

PROPERTIES OF STEADY STATE SOLUTIONS FOR SHALLOW WATER FLOWS

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September 3, 2007

Abstract

In this paper, we study the properties of the steady state solutions that arise from shallow water equations with changing channel geometry. We are able to examine the steady state features of the flow for given channel characteristics and boundary conditions and provide a simple framework to determine the occurrence and existence of these features. The most relevant ideas presented in this paper are: conditions for the existence of solutions, limiting values of the solutions as $Q \rightarrow \infty$, conditions for discontinuities in the solutions, and location of discontinuities within the solutions.

1 Introduction

We explore the properties of the steady state equations pertaining to the single layer quasi one dimensional shallow water model for channels with non-uniform rectangular cross sections presented in [3]. The steady state equations in [3] are obtained through a vertical averaging of the Euler equations of gas dynamics. They were manipulated to model changing channel topography and width for rectangular flow cross-sections. The shallow water equations are given in the form

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{1.1}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left(\sigma h u^2 + \frac{1}{2} g \sigma h^2 \right) = \frac{1}{2} g h^2 \sigma_x - g \sigma h B_x, \tag{1.2}$$

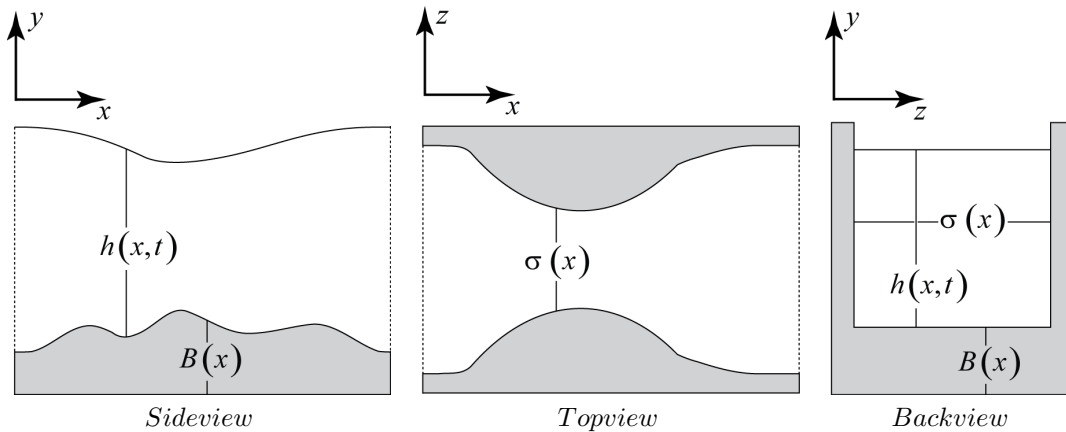


Figure 1: Different channel cross-sections, where x , y , and z represent the flow direction, the vertical direction, and the direction of the channel width, respectively.

where g is the acceleration of gravity, $\sigma(x)$ is the channel's width, $B(x)$ is the channel's topographical elevation, $h(x, t)$ is the height of the flow, $u(x, t)$ is the vertically averaged velocity of the flow, $A(x, t) = \sigma(x)h(x, t)$ is the channel's wet area, and $Q(x, t) = A(x, t)u(x, t)$ is known as the discharge rate.

From these equations, we concern ourselves only with the steady state solutions. The solution $v(x, t)$ is *steady state* if it is independent of time. That is,

$$\frac{\partial v}{\partial t}(x, t) = 0 \quad \forall t.$$

Consequently, due to this definition $\partial A/\partial t = \partial Q/\partial t = 0$ in equations (1.1) and (1.2). Applying these conditions to the hyperbolic system renders the steady state equations

$$Q = \sigma hu = C_1, \tag{1.3}$$

$$E = \frac{1}{2}u^2 + g(h + B) = C_2, \tag{1.4}$$

where C_1 and C_2 are constants in x and t that we define as Q , the *discharge*, and E , the *energy* of the system, respectively. Thus, we consider Q and E to be the conserved quantities of our steady state solution.

We will focus on properties and characteristics of equations (1.3) and (1.4) exclusively. Consequently, we extend the work illustrated in [3] by elucidating the interplay between the conserved quantities of the steady state equations, discharge (Q) and energy (E), and the channel geometry. From our analysis we introduce a framework that allows us to determine the occurrence and location of the most important phenomena for these types of flows, e.g., hydraulic jumps and critical points, given boundary conditions and the channel geometry.

The paper is structured in five sections. In § 2, we present useful definitions and the basic ideas that facilitate our study of the shallow water equations. In § 3, we present the study of the steady state shallow water equations in view of the energy of the system. The main results in this section are illustrated as conditions for the existence of real solutions for the equations (1.3) and (1.4). Then, in § 4, we continue to a more specific analysis that is limited to continuous flow. Here, we present some interesting properties of the steady state solutions. More specifically, we present conditions that must be satisfied given certain boundary information, the channel geometry, and as $Q \rightarrow \infty$. Finally, in § 5, we present a brief analysis for hydraulic jumps. The two main results of the section are the condition that must be satisfied for a jump to occur and the location at which this jump must occur.

Acknowledgment: The author expresses his deepest gratitude to both Dr. Jorge Balbás and Dr. Smadar Karni for their guidance and continued support throughout this R.E.U. (Research Experiences for Undergraduates) project.

