V (naming from left to right). The vertices marked by open circles are called outlets. One can form quivers from a number of blocks (repetition of blocks is allowed) by taking a partial matching of the outlets and identifying vertices according to the matching. An outlet cannot be matched to itself, and it cannot be matched to another outlet in the same block. We require that the resultant quiver be connected, and we eliminate any sets of arrows pointing in different directions and attached to the same vertices.

A quiver can be formed by these blocks as described above if and only if the quiver is of geometric origin.

We now introduce the notion of subquiver. We say that Q' is a subquiver of Q , if the set of vertices of Q' is a subset of the vertices of Q , and for every two vertices x, y of Q , the set of arrows between x and y in Q' is exactly the set of arrows between x and y in Q .

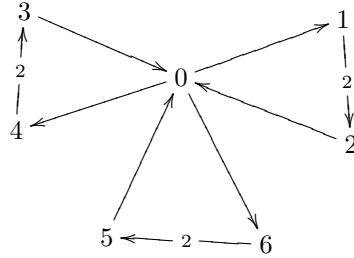
We now exhibit the nine known exceptional finite mutation classes that are not of geometric origin.



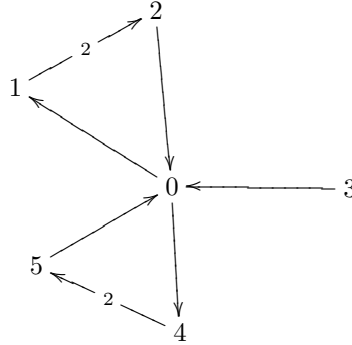


2 Proof of Finite Mutation Type

Let X_7 be the quiver.



Let X_6 be the quiver.

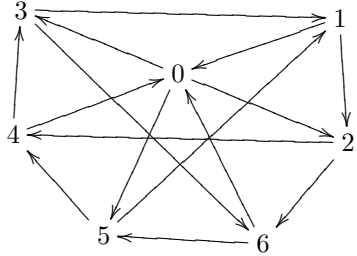


Lemma 1 *The quiver X_7 has finite mutation type. Its mutation class consists of two quivers (up to isomorphism).*

Proof

Mutation on vertex 1, 2, 3, 4, 5, or 6 yields quivers that are isomorphic to the original. For example, if one mutates on vertex 1, one must then change the position of 1 and 2 in order to get back to the original graph. This process is similar for all above mentioned vertices.

Mutation on vertex 0 produces the other quiver (call it X'_7):



All vertices other than the 0 vertex have two arrows pointing to them and two pointing away.

Mutation on any of the vertices other than 0 produces quivers which are isomorphic to X'_7 . Table 1 gives the vertex mutated on and the permutation of vertices to get to X'_7 .

vertex	swap	swap
1	2 and 0	4 and 6
2	1 and 0	4 and 6
3	4 and 0	2 and 5
4	3 and 0	1 and 6
5	6 and 0	2 and 3
6	5 and 0	2 and 3

Table 1: Mutations

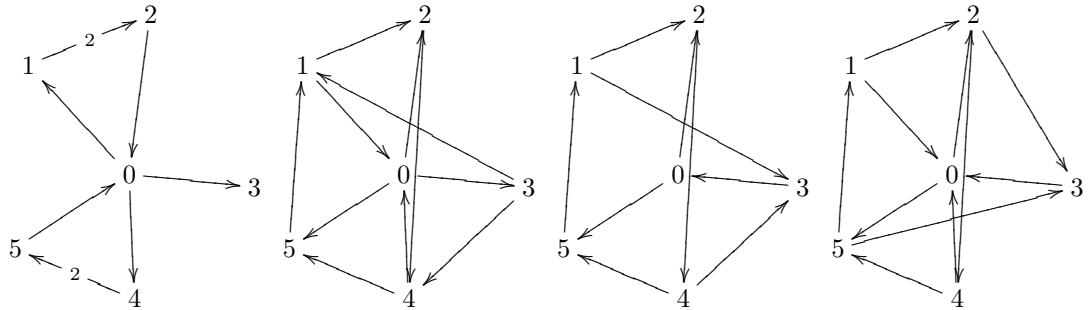
Mutating on the 0 vertex leads us back to X_7 , so all possible mutations are accounted for. Thus the quiver is of finite mutation class, and the class has only two elements.

QED

Since X_6 is a subgraph of X_7 , and all subgraphs of a quiver of finite mutation type are of finite mutation type, we get the following corollary.

