Some R-tree results

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Introduction

This paper is a summary and elaboration of some interesting results involving **R**-trees. None of the results in this paper are genuinely new; this paper is a collection and explanation of some of the papers listed in the bibliography. It is intended for a less experienced audience.

[CM], [CV], and [MKS] are cited, and a large part of this paper is based on results from them. [B], [MS], and [S] are very helpful for understanding the results in the cited papers, and provide many additional results.

1. Basic Definitions

A graph G is a set of vertices V together with a symmetric relation $\sim$ on V. When $v \sim w$, we say that v and w are connected by an edge.

A graph G is connected if $\forall v, w \in G$, there is a finite sequence of vertices $v_1, \ldots, v_n$ such that $v = v_1$, $v_i \sim v_{i+1}$ for all $1 \leq i < n$, and $v_n = w$. Such a sequence is called a path between v and w.

A graph contains a loop if there exists a finite sequence of vertices $v_1, \ldots, v_n$, such that $v_i \sim v_{i+1}$ for all $1 \leq i < n$ and $v_n \sim v_1$.

A graph is a tree if it contains at least one vertex, it is connected, and it does not contain a loop.

An **R**-tree is a generalization of a tree. A metric space (X, d) is an **R**-tree if:

1. Any two points $x, y \in X$ have a unique arc connecting them. An arc is the image of a topological embedding $f\colon [a, b] \to X$ where $f(a) = x$, $f(b) = y$.

2. The unique arc is a geodesic. A geodesic is the image of an isometric embedding $f\colon [a, b] \to X$ where $f(a) = x$, $f(b) = y$.

We will denote the unique arc connecting x and y by $\overline{x, y}$. Greek letters will be used as variables representing arcs.

We sometimes refer to a tree as V when it is actually $(V, \sim)$, and to an **R**-tree as X when the actual **R**-tree is (X, d).

Let G be a group and $(V, \sim)$ be a tree. We say G acts on $(V, \sim)$ iff $\exists \cdot : G \times V \to V$ such that:

1. $\forall g, h \in G \quad \forall v \in V \quad g(hv) = (gh)v$
2. $\forall v \in V \quad 1 \cdot v = v$
3. $\forall v, w \in V \quad \forall g \in G \quad v \sim w \iff gv \sim gw$

If G is a group and (X, d) is an $\mathbf{R}$-tree , we say G acts on (X, d) iff $\exists \cdot : G \times X \to X$ such that:

1. $\forall g, h \in G \quad \forall x \in X \quad g(hx) = (gh)x$
2. $\forall x \in X \quad 1 \cdot x = x$
3. $\forall x, y \in X \quad \forall g \in G \quad d(x, y) = d(g(x), g(y))$, i.e., g acts by isometries.

If G acts on T and g fixes no elements of T, g is called hyperbolic. Otherwise, g is called elliptic.

2. Invariant Lines

If G acts on an $\mathbf{R}$-tree T, define $|g|$ for $g \in G$ to be $\inf_{x \in T} d(x, gx)$. Thus, if g is elliptic, $|g| = 0$. The following result, an expansion of the proof of [CM 1.3], gives more information about $|g|$:

Theorem. *Define C_g to be $\{x \in T \mid d(x, gx) = |g|\}$. Then:*

a. $|g| = 0 \implies g$ is elliptic.

b. $|g| > 0 \implies C_g$ is isometric to $\mathbf{R}$ and g translates C_g by $|g|$.

Proof.

We need the following result [CM 1.1]:

> If T_1 and T_2 are disjoint non-empty closed subtrees of T, then there is a unique shortest geodesic, called the spanning geodesic, from a point in T_1 to a point in T_2.

[CM] proves this by taking a subset with the desired properties of a geodesic from a point in T_1 to a point in T_2.

Suppose g is hyperbolic. Let $x \in T$. Let $\alpha = \overline{x, gx}$. Let m be the midpoint of α.

Since $d(x, m) = d(gx, gm)$, $gm \in \alpha \implies gm = m$, contradicting the hypothesis. Thus $gm \notin \alpha$ and so $m \notin g^{-1}\alpha$. Similarly, $g^{-1}m \notin \alpha$ and so $m \notin g\alpha$.