2 Definitions

In this section we introduce, for convenience and reference, the definitions that we will continually use throughout this paper.

Definition 2.1. We denote our *domain of interest* as \mathcal{D} , where $\mathcal{D} \subset \mathbb{R}$.

Definition 2.2. $u(x) > 0$, $h(x) > 0$, $\sigma(x) > 0$, and $B(x) \geq 0 \quad \forall x \in \mathcal{D}$, where positive values of $u(x)$ are given by the positive direction of the x vector in Figure 1.

We denote Definition 2.2 as the *physicality condition*. The main point of the physicality condition is to emphasize that we will not directly treat the trivial steady state cases that follow $u(x) = 0$, $h(x) = 0$, or $\sigma(x) = 0$.

Definition 2.3. The *Froude number*, Fr , is a dimensionless velocity given in the form

$$Fr = \frac{u}{\sqrt{gh}},$$

where g is the acceleration of gravity.

Definition 2.4. A *critical point* is a point $x_n \in \mathcal{D}$ such that $Fr(x_n) = 1$.

Definition 2.5. We will denote a *critical point* by placing an asterisk, i.e.: x^* . That is, $Fr(x^*) \equiv 1$.

Definition 2.6. A water flow is said to be *subcritical* if $Fr(x) < 1 \forall x \in \mathcal{D}$. Similarly, a flow is said to be *supercritical* if $Fr(x) > 1 \forall x \in \mathcal{D}$. Furthermore, flows that are both supercritical and subcritical are said to be *transcritical*.

3 Energy

It is useful to re-write the energy of the system exclusively in terms of the boundary values. Common boundary values that are given for shallow water flows are $Q|_{x_{in}}$, discharge at the inlet, and $h|_{x_{out}}$, water height at the outlet. Once these boundary values are given, the entire flow structure is determined. That is, given a channel geometry, $\sigma(x)$ and $B(x)$, the values of $Q|_{x_{in}}$ and $h|_{x_{out}}$ completely determine the parameters E , Fr , u , and h for all $x \in \mathcal{D}$.

We can re-write E as a function of h , Q , and the channel geometry by solving for u in (1.3) and substituting into (1.4). For functional analysis, we label this new function Ξ . For sake of completeness, we also present Ξ in terms of Fr , Q , and the channel geometry. These manipulations render

$$\Xi(h, \sigma, B; Q) = \frac{Q^2}{2h^2\sigma^2} + g(h + B) \quad \text{and} \quad \Xi(Fr, \sigma, B; Q) = \left(\frac{gQ}{\sigma Fr}\right)^{2/3} \left(1 + \frac{Fr^2}{2}\right) + gB. \quad (3.1)$$

We can evaluate Ξ at the outlet, x_{out} , to obtain our flow energy dictated at this point. Then, since $Q|_{x_{in}} = Q|_{x_{out}}$ by definition, $E = \Xi|_{x_{out}}$ uniquely determines the energy of the flow regime. Thus, we obtain the solutions for the steady state shallow water equations by solving for the roots of the relationship

$$\Xi - \Xi|_{x_{out}} = 0 \quad (3.2)$$

directly.

3.1 Conditions for Existence: Roots of $\Xi - \Xi|_{x_{out}}$

It is important to note that $\sigma(x)$ and $B(x)$ are completely defined for all $x \in \mathcal{D}$. For this reason, we can view the function Ξ as dependent on h alone, because parameters σ , B , and Q are predetermined. Thus, through straightforward analysis we can show that the function $\Xi(h, \sigma, B; Q)$ allows a minimum value of energy in h for a given channel geometry at x . Let us denote this minimum as a function of x that we label $\xi(x)$ for convenience. We can obtain an explicit formula for $\xi(x)$ as shown in the following theorem.

Theorem 3.1. Let $\xi(x)$ denote the minimum allowable energy in h for $\Xi(h, \sigma, B; Q)$, then

$$\xi(x) = \frac{3}{2} \left(\frac{Qg}{\sigma(x)}\right)^{2/3} + gB(x) \quad (3.3)$$

for all $x \in \mathcal{D}$.

Proof. Let us begin by showing the following lemma.

Lemma 3.1. Let h_{min} denote the location of the global minimum of $\Xi(h, \sigma, B; Q)$ in h , then

$$h_{min} = \left(\frac{Q^2}{g\sigma^2}\right)^{1/3} \quad (3.4)$$

for all $x \in \mathcal{D}$.

Proof. For function Ξ we can take derivatives in terms of h . This renders

$$\frac{\partial \Xi}{\partial h}(h, \sigma, B; Q) = g - \frac{Q^2}{h^3 \sigma^2}.$$

Furthermore, $\lim_{h \rightarrow \infty} \Xi = \infty$ and $\lim_{h \rightarrow 0} \Xi = \infty$. Hence, the point satisfying $\frac{\partial \Xi}{\partial h} = 0$ must represent the global minimum of the function Ξ . Solving for h , we obtain (3.4) and prove the lemma. ■

Using the last lemma, we can substitute the value for h_{\min} in Ξ . That is, $\Xi|_{h_{\min}} = \xi$. ■

Locally, we can relate the previous theorem to the physicality of finding the roots of the relationship $\Xi - \Xi|_{x_{\text{out}}} = 0$.

Theorem 3.2. *The condition*

$$\Xi|_{x_{\text{out}}} \geq \xi(x_n) \tag{3.5}$$

is a sufficient condition for $\Xi - \Xi|_{x_{\text{out}}}$ to have at least one real root at $x_n \in \mathcal{D}$.

