

Sub-game Perfect Equilibria of the Simultaneous Ascending Auction

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1 Introduction

The simultaneous bid ascending auction has been used by the governments of the United States and the United Kingdom and other governments to auction off spectrum licenses. The success of this auction design in generating revenues has varied greatly between auctions. Auctions for radio licenses conducted by the United States government in 1994 produced over \$8 billion in revenues, far exceeding all expectations. However, an auction conducted in 1997 to sell wireless communication spectrum licenses produced only \$13.6 million in revenues, while projections had been that the auction would produce a revenue of \$1.8 billion (Engelbrecht-Wiggans and Kahn, 2005, p. 508).

Deviations of actual revenue from projected revenue maybe due to the fact that bidders' valuations were different from those that were projected; they could also be due to the fact that bidders played a different equilibrium of the auction than projected, or they could be due to the fact that bidders didn't play any equilibrium at all of the auction (assuming that the projection was based on the assumption of equilibrium behavior). Here, we will focus on an analysis of equilibria of simultaneous bid ascending auctions, and we will emphasize the multiplicity of equilibria.

The key paper in the formal analysis of the simultaneous ascending auction is Milgrom (2000). Milgrom investigates strategies to which he refers as "straightforward bidding." These strategies, sometimes also referred to as "naive bidding," or as a "competitive strategy," say that bidders bid myopically for the object that offers in the current round the largest surplus. Milgrom (2000) has shown that straightforward bidding yields a competitive equilibrium allocation provided that goods are substitutes. Remarkably, Milgrom (2000) does not contain any results indicating whether straightforward bidding is actually an equilibrium of the simultaneous ascending auction. The first question that we investigate in this paper is whether straightforward bidding is such an equilibrium. The second question that we address is whether there are other equilibria.

We focus on the very simple case that two objects are sold, that there are either two or three bidders, and that each bidder has single unit demand. For the case of two bidders we find that there is a large multiplicity of equilibria. Straightforward bidding is one of these equilibria. All equilibria predict the same, competitive equilibrium allocation of the two goods. Yet, the prices need not be competitive equilibrium prices. There is a large variety of prices

at which the auction can close. Straightforward bidding, and the associated prices, do play, however a special role: Straightforward bidding is the unique ex post equilibrium, that is, straightforward bidding is the only strategy that players can adopt without knowing the other players' valuations, and yet ex post it will always turn out to be the case that players strategies are equilibrium strategies.

In the case of three bidders, the multiplicity of equilibria that we found in the two bidders case can be reproduced. Instead of doing this, however, we present two specific examples in which other effects emerge. In the first example it turns out that straightforward bidding is actually *not* a sub-game perfect equilibrium. In the second example, we show that there may be sub-game perfect equilibrium that yield not only equilibrium prices, but also an equilibrium allocation that differ from those emerging from straightforward bidding.

Brusco and Lopomo (2002) study collusion via signaling in the simultaneous bid ascending auction when bidders have multiple object demand. They find that straightforward bidding is an equilibrium strategy, but that there are also low-revenue equilibria. We find similar results under the model developed in this paper in which bidders have single unit demand. A further connection between this paper and Brusco and Lopomo (2002) is that like Brusco and Lopomo we use the concept of straightforward bidding as a possible punishment for sustaining low price equilibria. In studying sub-game perfect equilibria we go further than Brusco and Lopomo because, for a very simple context, we are able to offer a complete characterization of sub-game perfect equilibria.

Engelbrecht-Wiggans and Kahn (2005) similarly use the concept straightforward bidding as a threat mechanism to sustain a low price equilibrium. Specifically, they show how that a strategy to which they refer as “stake, protect and revenge,” may sustain low price equilibria. We use a similar strategy to obtain in section 4.2 equilibria that are remarkable not primarily for the resulting prices, but for the resulting allocation of goods.

The remainder of this paper is organized as follows. Section 2 presents our model. Section 3 deals with the case of two bidders. Section 4 presents two examples involving three bidders. Section 5 concludes with some remarks about interesting open questions.