Corollary 1 *The quiver X_6 has finite mutation type.*

The following is the mutation class of X_6 (not including X_6 itself):



3 Proof that the quivers are not of geometric origin

In this section, we will give one proof that X_6 is not of geometric origin, and two different proofs that X_7 is not of geometric origin.

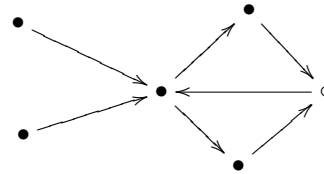
Proposition 1 X_6 is not of geometric origin.

Proof

To prove that X_6 cannot be formed from these blocks, one must note that the 0 vertex of X_6 has five arrows attached to it. Three of those arrows point toward the vertex and two point away. To form a quiver with this property from the blocks, we have the following possible combinations: I + V, II + IV, and IIIa + IV.

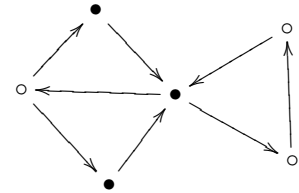
We can exclude any possibilities which contain V since the block V contains a four cycle (four vertices connected to one another forming a square) which cannot be eliminated by the attachment of any block to V. X_6 has no four cycle, so there is no way that it can be formed using V. The only other possibilities are II + IV, and IIIa + IV.

IIIa + IV leads to the following graph:



In order for this graph to be made into X_6 it is necessary for 2 of the 3 vertices with arrows pointing to the center to have a double arrow pointing to themselves. This cannot happen because there are not the correct number of outlets. Thus X_6 cannot be formed from IIIa + IV.

The same problem arises when one tries to make II + IV into X_6 :



Thus X_6 cannot be made from II + IV.

Therefore X_6 cannot be made from any of the blocks, so X_6 is not of geometric origin.

QED

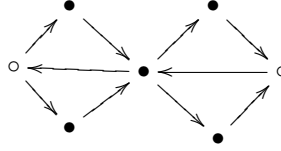
Proposition 2 X_7 is not of geometric origin.

Proof

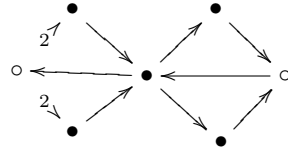
Note that vertex 0 of X_7 has six arrows attached to it. There are three pointing in and three pointing out. To form a quiver with this property from the blocks we have the following possibilities: II + V, and IV + IV.

We know that X_7 has no four cycles, and that the block V has a four cycle which cannot be eliminated, so X_7 cannot be made from any combinations of V. The only possibility left for constructing X_7 is IV + IV.

IV + IV yields the following graph:



In X_7 all of the vertices with arrows pointing to vertex 0 have a set of double arrows pointing to themselves. There is no way that IV + IV can be made to have this property since the result would be this:



But, due to there not being enough outlets, this cannot happen, so there is no way that X_7 can be formed from IV + IV.

Therefore X_7 cannot be formed from any of the blocks, thus it is not of geometric origin.

QED

Proposition 3 *If Q' is a connected subquiver of Q and Q is of geometric origin, then Q' is of geometric origin.*

Proof

By theorem 13.3, Q is obtained from a partial matching of the blocks I–V. If we introduce another trivial block VI, consisting of a single outlet with no arrows, then Q is obtained from a complete matching of the blocks I–VI. It follows that Q' is obtained from a complete matching of subquivers of the blocks I–VI. For each of the blocks I–V we can verify that any subquiver is obtained from a partial matching of blocks I–VI. For example, if we throw out one of non-outlet vertices of block V we get a graph that is obtained by glueing block IV and block VI (which is equivalent to changing one outlet in block IV to a non-outlet). Hence Q' is obtained from a complete matching of blocks of type I–VI, so Q' is also of geometric origin.

QED

4 Proof that the quivers are not subgraphs of quivers in the exceptional classes

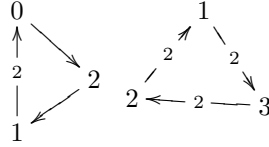
In doing this we will show that these quivers are indeed “new”, i.e. that they have not merely been overlooked in the study of the nine exceptional classes.

In order to prove the following propositions, one must bear in mind the following ideas concerning quivers of infinite mutation type.

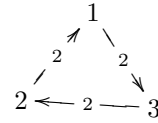
(1): If a connected quiver has at least 3 vertices and is of finite mutation type, then there are at most 2 arrows between any pair of vertices.