Proof. The function Ξ is continuous in h at $x_n \in \mathcal{D}$. From Theorem 3.1, we know that ξ represents the minimum allowable energy of the function Ξ in h . Thus, $\Xi|_{x_n} \geq \xi(x_n)$ by definition. This makes the condition $\Xi|_{x_{\text{out}}} \geq \xi(x_n)$ suitable to determine the existence of the roots for $\Xi - \Xi|_{x_{\text{out}}}$. The same argument can be made for variables Fr and u . ■

Once again we emphasize, the previous theorem is localized at x_n . A more general version of Theorem 3.1 will be presented later. From this localized version an interesting corollary arises that is included below for completeness.

Corollary 3.1. *Let $\Xi|_{x_{\text{out}}} \geq \xi(x_n)$ for some $x_n \in \mathcal{D}$ and let Fr_1 and Fr_2 be positive real roots of $\Xi - \Xi|_{x_{\text{out}}}$. Then,*

- $\Xi|_{x_{\text{out}}} = \xi(x_n)$ only if $Fr_1(x_n) = Fr_2(x_n) = 1$ and
- $\Xi|_{x_{\text{out}}} > \xi(x_n)$ only if $0 < Fr_1(x_n) < 1$ and $Fr_2(x_n) > 1$

for all $x_n \in \mathcal{D}$.

Proof. Let us prove the following lemma first.

Lemma 3.2. *If*

- $h < h_{\min}$, then $Fr(h) > 1$,
- $h = h_{\min}$, then $Fr(h) = 1$, and
- $h > h_{\min}$, then $0 < Fr(h) < 1$

for all $x \in \mathcal{D}$, where h_{\min} denotes the global minimum of Ξ in h .

Proof. We begin by re-writting the discharge, Q , in terms of Fr . We obtain the following equation:

$$Q^2 = g\sigma^2 h^3 Fr^2. \tag{3.6}$$

Upon solving this for h , the lemma is immediate. ■

Thus, since $\Xi|_{h_{\min}} = \xi$, Ξ possesses a double root which is $Fr = 1$. Similarly, since $\Xi \geq \xi$, then all other roots must follow from Lemma 3.2 when $h \neq h_{\min}$. ■

The previous theorems have linked the physicality of the solution of the steady state shallow water equations (1.3) and 1.4 with the properties of $\xi(x)$. For this reason, in this section, we generalize our analysis of $\xi(x)$ with the help of Theorem 3.1. In the case of continuous flow, we can extend the validity of Theorem 3.1 for all $x \in \mathcal{D}$.

Theorem 3.3. *The condition $\Xi|_{x_{out}} \geq \xi_{max}$ is a sufficient condition for $\Xi - \Xi|_{x_{out}} = 0$ to have at least one real solution for all $x \in \mathcal{D}$, with equality only if $\xi_{max} = \xi(x^*)$, where ξ_{max} represents the global maximum of $\xi(x)$.*

Proof. The first condition, $\Xi|_{x_{out}} \geq \xi_{max}$, is an extension of Theorem 3.2 for the domain \mathcal{D} , where we apply the same principles of Theorem 3.2 only now on all \mathcal{D} . The global maximum of ξ , ξ_{max} , is by definition $\xi_{max} \geq \xi$. Then, it follows that we can find roots for $\Xi - \Xi|_{x_{out}}$ for all $x \in \mathcal{D}$ only if $\Xi|_{x_{out}} \geq \xi_{max}$.

For the second part of theorem we do the following: from the work done in this section we can see that $\Xi|_{x_{out}} = \xi_{max}$ leads to the solution h_{min} for the equation $\Xi - \Xi|_{x_{out}} = 0$. To see that $\xi_{max} = \xi(x^*)$, we will use Lemma 3.2. We can see from this lemma that $h = h_{min}$ only if $x = x^*$. This completes our proof. ■

In the previous theorem, we took advantage of the fact that $\xi_{max} \geq \xi(x) \forall x \in \mathcal{D}$ by definition to state that the global the maximum of $\xi(x)$ will certainly satisfy the inequality $\Xi \geq \xi(x)$ for all values of x . This preserves the physicality of the flow and guarantees the existence of at least one positive root for all values of $x \in \mathcal{D}$. Figure 2 provides the general idea behind the last theorem and the outlook of the function Ξ , where we denote the global minimum of Ξ in h as h_{min} .

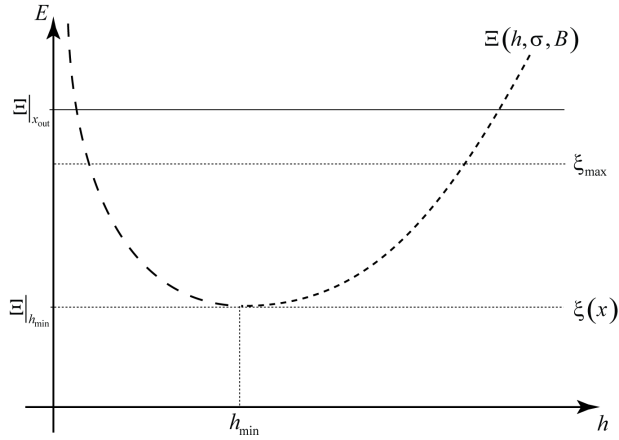


Figure 2: Ξ vs. h . We have labelled the subcritical and supercritical branches of Ξ , in accordance to Lemma 3.2, with dashed lines.