2 Model

We study a specific example of the simultaneous bid ascending auction as introduced by Milgrom (2000). There are two objects for sale: A and B . There are $M \geq 2$ bidders: $i = 1, 2, \dots, M$. Each bidder demands at most one object. Bidder i 's von Neumann Morgenstern utility if she obtains object X and pays p_X is: $\pi_i = v_X^i - p_X$ where v_X^i is bidder i 's valuation of object X and π_i is bidder i 's payoff. Bidder i 's von Neumann Morgenstern utility if she obtains no object and does not pay anything is $\pi_i = 0$. We assume that each player's specific valuations are common knowledge to all players.

The auction proceeds in $N + 1$ rounds: $n = 0, 1, 2, \dots, N$ where N will be assumed to be suitably large. In each round, each of the two objects will have a well-defined price: p_A and p_B . The price of each object is initially set to zero. In period $n = 0$ all players choose simultaneously. Each player can either bid for exactly one object, or drop out of the auction. If an object attracts exactly one bid, then the bidder who made that bid is the "leading bidder" for that object. If for some object there are multiple bidders, then one of these bidders is selected randomly as the "leading bidder" where each bidder has the same probability of being selected. In all rounds after the first round bidders who are currently the leading bidder for some object, and bidders who have dropped out of the auction, have no available choices. For objects that have received a bid during a previous period, the price of that object is raised by some increment, $\varepsilon > 0$ in comparison to the price at which the last bid for that object was placed. All remaining bidders then simultaneously either pick an object for which they wish to bid, or drop out of the auction. The auction closes once all players either hold a high bid on a particular object, or have dropped out. The auction also closes after round N . When the auction closes the currently leading bidders for each object obtain that object at the price at which they placed their bids.

Straightforward bidding, as defined in Milgrom (2000), will play a special role in our analysis.

Definition 1. *We say bidder j bids straightforwardly in period n , if the bidder bids on object X whenever $v_X - p_X > 0$ and $v_X - p_X > v_{X'} - p_{X'} \forall X'$, where we define v_X as player j 's valuation over object X , and p_X as the price of object X in period n . We define X' as any object such that $X' \neq X$. If $0 > v_X - p_X$ for any object, X , player j drops out of the auction under straightforward bidding.*

3 Two Bidders

3.1 Background

In this example of the simultaneous bid ascending auction, we assume that there are only 2 bidders: $i = 1, 2$. The bidders' valuations of the objects are given by the following table:

	A	B
1	v_A^1	v_B^1
2	v_A^2	v_B^2

We assume that the auction ends after a sufficiently large number of periods, N , such that $N\varepsilon > \max\{|v_A^1 - v_B^1|, |v_A^2 - v_B^2|\}$. We may also make the following assumptions:

1. $v_A^1 > v_B^1 > 0$.
2. $v_A^2, v_B^2 > 0$.
3. $v_A^1 - v_B^1 > v_A^2 - v_B^2$.

3.2 Analysis of sub-game perfect equilibrium bidding strategies

We assume that there exists an $N_1 \in \mathbb{N}$ such that

$$v_A^2 - v_B^2 < (N_1 - 1)\varepsilon < v_A^1 - v_B^1 < N_1\varepsilon.$$

This will hold whenever ε is sufficiently close to zero.

Proposition 1. *A pair of pure strategies of players 1 and 2 (s_1, s_2) is a sub-game perfect equilibrium if and only if the following conditions are met:*

- *If (A, A) was initially played by both players, player 1 bids on object A for all periods $\leq (N_1 - 1)$. For all periods $\geq N_1$ player 1 bids on object B. Player 2 bids on object A for the first k periods, and bids on object B for all later periods, where we define k as any number such that $0 < k \leq N_1 - 2$.*

- If (B, B) was initially played and $v_A^1 - v_B^1 > 0 > v_A^2 - v_B^2$, this is a symmetric case to the one above. If (B, B) was initially played and $v_A^1 - v_B^1 > v_A^2 - v_B^2 > 0$, both players 1 and 2 bid on object A in all following sub-games.
- If $v_A^1 - v_B^1 > 0 > v_A^2 - v_B^2$ in the initial period one of the three following strategies is played: (A, A) , (A, B) , (B, B) . If $v_A^1 - v_B^1 > v_A^2 - v_B^2 > 0$ in the initial period one of the two following strategies is played: (A, A) , (A, B) .