Since $x \in g^{-1}\alpha$ and $gx \in g\alpha$, $g^{-1}\alpha$ and $g\alpha$ are disjoint. Let β be the spanning arc from $g^{-1}\alpha$ to $g\alpha$. Note that β is a subarc of $\overline{x, gx} = \alpha$. Then β intersects $g^{-1}\alpha$ in one point. Call this point t.

By applying g, $g\beta$ intersects α in one point, gt. β also intersects $g\alpha$ in one point. Call this point r. Now,

$$\alpha = \overline{x, r} \ \cup \ \overline{r, gx}$$

$$g\alpha = \overline{gx, r} \ \cup \ \overline{r, g^2x}$$

Suppose (towards a contradiction) that $r \neq gt$. Then $g\beta$ does not touch α at r, so since $g\beta \subset g\alpha$, either $g\beta \subset \overline{gx, r}$ or $g\beta \subset \overline{r, g^2x}$.

If $g\beta \subset \overline{gx, r}$, then $g\beta \subset \alpha$, which contradicts that $g\beta$ and α intersect in one point.

If $g\beta \subset \overline{r, g^2x}$, then since $r \notin g\beta$, and α and $\overline{r, g^2x}$ touch only at r, α and $g\beta$ are disjoint, which also contradicts that they intersect in one point.

So $r = gt$, and $\beta = \overline{t, r} = \overline{t, gt}$, and $g\beta = \overline{gt, gr} = \overline{gt, g^2t}$.

Since $g\beta$ and α touch only at gt, β and $g\beta$ touch only at gt, and $g^i\beta$ and $g^{i+1}\beta$ touch only at $g^{i+1}t$. Since β is isometric to an interval in $\mathbf{R}$, this means $A := \bigcup_i \beta_i$ is isometric to $\mathbf{R}$, and g acts on A by translating it $d(t, gt)$.

If $p \in T$, let q be the point on A of minimal distance from p. Then gq is the point on A of minimal distance from gp, and

$$d(p, gp) = d(p, q) + d(q, gq) + d(gq, gp) = 2d(p, q) + d(q, gq) \geq d(q, gq)$$

The translation distance of A is $d(q, gq)$. Thus $|g| = d(q, gq)$, and $C_g = A$. Since g is hyperbolic, $d(q, gq) > 0$.

This proves the contrapositive of (a). To prove (b), suppose $|g| > 0$. Then g is hyperbolic, and (b) follows from the above. ∎

3. Free Group Automorphisms

A group G has property **FA** iff $\forall$ trees T (G acts on $T \implies \exists x \in T \ \forall g \in G \ gx = x$).

A group G has property **FR** iff $\forall$ $\mathbf{R}$-trees T (G acts on $T \implies \exists x \in T \ \forall g \in G \ gx = x$).

Section 3 of [CV] establishes that F_n, the free group on n elements, has property **FR**. In this section, we elaborate on parts of their discussion.

Define the following automorphisms of F_n: (Throughout this paper, i, j, k and l are all distinct and all range between 1 and n inclusive, and in an automorphism description, all variables not mentioned are fixed. We also write fg for the automorphism sending x to $g(f(x))$.

$\tau_{ij} : a_i \to a_j,\ a_j \to a_i$

$\tau_{ijk} : a_i \to a_j,\ a_j \to a_k,\ a_k \to a_i$

$e_i : a_i \to a_i^{-1}$

$e_{ij} : a_i \to a_{i,}^{-1}\ a_j \to a_j^{-1}$

$\rho_{ij} : a_i \to a_i a_j$ (from [CV])

$\lambda_{ij} : a_i \to a_j a_i$ (from [CV])

$\mu_{ij} : a_i \to a_j,\ a_j \to a_i^{-1}$

By Thm 3.2 of [MKS], the following automorphisms of F_n, which [MKS] calls "elementary automorphisms", generate $Aut(F_n)$:

1. $a_i \to a_{s_i}^{\epsilon_i}$, where s is a permutation of $\{1, 2, \ldots, n\}$ sending i to s_i, and $\epsilon_i = \pm 1$.
2. $a_i \to a_i a_j^n$
3. $a_i \to a_j^n a_i$
4. $a_i \to a_j^n a_i a_j^{-n}$

All automorphisms of type 1 can be written in terms of τ_{ij} and e_i. To do this, write the permutation s as a product of cycles:

$$s = \prod_{l=1}^{L} (A_l, B_l)$$

Then the automorphism can be written as:

$$(\prod_{l=1}^{L} \tau_{A_l B_l})(\prod_{i|\epsilon_i=-1} e_{s_i})$$

Automorphisms of type 2 can be written as ρ_{ij}^n, type 3 as λ_{ij}^n, and type 4 as $\lambda_{ij}^n \rho_{ij}^{-n}$.