In Figure 2 we can clearly see that the solutions to the steady state shallow water equations are obtained by finding the points of intersection of the curves of Ξ and $\Xi|_{x_{out}}$.

3.2 Locating the Global Maximum of ξ

Finding the location of the global maximum of $\xi(x)$, as seen from Theorem 3.3 and Figure 2, is central to determining the properties of the solutions of the steady state shallow water equations. We can locate this global maximum of $\xi(x)$ by finding the roots of the derivative of (3.1) in x and evaluating for the global maximum. This root finding procedure is equivalent to finding the roots of the function we denote as φ . We define

$$\varphi(x) = Q^2 \sigma_x^3 - g \sigma^5 B_x^3. \quad (3.7)$$

If φ has roots in x , we must choose the root corresponding to the global maximum. However, we can present one scenario in which finding the root of $\varphi(x)$ is sufficient to determine the location of the global maximum.

Theorem 3.4. *Let x_n be a root of $\varphi(x)$. If $\sigma_x(x_n) \neq 0$ and $B_x(x_n) \neq 0$, then $x_n = x^*$.*

Proof. Let us prove the following lemma first.

Lemma 3.3. $h(x^*)\sigma_x(x^*) = \sigma(x^*)B_x(x^*)$

Proof. We begin by taking derivatives in x of (1.3) and (1.4), rendering

$$0 = uu_x + g(h_x + B_x) \quad \text{and} \quad 0 = \frac{\sigma_x}{\sigma} + \frac{h_x}{h} + \frac{u_x}{u}$$

respectively. We now substitute one equation into the other for u_x and replace u with the definition of the Froude Number. This gives

$$(1 - Fr^2) h_x \sigma = Fr^2 h \sigma_x - \sigma B_x \quad \text{or} \quad \sigma w_x = Fr^2 A_x \quad (3.8)$$

after some simplification, where $w = h + B$ and $A = \sigma h$. Evaluating the first equation at x^* , $Fr = 1$, proves the lemma. ■

We begin by substituting Q in (3.7) using (3.6). Then, using Lemma 3.3 we can write φ in two ways:

$$\varphi(x) = g\sigma^2 h^3 \sigma_x^3 (Fr^2 - 1) \quad \text{or} \quad \varphi(x) = g\sigma^5 B_x^3 (Fr^2 - 1). \quad (3.9)$$

We used Lemma 3.3 because this equality must be satisfied at the critical point. Hence, in locations where $h\sigma_x = \sigma B_x$ we know that the root of φ , x_n , is $x_n = x^*$ if $\sigma_x(x_n) \neq 0$ and $B_x(x_n) \neq 0$. ■

Therefore, finding the energy associated with transcritical flow is equivalent to finding which root of φ allows for the global maximum of ξ . In the cases where Theorem 3.4 can be applied, merely finding the root of φ suffices to determine the global maximum of ξ . In conclusion, for locally continuous flow we expect the energy, given by $\Xi|_{x_{\text{out}}}$, to satisfy Theorem 3.2. There is no guarantee that $\Xi|_{x_{\text{out}}}$ can actually satisfy the global condition obtained in Theorem 3.3. If $\Xi|_{x_{\text{out}}}$ is unable to satisfy Theorem 3.3, the flow experiences discontinuities in the solutions. See Figure 3 for graphical representation.

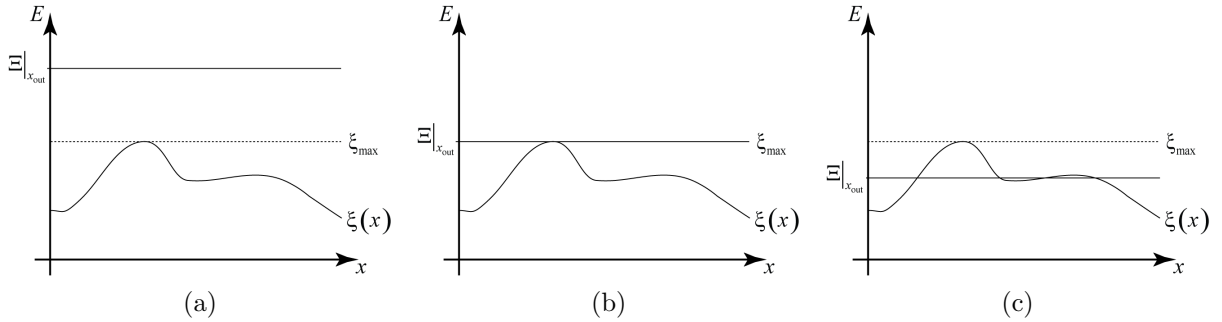


Figure 3: (a) Subcritical and Supercritical Flow, (b) Transcritical Flow, (c) Discontinuous Flow

4 Continuous Flow

In this section, we present analysis of the steady state equations for flows that are subcritical, supercritical, and transcritical without discontinuities in their solutions. It is important to note that steady state flow must match the boundary condition continuously or through a discontinuity such as a hydraulic jump. In the case of exclusively subcritical, supercritical, and transcritical flow, there is no need for such a discontinuity because condition Theorem 3.3 is satisfied and every solution can be followed continuously throughout \mathcal{D} . Therefore, it is the boundary, in this case $h|_{x_{\text{out}}}$, that determines whether the flow structure is subcritical, supercritical, or transcritical. This idea can be verified from Figure 2 directly, where $\Xi|_{x_{\text{out}}}$ has two roots for $\Xi|_{x_{\text{out}}} > \xi_{\text{max}}$ and one root for $\Xi|_{x_{\text{out}}} = \xi_{\text{max}}$.