This proposition gives a complete characterization of all possible sub-game perfect equilibria in pure strategies in the 2 player 2 object case. Notice that there are a multiplicity of sub-game perfect equilibria. However, despite this, there is only one sub-game perfect equilibrium allocation: player 1 obtains object A and player 2 obtains object B. Different equilibria may lead to different prices, but they will not lead to a different allocation. This result shows that, similar to the results shown in Brusco and Lopomo (2002), there are non-cooperative, low-price equilibria in which the firms correctly anticipate the final allocation of goods and cease bidding early. There are also high price equilibria in which one firm drives up the price that the other firm has to pay up to the point at which this firm is indifferent between the good that it obtains and the other good. These equilibria are reminiscent of the bidding to raise competitors' prices that has been documented by Börgers and Dustmann (2005) for a spectrum auction in the UK. There is also a range of sub-game perfect equilibria that are between these two extremes.

Proof.

Claim 1. *Assume (A, A) was initially played. In any period n , such that: $N_1 \leq n \leq N$, if the auction has yet to end, in all sub-game perfect equilibrium, either player will bid on B.*

Proof. Prove true in period N:

If we are now in period N and the auction has not ended, it must be the case that each player bid on A for all of the previous periods.

$$\implies p_A = N\varepsilon, p_B = 0$$

Assume player 1, is bidding at period N. Player 1 knows that the game will end after period N. Thus the payoffs of bidding are as follows:

- A: $v_A^1 - p_A = v_A^1 - N\varepsilon$
- B: v_B^1

Recall:

$$N\varepsilon > v_A^1 - v_B^1 \implies v_B^1 > v_A^1 - N\varepsilon.$$

The payoff of bidding on B is greater than the payoff of bidding on A. It follows that in a sub-game perfect equilibrium player 1 bids on object B. A symmetric argument follows for player 2.

Assume that player 1 bids in period n where $N_1 \leq n < N$. Assume if the auction does not end by period n , in period $n+1$ player 2 will play B. Prove that in a sub-game perfect equilibrium, player 1 will play B in period n .

If the auction has yet to end by period n and initially both players bid on A, this implies all bids have been on A for the periods preceding n .

$$\implies p_A = n\varepsilon, p_B = 0$$

We assume player 2 will play B in period $n+1$. Thus if player 1 bids on A in period n , he will win A at price $n\varepsilon$. If he bids on B the auction will end at period n , and he will pay 0 for object B. The payoffs of bidding are then as follows:

- A: $v_A^1 - n\varepsilon$
- B: v_B^1

Recall:

$$v_A^1 - v_B^1 < N_1\varepsilon \leq n\varepsilon \implies v_A^1 - n\varepsilon \leq v_B^1.$$

The payoff of bidding on B is greater than the payoff of bidding on A. It follows that in a sub-game perfect equilibrium player 1 bids on object B. A symmetric argument follows for player 2. \square

Claim 2. *Assuming A, A was initially played, in any sub-game perfect equilibrium, in period $N_1 - 1$ player 1 will bid on object A and player 2 will bid on object B.*

Proof. Recall we have assumed both players initially bid on object A, since the auction has yet to end for $N_1 - 1$ periods, it must be that each player previously bid on object A in every preceding period. $\implies p_A = (N_1 - 1)\varepsilon, p_B = 0$.

First assume player 1 bids in period $N_1 - 1$. Player 1 knows, as proved earlier, that in period N_1 player 2 will bid on object B. Thus if player 1 bids on object A he will win object A at price $p_A = (N_1 - 1)\varepsilon$. If he bids on object B, the auction will end and player 1 will win object B at price $p_B = 0$. Thus the payoffs from bidding are as follows:

- A: $v_A^1 - (N_1 - 1)\varepsilon$
- B: v_B^1

Recall:

$$(N_1 - 1)\varepsilon < v_A^1 - v_B^1 \implies v_B^1 < v_A^1 - (N_1 - 1)\varepsilon$$

The payoff of bidding on A is greater than the payoff of bidding on B. It follows that in a sub-game perfect equilibrium player 1 bids on object A.