(2): We know that the subgraphs of all quivers of finite mutation type are themselves of finite mutation type. This leads to the conclusion that if a subgraph of a quiver is of infinite mutation type then the quiver itself is of infinite mutation type.

(3): The only connected graphs up to isomorphism with 3 vertices of finite mutation type and double arrows are (see [3]):



Any connected quiver with ≥ 4 vertices containing



is not of

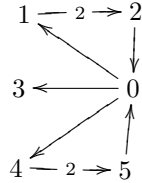
finite mutation type. Perhaps this need a bit of an argument, but it follows from (3): Assume that there is another vertex, and we can assume without loss of generality that it is connected to 1. From (3) follows that there must be a single or double arrow from 4 to 1 by considering the quiver with vertices 1,2,4. But then the subquiver with vertices 1,3,4 cannot be of finite mutation type.

The following proofs will present which mutations to take on what vertices to arrive at one of the above mentioned conditions (called the sequence), thus giving a proof that the quiver in question is of infinite mutation type.

Proposition 4 *The only connected mutation-finite quiver with 7 vertices containing X_6 is X_7 .*

Proof

We will redraw X_6 so that we may take advantage of the natural symmetry of the graph thereby saving some work (note that the arrow from vertex 3 to vertex 0 has been reversed, this was done to make the graph fit the system used in the proof):



There are now 6 cases to deal with (the added vertex will be referred to as “vertex 6”).

Case 1: 1 attachment

This is the case where vertex 6 is attached to X_6 by only one set of arrows. Note that vertex 6 cannot be connected to vertex 1,2,4, or 5.

If we connect vertex 6 to vertex 3 with one arrow from 3 to 6, then 3,0,2,1,3 is the sequence (if we take successive mutations on vertex 3, then on vertex 0,

and so on, we will arrive at one of the situations mentioned above which proves that the quiver is of infinite mutation type, use [4]).

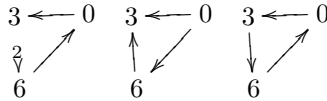
If we connect vertex 6 to vertex 0 with one arrow from 0 to 6, then 0,3,2,5,4 is the sequence.

Thus we have exhausted all possibilities for one attachment (up to isomorphism).

Case 2: 2 attachments

If vertex 6 is connected to vertex 1 and vertex 2 in the only way possible (see argument above), then 0,1,0,1,2 is the sequence. Using the symmetry of X_6 , this is also the same as connecting vertex 6 to vertices 4 and 5.

Vertex 6 can only be attached to vertices 3 and 0 in the following ways (these are the only finite subgraphs which can occur in the given circumstances), call them sub-cases 1, 2 and 3:

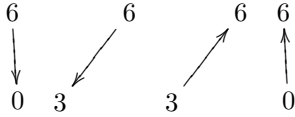


Sub-case 1 gives us X_7 . Sub-case 2 has sequence 6,0. Sub-case 3 has sequence 3 (after mutation on 3 we arrive at one of the single attachment cases already discussed).

Thus we have exhausted all possibilities for two attachments (up to isomorphism).

Case 3: 3 attachments

Vertex 6 must be attached to either vertices 1 and 2 or vertices 4 and 5. Again because of the symmetry it does not matter which. Vertex 6 must then attach to either vertex 0 or 3 in the following four possible ways (having already eliminated subgraphs which are not of finite mutation type), and again labeling them as sub-cases 1 through 4:



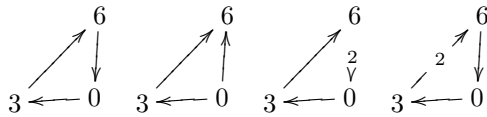
Sub-case 1 has sequence 6,0. Sub-case 2 has sequence 6,0,5. Sub-case 3 has sequence 0,3,6,2. Sub-case 4 has sequence 6,3,0.

Thus we have exhausted all possibilities for 3 attachments (up to isomorphism).

Case 4: 4 attachments

Here vertex 6 may be attached to vertices 1,2,4, and 5 in only one way, and the sequence for that is 6,0,3,5.

The other possible attachments are as follows (using the same template as the other cases):

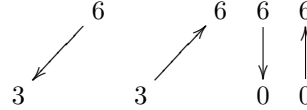


Sub-case 1 has sequence 6,0. Sub-case 2 has sequence 6,0. Sub-case 3 has sequence 6. Sub-case 4 has sequence 6.

Thus we have exhausted all possibilities for four attachments (up to isomorphism).