4.1 Regimes

The focus of this section is to introduce a few equations that provide insight into the characteristics of the steady state shallow water flow in general and provide some properties of equations that have been previously introduced. As is expected for these kinds of flows, most of the equations that we will derive have significant changes in their features at the critical point. We will divide the analysis in roughly two ways. First, we will begin analyzing for general trends that are independent of the boundary information. Then, we provide a full analysis including boundary information.

We can arrive at the following general properties for (1.3) and (1.4) directly from Lemma 3.3 and Definition 2.2.

Properties 4.1. At the critical point, x^* , one of the following conditions must hold.

- $\sigma_x(x^*) = 0$ and $B_x(x^*) = 0$
- $\sigma_x(x^*) > 0$ and $B_x(x^*) > 0$
- $\sigma_x(x^*) < 0$ and $B_x(x^*) < 0$

Therefore, a *critical point* can only occur in regions of \mathcal{D} where at least one of the previously stated properties is met. If specific boundary values are not available, this table can provide us with a general idea for the location of the critical point. Moreover, if $\sigma_x(x^*) \neq 0$ and $B_x(x^*) \neq 0$, we can use Lemma 3.3 to obtain the value of h that must be satisfied at x^* . On the other hand, if $\sigma_x(x^*) = 0$ and $B_x(x^*) = 0$ we can obtain the value of h at x^* by using Theorem 3.2.

In the previous section, we briefly introduced the relationship (3.8). Now, we make note of a very useful relationship that can be obtained from (3.8), which offers more understanding of the flow structure. We denote this new relationship as η , where

$$\eta(x) = 3B_x - 2 \left(\frac{E}{g} - B \right) \frac{\sigma_x}{\sigma} = 2(1 - Fr^2) h \frac{Fr_x}{Fr}. \quad (4.1)$$

Using both (4.1) and (3.8), we can identify general trends of the flow that are dependent on the channel geometry, but independent of specific boundary values. From (3.8) and (4.1) directly, we can provide the following analysis:

Properties 4.2. The following properties hold for all $x \in \mathcal{D}$.

- For $\sigma_x \leq 0$ and $B_x > 0$ or $\sigma_x < 0$ and $B_x \geq 0$.
 - If $Fr > 1$, then $h_x > 0$ and $Fr_x < 0$.
 - If $Fr < 1$, then $h_x < 0$ and $Fr_x > 0$.
- For $\sigma_x = 0$ and $B_x = 0$.
 - If $Fr = 1$, then h_x and Fr_x may take any value.
 - If $Fr \neq 1$, then $h_x = 0$ and $Fr_x = 0$.
- For $\sigma_x \geq 0$ and $B_x < 0$ or $\sigma_x > 0$ and $B_x \leq 0$.
 - If $Fr > 1$, then $h_x < 0$ and $Fr_x > 0$.
 - If $Fr < 1$, then $h_x > 0$ and $Fr_x < 0$.

Table of Properties 4.2 provides us with useful conditions of our solutions h and Fr , regardless of the boundary values. However, if boundary information is available, it is far more convenient to analyze η directly. This analysis renders:

Properties 4.3. The following properties of $\eta(x)$ hold for all $x \in \mathcal{D}$.

- For $\eta(x) < 0$.
 - If $Fr > 1$, then $Fr_x > 0$.
 - If $Fr < 1$, then $Fr_x < 0$.
- For $\eta(x) = 0$.
 - If $Fr = 1$, then Fr_x may take any value.
 - If $Fr \neq 1$, then $Fr_x = 0$.
- For $\eta(x) > 0$.
 - If $Fr > 1$, then $Fr_x < 0$.
 - If $Fr < 1$, then $Fr_x > 0$.

From this table of properties we can deduce the following theorems directly.

Theorem 4.1. *Let $\mathcal{P} \subset \mathcal{D}$ and suppose that the flow changes from subcritical to supercritical, or vice-versa, on \mathcal{P} . If $Fr_x \neq 0$ on \mathcal{P} , then η must change signs on \mathcal{P} .*

Proof. If $Fr_x = 0$ on \mathcal{P} , then the sign of η , as given by equation (4.1), is dictated by the term $(1 - Fr^2)$. Hence, a change of Froude Number from $0 < Fr < 1$ to $Fr > 1$, or vice-versa, must be accompanied by a change in sign of η . ■

Theorem 4.2. *If $\eta(x) \neq 0$, then $Fr(x) \neq 1$ and $Fr_x(x) \neq 0$.*

Proof. This is immediate. ■

As an independent point, we bring to attention that for centered channels, i.e., $\sigma_x(x_n) = B_x(x_n) = 0$ for some $x_n \in \mathcal{D}$, $Fr_x(x_n) = 0$ implies that $u_x(x_n) = 0$, which in turn implies that $h_x(x_n) = 0$. This need not be the case for un-centered channels.