Now assume that player 2 bids in period $N_1 - 1$. Player 2 knows player 1 will bid on object B in period N_1 and the auction will end. Thus if player 2 bids on object A he will win object A at price $p_A = (N_1 - 1)\varepsilon$. If he bids on object B, the auction will end and player 2 will win object B at price $p_B = 0$. Thus the payoffs from bidding are as follows:

- A: $v_A^2 - (N_1 - 1)\varepsilon$
- B: v_B^2

Recall:

$$v_A^2 - v_B^2 < (N_1 - 1)\varepsilon \implies v_A^2 - (N_1 - 1)\varepsilon < v_B^2$$

The payoff of bidding on B is greater than the payoff of bidding on A. It follows that in a sub-game perfect equilibrium player 2 bids on B. \square

Claim 3. *Assume (A, A) was initially played. For any period, n , such that $0 < n < N_1 - 1$, in any sub-game perfect equilibrium, if the auction has yet to end, player 1 always bids on object A and player 2 is indifferent between object A and object B.*

Proof. First prove true for period $N_1 - 2$:

Recall we have assumed both players initially bid on object A, since the auction has yet to end for $N_1 - 2$ periods, it must be that each player previously bid on object A in every preceding period. $\implies p_A = (N_1 - 2)\varepsilon, p_B = 0$.

First assume player 1 bids in period $N_1 - 2$. Player 1 knows, as proved earlier, that in period $N_1 - 1$ player 2 will bid on object B. Thus if player 1 bids on object A he will win object A at price $p_A = (N_1 - 2)\varepsilon$. If he bids on object B, the auction will end and player 1 will win object B at price $p_B = 0$. Thus the payoffs from bidding are as follows:

- A: $v_A^1 - (N_1 - 2)\varepsilon$
- B: v_B^1

Recall:

$$(N_1 - 2)\varepsilon < v_A^1 - v_B^1 \implies v_B^1 < v_A^1 - (N_1 - 2)\varepsilon$$

The payoff of bidding on A is greater than the payoff of bidding on B. It follows that in a sub-game perfect equilibrium player 1 bids on object A.

Now assume player 2 bids in period $N_1 - 2$. If player 2 bids on object A, we know in period $N_1 - 1$ player 1 will bid on object A, as proved earlier. The auction will continue to period N_1 where player 2 will bid on object B and receive a payoff v_B^2 as we proved earlier.

If player 2 bids on object B, the auction will end and he will receive a payoff of v_B^2 as $p_B = 0$. Thus in period $N_1 - 2$: Payoff from bidding on A = Payoff from bidding on B. Player 2 is indifferent between bidding A or B.

Assume that the claim holds true for all periods, k such that $N_1 - 2 > k \geq n + 1$. Prove the claim holds true in period n :

First assume player 1 bids in period n . If player 1 bids on object B his payoff is v_B^1 and the auction ends. If player 1 bids on object A, in the following period we assume that player 2 is indifferent between bidding on object A and object B. If player 2 bids on object B the auction is over and player 1's payoff is $v_A^1 - n\varepsilon$. If player 2 continues to bid on object A however, he will only do so up till at most period $N_1 - 2$, as we have already proved for any period $\geq N_1 - 1$ player 2 will bid on object B. Thus the highest price level

possible for object A to reach is $p_A = (N_1 - 1)\varepsilon$. Therefore the payoff from bidding on object A for player 1 in period n is: $\geq v_A^1 - (N_1 - 1)\varepsilon$ Recall we know:

$$(N_1 - 1)\varepsilon < v_A^1 - v_B^1 \implies v_B^1 < v_A^1 - (N_1 - 1)\varepsilon$$

The payoff of bidding on A is greater than the payoff of bidding on B. It follows that in a sub-game perfect equilibrium player 1 bids on object A.

Now assume player 2 bids in period n. If player 2 bids on object B, his payoff is v_B^2 and the auction ends. If player 2 bids on object A in period n, we assume player 1 will bid on object A in period n+1. We also assume that in all later periods player 2 either strictly prefers object B or is indifferent between bidding on objects A and B. Therefore in period n+2, player 2 either bids on object B and receives a payoff of v_B^2 or player 2 is indifferent between bidding on A or B. Since the payoff of bidding on B is v_B^2 this implies the payoff in period n+2 is v_B^2 . Thus the payoff of bidding on object A is v_B^2 . Since the payoff of bidding on A = payoff of bidding on B in period n, player 2 is indifferent between A and B. \square

If we assume $0 > v_A^2 - v_B^2$ symmetric results to the claims above can be shown when B, B is initially played by both players. Simply replace player 1 with player 2 and object A with B.