Thus $\langle \tau_{ij}, e_i, \lambda_{ij}, \rho_{ij} \rangle = Aut(F_n)$.

The following is a theorem of group theory that can be applied here:

Theorem. *Given an automorphism f on a group G, and a normal subgroup $N \triangleleft G$ where $f(N) = N$, the function $\tilde{f}(Ng) = Nf(g)$ is well-defined, and $\tilde{f}$ is an automorphism on G/N.*

Proof.

To show $\tilde{f}$ is well-defined, we need to show $Ng_1 = Ng_2 \Longrightarrow \tilde{f}(Ng_1) = \tilde{f}(Ng_2)$.

If $Ng_1 = Ng_2$, $\exists n_1, n_2 \in N n_1 g_1 = n_2 g_2$. Then $n_1 g_1 g_2^{-1} = n_2$ and $g_1 g_2^{-1} = n_1^{-1} n_2 \in N$, so $f(g_1 g_2^{-1}) \in f(N) = N$. Since f is an automorphism, $f(g_1 g_2^{-1}) = f(g_1) f(g_2)^{-1} \in N$, which means $f(g_1) \in Nf(g_2)$, so $Nf(g_1) \subset Nf(g_2)$.

Similarly, $f(g_2)^{-1} \in f(g_1)^{-1} N$, so $f(g_2) \in Nf(g_1)$, so $Nf(g_2) \subset Nf(g_1)$.

Thus $Nf(g_1) = Nf(g_2)$, that is, $\tilde{f}(Ng_1) = \tilde{f}(Ng_2)$.

To show $\tilde{f}$ is 1-1, we need to show $\tilde{f}(Ng_1) = \tilde{f}(Ng_2) \Longrightarrow Ng_1 = Ng_2$.

If $\tilde{f}(Ng_1) = \tilde{f}(Ng_2)$, $Nf(g_1) = Nf(g_2)$. Then $f(g_1) f(g_2)^{-1} \in N$, as above with g_1 and g_2. Then $f(g_1 g_2^{-1}) \in N$. Since f is 1-1 and $f(N) = N$, $g_1 g_2^{-1} \in N$, so $Ng_1 = Ng_2$.

$\tilde{f}$ is onto because $\tilde{f}(Nf^{-1}(g)) = Nf(f^{-1}(g)) = Ng$, so Ng is in the image of $\tilde{f}$ for all $g \in G$.

To show $\tilde{f}$ is a homomorphism, we need to show $\tilde{f}(Ng_1 N g_2) = \tilde{f}(Ng_1)\tilde{f}(Ng_2)$. $\tilde{f}(Ng_1 N g_2) = \tilde{f}(Ng_1 g_2) = Nf(g_1 g_2) = Nf(g_1) f(g_2) = Nf(g_1) N f(g_2) = \tilde{f}(Ng_1)\tilde{f}(Ng_2)$.

We now apply this to F_n and the commutator subgroup $[F_n, F_n] := \{aba^{-1}b^{-1} | a, b \in F_n\}$. To show $[F_n, F_n] \triangleleft F_n$, we need to show $a[F_n, F_n]a^{-1} = [F_n, F_n]$ $\forall a \in F_n$.

$[F_n, F_n] \subset a[F_n, F_n]a^{-1}$ (take $a = 1$). To show the other inclusion, note that $axyx^{-1}y^{-1}a^{-1} = [axa^{-1}, aya^{-1}]$.