4.2 Froude Number Derivative Fr_x

It is of great importance for us to determine if $Fr_x = 0$ can occur within a flow structure. In the case of the critical point, $Fr_x(x^*) = 0$ can pose a problem of non-uniqueness for the solutions, especially in centered channels. In this section, we will not concentrate on any non-uniqueness problems associated with $Fr_x(x^*) = 0$, but rather on a specific method for determining if the condition $Fr_x(x^*) = 0$ can actually occur given a channel geometry. This leads us to our next theorem.

Theorem 4.3. *If*

$$3 \left(\frac{Q^2}{g\sigma^2} \right)^{1/3} \sigma_{xx} - 3\sigma B_{xx} - 5\sigma_x B_x = 0 \quad (4.2)$$

at x^ , then $Fr_x(x^*) = 0$ and $\eta_x(x^*) = 0$.*

Proof. Let us begin by taking derivatives of (4.1). Directly this renders

$$\eta_x(x) = 3B_{xx} + 2 \frac{B_x \sigma_x}{\sigma} - 2 \left(\frac{E}{g} - B \right) \left(\frac{\sigma_{xx}}{\sigma} - \frac{\sigma_x^2}{\sigma^2} \right) = -4hFr_x^2 + 2(1 - Fr^2) \left(h_x \frac{Fr_x}{Fr} + h \frac{Fr_{xx}}{Fr} - h \frac{Fr_x^2}{Fr^2} \right).$$

Now, to simplify the previous equation, we can evaluate at the critical point x^* . We can use Lemma 3.3 and (3.1) as the energy of the system to obtain the following form:

$$-\frac{\eta_x(x^*)}{4h} = Fr_x^2(x^*) = \frac{1}{4\sigma h} \left(3 \left(\frac{Q^2}{g\sigma^2} \right)^{1/3} \sigma_{xx} - 3\sigma B_{xx} - 5\sigma_x B_x \right). \quad (4.3)$$

This completes our proof. ■

We can also present a very convenient form of 4.2 that is not dependent on the boundary information.

Corollary 4.1. *Let $\sigma_x(x^*) \neq 0$ and $B_x(x^*) \neq 0$. If*

$$3 \frac{B_x \sigma_{xx}}{\sigma_x} - 3B_{xx} - 5 \frac{B_x \sigma_x}{\sigma} = 0 \quad (4.4)$$

at x^ , then $Fr_x(x^*) = 0$ and $\eta_x(x^*) = 0$.*

Proof. We can use Lemma 3.1 and Theorem 3.2 to note that $h_{\min} = \left(\frac{Q^2}{g\sigma^2}\right)^{1/3}$. In this case h_{\min} is at x^* , so, we may also use Lemma 3.3 to substitute this value of h . Dividing by σ proves the theorem. ■

4.3 Limiting Values

We can determine some general behavior of the steady state shallow water equations, at very large values of Q , by determining the properties of (3.1) as $Q \rightarrow \infty$. We denote this new function as ω and present it in the following theorem.

Theorem 4.4. *We can obtain the values of Fr at x , as $Q \rightarrow \infty$, by finding the roots of*

$$\omega(Fr, \sigma; Fr|_{x_n}, \sigma|_{x_n}) = 1 + \frac{Fr^2}{2} - \left(\frac{\sigma Fr}{\sigma|_{x_n} Fr|_{x_n}}\right)^{2/3} \left(1 + \frac{Fr^2|_{x_n}}{2}\right), \quad (4.5)$$

for some value $x_n \in \mathcal{D}$.

Proof. Let us begin by using (3.1) in terms of the Froude Number. For the system we are analyzing, we take advantage of the fact that the energy does not change through the channel. Therefore, we take the difference $\Xi - \Xi|_{x_n} = 0$, where x_n is our arbitrary point of interest, noting that $\Xi|_{x_n} = \Xi|_{x_{out}}$. From this, we can group the terms in the following manner:

$$1 + \frac{Fr^2}{2} - \left(\frac{\sigma Fr}{\sigma|_{x_n} Fr|_{x_n}}\right)^{2/3} \left(1 + \frac{Fr^2|_{x_n}}{2}\right) + g(B - B|_{x_n}) \left(\frac{\sigma Fr}{gQ}\right)^{2/3} = 0.$$

We can take the limit $Q \rightarrow \infty$ of this previous equation and define this new relationship as ω . ■

Theorem 4.5. *The condition*

$$\left(\frac{2 + Fr^2|_{x_n}}{3}\right)^{3/2} \frac{1}{Fr|_{x_n}} \geq \frac{\sigma|_{x_n}}{\sigma}, \quad (4.6)$$

for some value $x_n \in \mathcal{D}$, is a sufficient condition for ω to have at least one real root at x .

Proof. Let us begin by taking derivatives in terms of Fr of ω . We can obtain

$$Fr_{1,2} = \pm \left(\frac{\sigma}{\sigma|_{x_n} Fr|_{x_n}}\right)^{1/2} \left(\frac{2 + Fr^2|_{x_n}}{3}\right)^{3/4}$$

as the value at which the derivative of ω is zero, i.e., $\frac{d\omega}{dFr} = 0$. Furthermore, a quick analysis of (4.5) shows that $\omega(0, \sigma) = 1$ and as $Fr \rightarrow \infty$, $\omega(Fr, \sigma) \rightarrow \infty$. If we consider the analysis only for $Fr > 0$, then we see that in order to have one or more roots, the condition $\omega(Fr_1, \sigma) \leq 0$ for all $x \in \mathcal{D}$ must be satisfied. Thus, we substitute Fr_1 into (4.5) and, after some simplification, obtain

$$\left(\frac{2 + Fr^2|_{x_n}}{3}\right)^{3/2} \frac{1}{Fr|_{x_n}} \geq \frac{\sigma|_{x_n}}{\sigma}$$

as our condition for (4.5). ■

Corollary 4.2. *The condition*

$$\sigma \geq \sigma|_{x_n},$$

for some value $x_n \in \mathcal{D}$, is a weak lower bound for ω to have at least one real root at x .