Claim 4. *Assume $v_A^2 - v_B^2 > 0$ and (B, B) was initially played. For any period n, such that $N \geq n > 0$, if the auction has yet to end, in all sub-game perfect equilibrium, each player bids on object A.*

Proof. Prove true in period N:

If the auction has yet to end by period N and B, B was initially played, it must be the case that $p_A = 0$ and $p_B = N\varepsilon$. Without loss of generality assume player 1 bids in period N. Since the auction will end in the following period player 1 will win the object he/she bids on in period N. Thus the payoff of bidding is as follows:

- A: v_A^1
- B: $v_B^1 - N\varepsilon$

Recall:

$$v_A^1 > v_B^1 > v_B^1 - N\varepsilon \implies v_A^1 > v_B^1 - n\varepsilon$$

Thus in a sub-game perfect equilibrium player 1 bids on object A. A symmetric argument follows for player 2.

Assume the claim holds true for all periods $\geq n + 1$ where $n > 0$. Prove the claim holds in period n :

If the auction has yet to end by period n and B, B was initially played, it must be the case that $p_A = 0$ and $p_B = n\varepsilon$. Without loss of generality assume player 1 bids in period n . Since we assume if the auction does not end in period n , in the following period player 2 will bid on object A, player 1 knows if he/she bids on object B, he will win object B. Thus the payoff of bidding is as follows:

- A: v_A^1
- B: $v_B^1 - n\varepsilon$

Recall:

$$v_A^1 > v_B^1 > v_B^1 - n\varepsilon \implies v_A^1 > v_B^1 - n\varepsilon$$

Thus in a sub-game perfect equilibrium player 1 bids on object A. A symmetric argument follows for player 2. \square

Claim 5. *If $v_A^1 - v_B^1 > v_A^2 - v_B^2 > 0$ in the initial period players 1 and 2 play either (A, A) or (A, B) in a sub-game perfect equilibrium.*

Proof. From the above claims we can determine the payoffs from bidding (A, A) and (B, B) in the initial period, in a sub-game perfect equilibrium. It follows that both players play the following game in the initial period:

	A	B	Drop
A	$v_A^1 - k\varepsilon, v_B^2$	v_A^1, v_B^2	$v_A^1, 0$
B	v_B^1, v_A^2	$\frac{v_A^1 + v_B^1}{2}, \frac{v_A^2 + v_B^2}{2}$	$v_B^1, 0$
Drop	$0, v_A^2$	$0, v_B^2$	$0, 0$

Where $k \in \mathbb{N}$ and $0 < k \leq N_1 - 1$. In all sub-game perfect equilibrium, player 1 will always play A in the initial period, while player 2 is indifferent between A and B. \square

Claim 6. *If $v_A^1 - v_B^1 > 0 > v_A^2 - v_B^2$, in the initial period players 1 and 2 play either (A, A), (A, B), or (B, B) in a sub-game perfect equilibrium.*

Proof. Under the above assumptions, we know from the claims proved earlier, both players play the following game in the initial period:

	A	B	Drop
A	$v_A^1 - k\varepsilon, v_B^2$	v_A^1, v_B^2	$v_A^1, 0$
B	v_B^1, v_A^2	$v_A^1, v_B^2 - j\varepsilon$	$v_B^1, 0$
Drop	$0, v_A^2$	$0, v_B^2$	$0, 0$

Where k is the same as defined above and $j \in \mathbb{N}$ such that $0 < j\varepsilon \leq v_A^2 - v_B^2$. There are three possible sub-game perfect equilibrium strategy combinations for players 1 and 2: (A, B); (A, A); and (B, B). \square

This completes the proof of Proposition 1. \square

3.3 Equilibrium selection and straightforward bidding

Are some of the equilibria in Proposition 1 more plausible than others? Informally, it seems that different equilibrium selections can be advocated. Suppose, for example, that participation in the auction is costly, and each further round of bidding raises the participation costs by a small amount. Then it seems likely that bidders will settle for a low price equilibrium in which bidding ceases early. Suppose, alternatively, that bidders have a small incentive to financially damage other bidders, for example because bidders are firms that compete in the same market, and each firm wants to weaken its rivals. Then it seems plausible that firms play a high price equilibrium.

In this subsection we focus on straightforward bidding which has received special attention in the literature (Milgrom, 2000). We defined straightforward bidding in Section 2. We now first verify that straightforward bidding is a sub-game perfect equilibrium.