Thus, a free group automorphism induces an automorphism on $F_n/[F_n, F_n]$. Dividing the free group on n elements by its commutator gives the free abelian group on n elements, $\mathbf{Z}_n$. Given a free group automorphism f, $\overline{f}$ will denote the matric representation of the induced automorphism on $\mathbf{Z}_n$ (using the standard basis). ∎

If $f \in Aut(F_n)$, then $\overline{\tau_{ij} f}$ is the matrix obtained by swapping two rows of $\overline{f}$, since $\tau_{ij} f$ sends a_i to $f(a_j)$ and a_j to $f(a_i)$. Making similar observations for $e_i f$, $\lambda_{ij} f$, and $\rho_{ij} f$ leads to the observation that

$$\det(\overline{\tau_{ij} f}) = -\det \overline{f} \qquad \det(\overline{\lambda_{ij} f}) = \det \overline{f}$$
$$\det(\overline{e_i f}) = -\det \overline{f} \qquad \det(\overline{\rho_{ij} f}) = \det \overline{f}$$

Since these four types of automorphisms generate $Aut(F_n)$, $\det \overline{f} = \pm 1$ $\forall f \in Aut(F_n)$.

The "special automorphisms" of [CV] are those f with $\det \overline{f} = 1$.

Theorem. *$f \mapsto \overline{f}$ is a homomorphism.*

Proof.

We need to show that $\overline{fg} = \overline{f}\,\overline{g}$. Consider the element of $\overline{fg}$ in row i and column j. This is just the total number of a_j that appear in $(fg)(a_i)$, where we count a_j^{-1} as -1 occurrence. The element of $\overline{f}\,\overline{g}$ in row i and column j is the dot product of $\overrightarrow{v}$, the ith row of $\overline{f}$, and $\overrightarrow{w}$, the jth column of $\overline{g}$.

The kth element of $\overrightarrow{v}$ is the number of a_k in $f(a_i)$, and the kth element of $\overrightarrow{w}$ is the number of a_j in $g(a_k)$. Thus $\overrightarrow{v} \cdot \overrightarrow{w}$ is the number of a_j in $(fg)(a_i)$, and so $\overline{fg} = \overline{f}\,\overline{g}$. ∎

Theorem. *The special automorphism subgroup $\mathbf{SA}_n$ of F_n is an index two subgroup of $Aut(F_n)$.*

Proof.

The mapping $f \mapsto det(\overline{f})$ is a homomorphism from $Aut(F_n)$ to $\mathbf{Z_2}$. The special homomorphisms are the kernel of this homomorphism. ∎

The following relations hold $\forall i, j, k, l$, where γ represents either λ or ρ. These relations are essentially a subset of Nielsen's 1924 presentation of $Aut(F_n)$ given in [MKS].

$\tau_{ij}\gamma_{jk} = \gamma_{ik}\tau_{ij}$	$\tau_{ij}\tau_{ik} = \tau_{kij}$
$\tau_{ij}\gamma_{ik} = \gamma_{jk}\tau_{ij}$	$\tau_{ij} = \tau ji$
$\tau_{ij}\gamma_{ki} = \gamma_{kj}\tau_{ij}$	$\tau_{ij}\tau_{kl} = \tau_{lij}\tau_{kli}$
$\tau_{ij}\gamma_{kj} = \gamma_{ki}\tau_{ij}$	$e_i e_j = \mu_{ij}^2$
$\tau_{ij}\gamma_{ij} = \gamma_{ji}\tau_{ij}$	$e_i\tau_{ij} = \mu_{ij}^{-1}$
$\tau_{ij}\gamma_{ji} = \gamma_{ij}\tau_{ij}$	$\tau_{ij}e_i = \mu_{ij}$
$e_i\rho_{ij} = \lambda_{ij}^{-1}e_i$	$e_i\tau_{jk} = e_ie_je_j\tau_{jk} = \mu_{ij}^2\mu_{kj}$
$e_i\lambda_{ij} = \rho_{ij}^{-1}e_i$	$\tau_{jk}e_i = \tau_{jk}e_je_je_i = \mu_{jk}\mu_{ij}^2$
$e_i\rho_{ji} = \rho_{ji}^{-1}e_i$	$\tau_{ij}^2 = e_{ij}^2 = 1$
$e_i\lambda_{ji} = \lambda_{ji}^{-1}e_i$	$\mu_{ij} = \rho_{ij}\lambda_{ji}^{-1}\lambda_{ij}$
	$\tau_{ijk} = \mu_{jk}\mu_{ij}$

Theorem [stated without proof in CV]. *Let $l_i = \lambda_{i,i+1}$ for $i \neq n$ and $\lambda_{n,1}$ for $i = n$. Let $r_i = \rho_{i,i+1}$ for $i \neq n$ and $\rho_{n,1}$ for $i = n$. The set $\{l_i, r_i | 1 \leq i \leq n\}$ generates $\mathbf{SA}_n$.*

Proof.