Proof. Let us examine the left side of (4.6) and denote it as λ . Then, we can show that $\lambda \geq 1$ for all values $Fr|_{x_n} > 0$. As in the previous theorem, let us take derivatives in Fr and evaluate to zero. We obtain

$$Fr_{1,2} = \pm 1,$$

where we can ignore the negative value of Fr . Now, we know that as $Fr \rightarrow 0$, $\lambda \rightarrow \infty$. Furthermore, as $Fr \rightarrow \infty$, $\lambda \rightarrow \infty$. Since $\lambda(1) = 1$, we can see that the minimum value that lambda takes for $Fr > 0$ is 1. Thus, we can replace the left side of (4.5) and obtain the condition for the corollary. ■

Furthermore, as $Q \rightarrow \infty$, the location of the global maximum of ξ is significantly simplified.

Theorem 4.6. *As $Q \rightarrow \infty$, then $x_{\xi_{max}} \rightarrow x_{\sigma_{min}}$, where $x_{\xi_{max}}$ denotes the location of the global maximum of ξ and $x_{\sigma_{min}}$ denotes the location of the global minimum of σ .*

Proof. Direct evidence of this fact can be obtained from Theorem 3.3 and (3.7). From our discussing earlier, we found that $x_{\xi_{max}}$ was a root of φ . Thus, from (3.7), we see that as $Q \rightarrow \infty$ the value of $\sigma_x \rightarrow 0$ to preserve $\varphi(x_{\xi_{max}}) = 0$. This occurs in order to preserve the root $x_{\xi_{max}}$ of φ since the term $g\sigma^5 B_x^3$ suffers only moderate changes as we increase Q and B_x changes with respect to x . We can further show, from (3.3), that the value which σ approaches must be the global of the minimum of $\sigma(x)$. From (3.3) we see that as Q increases, the first term becomes significantly more relevant than the second. This is due to the fact that Theorem 3.3 must still be satisfied under these conditions. Thus, the value of $\sigma(x)$ becomes the determining factor of the maximum value of $\xi(x)$. This, in part, makes the location of the global maximum of ξ approach the location of the global minimum of $\sigma(x)$. ■

We can obtain interesting insight into the nature of these limit values for Fr by taking derivatives with respect to Q of (3.1) and condition $\Xi - \Xi|_{x_n} = 0$. The relationship that is obtained is

$$\frac{dFr}{dQ} (Fr^2 - 1) = g (B - B|_{x_n}) \left(\frac{\sigma}{g}\right)^{2/3} \left(\frac{Fr}{Q}\right)^{5/3}. \quad (4.7)$$

Directly from this equation we are able to obtain the following table of properties.

Properties 4.4. The following properties hold for all $x \in \mathcal{D}$.

- For $B > B|_{x_n}$.
 - If $Fr > 1$, then $\frac{dFr}{dQ} > 0$.
 - If $Fr < 1$, then $\frac{dFr}{dQ} < 0$.
- For $B = B|_{x_n}$.
 - If $Fr = 1$, then $\frac{dFr}{dQ}$ can take any value.
 - If $Fr \neq 1$, then $\frac{dFr}{dQ} = 0$.
- For $B < B|_{x_n}$.
 - If $Fr > 1$, then $\frac{dFr}{dQ} < 0$.
 - If $Fr < 1$, then $\frac{dFr}{dQ} > 0$.

We can use this information to see what effect increasing Q has on the Froude number solution. The direct conclusion from the previous table of properties is that if $B < B|_{x_n}$ for all Q , the limiting supercritical and subcritical values of the Froude number, obtained from (4.5), represent the minimum and maximum Froude numbers possible in that location, respectively. Similarly, if $B > B|_{x_n}$ for all Q , the supercritical and subcritical Froude numbers obtained from (4.5) represent the maximum and the minimum Froude numbers possible in that location.

5 Discontinuous Flow

In this section, we present analysis of the steady state equations for discontinuous flow. As is given by physical entropy conditions of fluid dynamics, we will concern ourselves only with discontinuous solutions such that $Fr > 1$ before and $Fr < 1$ after the discontinuity. Contrary to the section on continuous flow, discontinuous flow does not satisfy Theorem 3.3. We will treat discontinuous flow as continuous flow between the discontinuous sections. That is, we must only take into special consideration the discontinuities within the flow. For all other locations of the flow the analysis presented in the previous section is still valid, within the parameters we present in this section. In other words, for subcritical and supercritical flow there is always a continuous solution that will satisfy $\Xi|_{x_{\text{out}}}$. On the other hand, discontinuous flow is, by definition, transcritical before the jump, because $\Xi|_{x_{\text{out}}} < \xi(x^*)$ leads to problems with the existence of roots as previously explained. This point can be directly discerned from Figure 2. Thus, we treat discontinuous flow as transcritical flow that allows a jump in it's solution to match the downstream boundary condition only if the flow is capable of satisfying specific jump conditions.