Proposition 2. *Straightforward bidding is a sub-game perfect equilibrium.*

Proof. First assume: $v_A^1 - v_B^1 > 0 > v_A^2 - v_B^2$.

Without loss of generality assume player 1 is bidding in some period n . If we assume (A, A) was initially played it is a sub-game perfect equilibrium for player 1 to bid on object A for all periods $\leq N_1 - 1$ and bid on object B for all periods $\geq N_1$ as proved in proposition 1. It follows that if (A, A) was initially played, player 1 bids on object A for all periods such that $v_A^1 - p_A > v_B^1$ and object B for all periods such that $v_A^1 - p_A < v_B^1$. Recall we also know from proposition 1 that if (B, B) was initially played, it is always a sub-game perfect equilibrium for player 1 to bid on object A. Thus it is a sub-game perfect equilibrium for player 1 to bid on object B in any sub-game such that $p_B > 0$. Thus player 1 bids on object B while $v_A^1 > v_B^1 - p_B$. It is also shown in proposition 1 that it is sub-game perfect for player 1 to bid on object A in the initial period. It follows that player 1 bids straightforwardly in all possible sub-games. The argument for player 2 is symmetric.

Now assume: $v_A^1 - v_B^1 > v_A^2 - v_B^2 > 0$.

First assume that player (A, A) was initially played. If player 1 bids, we know he bids on object A for all periods $\leq N_1 - 1$ and bids on object B for all periods $\geq N_1$ as shown by proposition 1. Thus player 1 bids on object A for all periods such that $v_A^1 - p_A > v_B^1$ and object B for all periods such that $v_A^1 - p_A < v_B^1$. If player 2 bids, we know it is a sub-game perfect equilibrium bidding strategy for him to bid on object A for the first k periods and B for all proceeding periods, where we define k as such: $v_A^2 - k\varepsilon > v_B^2$ and $v_A^2 - (k + 1)\varepsilon < v_B^2$. Thus player 2 bids straightforwardly. Now assume (B, B) was initially played. We have proved in proposition 1 that for any period where $p_B > 0$, in a sub-game perfect equilibrium each player bids on object A. Thus each player bids on object A while $v_A^1 > v_B^1 - p_B$ or $v_A^2 > v_B^2 - p_B$. Recall we know in the initial period straightforward bidding is a sub-game perfect equilibrium, as we know (A, A) is a part of a sub-game perfect equilibrium. It follows that both players bid straightforwardly. \square

Straightforward bidding leads to prices that are intermediate among the equilibrium prices described in Proposition 1. There are equilibria that lead to higher prices, and there are equilibria that lead to lower prices. We now ask whether there is a reason to select straightforward bidding among all sub-game perfect equilibria. The argument that we shall offer is that straightforward bidding is informationally robust. Each bidder's strategy does not require knowledge of the other player's valuation. If all bidders bid straight-

forwardly, strategies form a sub-game perfect equilibrium ex post for all possible valuation profiles. To make this formal we need the following definition.¹

Definition 2. *A pair of pure strategies (s_1, s_2) where for each player i the strategy s_i depends on the player's own valuations, but not on the other player's valuations, form an ex post equilibrium if, for each specification of valuations, the strategies form a sub-game perfect equilibrium of the auction game in which the valuations are common knowledge.*

Proposition 3. *Straightforward bidding is the only ex-post equilibrium.*

Proof. We have already shown that straightforward bidding is always sub-game perfect equilibrium. Since straightforward bidding depends only on a player's own valuation, but not on the other player's valuation, it follows that straightforward bidding is an ex-post equilibrium.

To prove uniqueness we first prove that in an ex-post equilibrium, each player must bid straightforwardly in the initial period. Assume there exists an ex-post equilibrium such that player 1 bids on object B in period 0. If player 2's valuations are such that $v_A^1 - v_B^1 > v_A^2 - v_B^2 > 0$ we have showed that this can never be part of a sub-game perfect equilibrium. Therefore, bidding for B cannot in period 0 cannot be part of an ex post equilibrium for player 1. The argument for player 2 is analogous.