Let $f \in \mathbf{SA}_n$. Write $f = \beta_1\beta_2 \cdots \beta_N$ where each β_i is a τ, e, λ, or ρ. Since $1 = \det \overline{f} = \prod_i \det(\overline{\beta}_i)$, there are an even number of factors β_i with $\det(\overline{\beta}_i) = -1$.

The first set of relations above allows f to be written in a form with all λ and ρ on the left and an even number of τ and e on the right. To write f in this form, use the relations to repeatedly replace each pair of generators where only the right generator is special with a pair where only the left generator is special.

The second set allows the product of any two automorphisms that are not special to be written as the product of special automorphisms.

Thus the set $\{\lambda_{ij}, \rho_{ij}\}$ generates $\mathbf{SA}_n$.

$[\lambda_{ij}, \lambda_{jk}] = \lambda_{ik}$. This allows us to write any λ_{ij} in terms of l_i by induction on $j-i$. The same is true for ρ and r_i, which proves the result. ∎

Theorem [stated without proof in CV]. *All elements in the above generating set for* $\mathbf{SA}_n$ *are conjugate.*

Proof.

Let $\kappa_{ij} = \tau_{ij}\tau_{i+1,j+1}$. This swaps a_i with a_j and a_{i+1} with a_{j+1}. Then $\kappa_{ij} l_j \kappa_{ij}^{-1} = l_i$, and similarly for r_i and r_j. Since $e_{i,i+1}\ l_i\ e_{i,i+1}^{-1} = r_i$, all l_i and r_i are conjugate in $\mathbf{SA}_n$. ∎

The main result of [CV] is to give a criterion for property **FR**. To do this, they define the following:

A minipotent word in g and h is a word of the form $g^{\epsilon_1}h^{\epsilon_2}\cdots g^{\epsilon_{2n-1}}h^{\epsilon_{2n}}$ or the form $h^{\epsilon_1}g^{\epsilon_2}\cdots h^{\epsilon_{2n-1}}g^{\epsilon_{2n}}$, where each $\epsilon_i = \pm 1$.

If G is a group and $S = \{s_1, \cdots, s_n\}$ is a set of generators for G, then let $\Delta(G,S)$ denote the graph with vertex set S and relation $s_i \sim s_j \iff$ some minipotent word commutes with either s_i or s_j. Let $\Delta'(G,S)$ denote the graph with vertex set S and relation $s_i \approx s_j \iff$ some word of the form $[s_i, s_j^{(k)}]$ commutes with s_j, where $[a, b^{(0)}] = a$ and $[a, b^{(k)}] = [[a, b^{(k-1)}], b]$. (In [CV], $\Delta'(G,S)$ is directed, but for these purposes it doesn't matter if we ignore the directions.)

[CV 2.4] gives the following criterion for property **FR**:

CV 2.4. *If all the generators in* S *are conjugate,* $\Delta(G,S)$ *is complete,* $\Delta'(G,S)$ *is connected, and* $G/[G,G]$ *is finite, then* G *has property* **FR**.

[CV] then uses the above results to apply this to $G = \mathbf{SA}_n$, where $n \geq 3$ and S is $\{l_i, r_i\}$. In this case, $\Delta'(G,S)$ is complete. To see this, let s_i stand for l_i or r_i. s_i commutes with s_j if i, $i+1$, j, and $j+1$ are all distinct. If $i \equiv j+1 \pmod{n}$, then s_i and $[s_j, s_i]$ commute if $n \geq 3$, and so $s_i \approx s_j$. Since $l_i r_i = r_i l_i$, $s_i \approx s_i$. Thus $s_i \approx s_j\ \forall i,j$.

$G/[G,G]$ is the trivial group here, since $l_i = [\lambda_{i,i+2}, \lambda_{i+2,i+1}]$ and $r_i = [\rho_{i,i+2}, \rho_{i+2,i+1}]$. Thus $\mathbf{SA}_n$ has property **FR** for $n \geq 3$. See [CV] for the proof of 2.4 and additional examples.

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