5.1 Hydraulic Jump Conditions

The *Rankine-Hugoniot jump condition* states that for a hyperbolic system of equations, the flux function is conserved across a jump. Our system of equations, (1.1) and (1.2), is hyperbolic for $h > 0$ which satisfies the the physicality condition. In the case of our model, conservation of flux renders

$$\begin{bmatrix} Q_r - Q_l \\ \sigma_l h_l u_l^2 + \frac{1}{2} g \sigma_l h_l^2 - \sigma_r h_r u_r^2 - \frac{1}{2} g \sigma_r h_r^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (5.1)$$

where the subscripts l and r denote the flow states left and right of the jump, respectively. We bring to attention that we assume hydraulic jumps take place over negligible distances, i.e., $x_l - x_r \approx 0$. Therefore, we will treat $\sigma_l = \sigma_r$ and $B_l = B_r$ for simplicity. Using the previous considerations we can present the following theorem.

Theorem 5.1. *Across a discontinuity, the condition*

$$Fr_l = \frac{\left(1 + \sqrt{1 + 8Fr_r^2}\right)^{3/2}}{8Fr_r^2} \quad (5.2)$$

must be satisfied, where we use the subscripts 'l' and 'r' denote the locations to the left and right of the discontinuity, respectively.

Proof. Using conservation of discharge, Q , we can re-write the bottom equation of the system (5.1) as

$$u_l u_r (h_r - h_l) = \frac{g}{2} (h_r - h_l) (h_r + h_l). \quad (5.3)$$

Across a jump, it is reasonable to assume that $h_l \neq h_r$. By dividing through by $(h_r - h_l)/2$, solving for h in (3.6), re-writing (3.6) in terms of u and solving for u , substituting, and simplifying, we can obtain

$$2Fr_l^{2/3} Fr_r^{2/3} = \frac{1}{Fr_l^{2/3}} + \frac{1}{Fr_r^{2/3}} \quad (5.4)$$

as our jump condition. We can further write this as a simple quadratic polynomial by using Definition 2.2 and applying the substitution $x = Fr_l^{2/3}$ and $y = Fr_r^{2/3}$. The relationship amounts to

$$2x^2 y^2 - x - y = 0, \quad (5.5)$$

with no loss of generality. This equations admits the solution

$$y = \frac{1 \pm \sqrt{1 + 8x^3}}{4x^2}.$$

We may discard one of these solutions by imposing the physical condition that $y \geq 0$ with no loss of generality on Fr_l . Let us impose this condition on the solution $1 - \sqrt{1 + 8x^3}$ first. The positivity constraint on this solution contradicts the positivity on the right hand side of the jump. That is, the constraint states that if $y \geq 0$, then $-\frac{1}{2} \leq x \leq 0$. Thus, we discard this non-physical solution and conserve the positive solution. Substituting back to Fr_l and Fr_r from x and y proves the theorem. ■

We can show that (5.2) represents a jump condition similar to incompressible flow in gas dynamics by showing some of its properties. Let us impose the condition $y \leq 1$ to (5.2). After some algebraic manipulation we can arrive to the relationship

$$(2x + 1)(1 - x) \leq 0,$$

where x and y are as stated in the theorem. This condition states that if $y \leq 1$, then $x \geq 1$ is the only possible solution. By solving for x in (5.5) and applying the same analysis to x , we can generalize our statement to the following: if and only if $y \leq 1$, then $x \geq 1$.

We will treat the energy across a jump has given by $E_l = \xi(x^*)$ and $E_r = \Xi|_{x_{out}}$, where $E_r \leq E_l$. For deeper insight into physical entropy conditions, we refer the reader to [2].

5.2 Hydraulic Jump Location

In the previous section we discussed the conditions that must be satisfied for the occurrence of a hydraulic jump. We will now focus our attention on finding the location location of the hydraulic jump. We denote the function we will use to find this location as ν .

Theorem 5.2. *Let $h|_{x_{out}} \geq h_{min}$, where h_{min} denotes the global minimum of Ξ in h . If*

$$\nu(x) = 2Fr_l^{2/3}Fr_r^{2/3} + \frac{1}{2}Fr_l^{4/3} + \frac{1}{2}Fr_r^{4/3} + (2gB - E_l - E_r) \left(\frac{\sigma}{Qg} \right)^{2/3}, \quad (5.6)$$

then finding the location of a discontinuity is equivalent to finding the roots of ν in x .

Proof. We offer this proof as an outline. In order to find the location of a jump, we can use (5.3). We begin by dividing through by $(h_r - h_l)/2$, solving for h in (3.6), re-writing (3.6) in terms of u and solving for u , and using (1.4), respectively. We can arrive at the relationship given in the theorem directly by substituting all of these relationships into (5.3). ■

We can obtain the location of a hydraulic jump using (5.6), because Fr_r is given by Fr_l using Theorem 5.1, $E_r = \Xi|_{x_{out}}$, and $E_l = \xi(x^*)$. Thus, given a channel geometry we must compute the values of Fr for that given flow, then we can apply (5.6) to obtain the hydraulic jump location. This is assuming that $Fr_l \geq 1$, of course.

The case where $h|_{x_{out}} < h_{min}$ in discontinuous flow is not discussed in this paper. This case can be simply treated as though the discontinuity is matched outside of \mathcal{D} . Some insight into this idea is given in [2].

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