Now prove: In an ex-post equilibrium, each player bids straightforwardly in all periods greater than zero. Assume there exists an ex-post equilibrium such that player 1 does not bid straightforwardly. Thus in some period, n , such that $n > 0$, player 1 either:

1. Bids on object B while $v_A^1 - p_A > v_B^1$
2. Bids on object A while $v_B^1 > v_A^1 - p_A$

First we analyze case 1: If $v_A^1 - v_B^1 > v_A^2 - v_B^2$ we have shown that it can never be a sub-game perfect equilibrium to bid on object B while $v_A^1 - p_A > v_B^1$. This directly follows from proposition 1.

¹Strictly speaking, the following definition describes an "ex post sequential equilibrium" in the sense of Börgers and McQuade (2007). Note that Börgers and McQuade (2007, Proposition 5) shows that an "ex post sequential equilibrium" also has the stronger property of being a "strongly information invariant sequential equilibrium."

Now case 2: Since $p_A > 0 \Rightarrow (A, A)$ was initially played. Yet here, player 1 bids on object A in some period $\geq N_1$. It directly follows from proposition 1 that his is not a sub-game perfect equilibrium. \square

4 Three Bidders

We now show that the analysis of sub-game perfect equilibria changes in two interesting ways when we introduce three rather than two bidders. Firstly, unlike in the two bidder case (see Proposition 2), straightforward bidding need not be a sub-game perfect equilibrium. This is shown in the example in subsection 4.1. Secondly, unlike in the two bidder case (see Proposition 1), it is not the case that sub-game perfect equilibrium outcomes differ only with respect to prices, but not with respect to the allocation of goods to bidders. This is shown in the example in subsection 4.2. In both examples, we assume that the increment size ε equals 1.

4.1 Straightforward Bidding

In the first example we assume that the bidders' valuations of the two objects are described as follows:

	A	B
1	4.5	2.1
2	2.3	1.6
3	0.9	0.7

We assume that N , the index of the last period, is at least: $N \geq 5$. This implies that if the auction has yet to end by period N , $p_A + p_B \geq N \geq 5$.

Proposition 4. *Straightforward bidding is not a sub-game perfect equilibrium in the initial period.*

Proof. We shall show that player 2 in period $n = 0$ has an incentive to deviate from straightforward bidding. To prove this, we start by calculate an upper bound for the expected utility that player 2 obtains if all players follow straightforward bidding. If all players bid straightforwardly, they will

all bid on object A in period 0. One of the three players is given the high bid on object A. With a $\frac{1}{3}$ probability, player 1 is given the high bid on object A. In the following period both players 2 and 3 bid on object B. There is a .5 probability player 3 is given the high bid on object A and player 2 bids again in period 2. There is therefore a probability of at least $\frac{1}{6}$ that the price of B will rise to 1 or more. From Milgrom (2000, Theorem 2) we can infer that player 2 will eventually win object B. Therefore, an upper bound for player 2's expected utility is: $\frac{1}{6}(v_B^2 - 1) + \frac{5}{6}(v_B^2)$.

Now suppose player 2 deviates and bids for object B in period 0. Then player 2 will become the leading bidder for B, and either 1 or 3 will be the leading bidder for A. If 1 is the leading bidder for A, then 3 drops out in the next round and the auction ends. If 3 is the leading bidder for A, then 1 bids in the next round for A, and in the subsequent round 3 drops out. In either case, the price of B will remain at zero, and therefore player 2's utility will be v_B^2 . This is larger than the upper bound derived in the previous paragraph. \square

Proposition 5. *The following strategy profile is a sub-game perfect equilibrium: Players 1, and 3 bid straightforwardly. Player 2 bids on object B in the initial period. In all later periods, player 2 bids straightforwardly.*

Proof. We proceed in a number of steps.

Claim 7. *In period N, straightforward bidding is a sub-game perfect equilibrium.*

Proof. If 2 players are bidding in period n, by the rules of the auction, it must be the case that one object remains priced at 0. Without loss of generality assume $p_A = 0$. Recall that in period N, $p_A + p_B \geq 5$. Thus $p_B \geq 5$. Using the one deviation principle, we can analyze the game from the perspective of player i. By assumption the other bidder in the period will bid on object A. If player i deviates and bids on object B, he will win object B at price $p_B \geq 5$ as the auction ends in the following period. Thus $\pi_i = v_B^i - p_B \leq v_B^i - 5 < 0$. Recall straightforward bidding yields a non-negative payoff, thus this can not be a profitable deviation.