Case 5: 5 attachments

We must attach vertex 6 to vertices 1,2,4, and 5, and we must also attach it to vertex 3, or 0 in the following way:

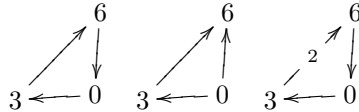


Sub-case 1 has sequence 6,0. Sub-case 2 has sequence 6,0,3. Sub-case 3 has sequence 6. Sub-case 4 has sequence 6.

Thus we have exhausted all possibilities for five attachments (up to isomorphism).

Case 6: 6 attachments

We must attach vertex 6 to all other vertices, and there are but 3 sub-cases:



In all three sub-cases the sequence is the same, namely 6.

Thus we have exhausted all possibilities for six attachments (up to isomorphism).

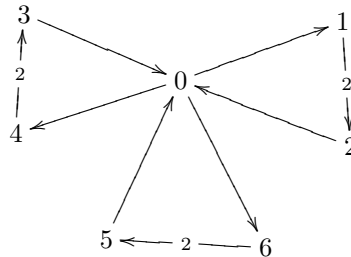
Therefore the only connected mutation-finite quiver with 7 vertices containing X_6 is X_7 .

QED

Corollary 2 *There is no connected mutation-finite quiver with 8 vertices containing X_7 .*

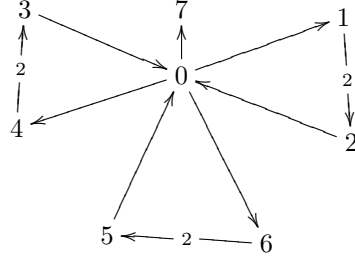
Proof

Here is a refresher of X_7 :

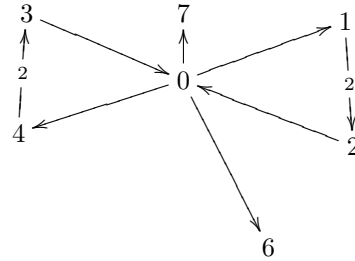


We will call the added vertex “vertex 7”. Now any way in which we attach vertex 7 to X_7 we can reduce the resulting quiver down to one of the cases in

Proposition 2. For example, if we attach vertex 7 to X_7 by one attachment we get the following:



Now if we eliminate vertex 1, 3, or 5, we are left with a subgraph which has been seen in the proof of Proposition 2:



Since we have already proven that all of the graphs in Proposition 2 are of infinite mutation type (with the exception of X_7), and that all of these are the subgraphs of vertex 7 attached to X_7 , there is no way that any attachment of vertex 7 to X_7 can produce a quiver of finite mutation type. Thus there is no connected mutation-finite quiver with 8 vertices containing X_7 .

QED

We now only need to show that neither X_6 nor X_7 are a subquiver of a quiver in the nine exceptional classes.

Lemma 2 X_6 is not isomorphic to the nine exceptional classes.

Proof

If X_6 were isomorphic to one of the nine, then it would need to be isomorphic to E_6 , since both have the same number of vertices. If X_6 were isomorphic to E_6 , then both must have the same mutation class. If this were the case, then E_6 would appear in the mutation class of X_6 . Since it does not, we can say that X_6 is not isomorphic to E_6 , and consequently, to the nine exceptional classes.

QED

Lemma 3 X_7 is not isomorphic to the nine exceptional classes.

Proof

If X_7 were isomorphic to one of the nine exceptional classes, then it must be isomorphic to either E_7 or \tilde{E}_6 , since they all have the same number of vertices. If X_7 were isomorphic to E_7 (\tilde{E}_6 respectively), then they both, X_7 and E_7 (\tilde{E}_6), must have the same mutation class. If this were so, then E_7 (\tilde{E}_6) would be in

the mutation class of X_7 . Since this is not the case, we know that X_7 is not isomorphic to any of the nine exceptional classes.

QED

Now using Lemmas 2 and 3, Corollary 2, and Proposition 2 we have the following.

Corollary 3 *X_6 and X_7 are not subgraphs of a quiver of the nine exceptional types.*

5 Conclusion

We have exhibited two quivers which are of finite mutation type, and which do not fit into the block decomposition model put forward by [1]. These quivers also are not isomorphic to the nine exceptional types, nor are they subgraphs of any of the nine exceptional types. Therefore, these quivers are two new exceptional quivers of finite mutation type.

Acknowledgments

I would like to thank Tracy Payne (Idaho State University), Ralf Spatzier (U of M), Christine Betz Bolang (U of M), and most of all Harm Derksen (U of M).

References

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