If only player 1 bids in period n, whichever object player i bids on in period N, he will win. By this logic, there can never be a profitable deviation from straightforward bidding. \square

Now we analyze the auction in some period n where $n < N$.

Claim 8. *If in some period, n , $p_A \geq 1$ and $p_B \geq 1$, straightforward bidding is a sub-game perfect equilibrium.*

Proof. Under the one-deviation principle we may assume that in all periods following n , all players bid straightforwardly. First assume that player 1 bids in period n (a symmetric argument follows for player 2). Without loss of generality assume player 3 holds object A.

Case 1: $v_A^1 - p_A > v_B^1 - p_B$ and $v_A^1 - p_A > 0$.

If player 1 bids on object A, we assume that in the following period player 3 will drop out. Thus player 1 would win object A at price p_A . If player 1 deviates to bidding on object B, he may either win object A or B, at prices greater than or equal to the current prices, or drop out of the auction. This follows because prices can never fall in the auction. Thus the payoffs from bidding can be characterized as follows:

- A: $v_A^1 - p_A$.
- B: $\leq v_A^1 - p_A$.
- Drop out: 0.

It follows there does not exist a profitable deviation from bidding on object A.

Case 2: $v_B^1 - p_B > v_A^1 - p_A$ and $v_B^1 - p_B > 0$. We know if player 1 bids on object A in the following period player 3 will drop out. Thus player 1 would win object A at price p_A . If player 1 bids on object B, we know that for all following periods all players bid straightforwardly. If either player 1 or 2 ever bid on object A in any of the following periods, by our assumption, player 3 will drop out and the auction will end. Thus, under straightforward bidding, player 1 will continue to bid on object B for all periods such that $v_B^1 - p_B > \max\{v_A^1 - p_A, 0\}$. Thus if player 1 wins object B, we know his payoff must be such that $v_B^1 - p_B > \max\{v_A^1 - p_A, 0\}$. If does not win object B, his payoff must be such that $\pi_1 = \max\{v_A^1 - p_A, 0\}$. Thus the payoffs from bidding can be categorized as follows:

- A: $v_A^1 - p_A$.

- B: $\geq \max\{v_A^1 - p_A, 0\}$.
- Drop out: 0.

It follows there does not exist a profitable deviation from bidding on object B.

Case 3: $0 > v_A^1 - p_A$ and $0 > v_B^1 - p_B$.

If player 1 bids on object A he may or may not be outbid. If he is outbid, we know in the following period he drops out as we assume he bids straightforwardly. If he wins object A his payoff is $0 > v_A^1 - p_A$. A symmetric argument follows for bidding on object B. Thus the payoffs from bidding can be categorized as follows:

- A: ≤ 0 .
- B: ≤ 0 .
- Drop out: 0.

It follows there does not exist a profitable deviation from dropping out.

Now assume player 3 bids in period n . If player 3 bids on object A he may or may not be outbid. If he is outbid, we know in the following period he drops out, as we assume he bids straightforwardly. If he wins object A his payoff is $0 > v_A^3 - p_A$. A symmetric argument follows for bidding on object B. Thus the payoffs from bidding can be categorized as follows:

- A: ≤ 0 .
- B: ≤ 0 .
- Drop out: 0.

It follows there does not exist a profitable deviation from dropping out. \square

Claim 9. *For all periods n such that $p_A \geq 3$ and $p_B = 0$, straightforward bidding is a sub-game perfect equilibrium. .*

Proof. Suppose only two players are present in the auction. Then straightforward bidding is a sub-game perfect equilibrium by Proposition 1. We therefore assume in this proof that no player has dropped out of the auction.

Assume player 1 bids in period n . Does there exist a profitable deviation for player 1 from straightforward bidding if we assume straightforward bidding is played for the remainder of the auction? We know that under straightforward bidding all players bid on object B in any such sub-game characterized as above. If player 1 deviates and bids on object A, by assumption, no other player will bid on object A again for the remainder of the auction. Thus player 1 will win object A at price p_A . If player 1 follows straightforward bidding he will bid on object B for all periods such that $v_B^1 - p_B > \max\{v_A^1 - p_A, 0\}$. If he eventually wins object B we know his payoff is $\pi_1 = v_B^1 - p_B > \max\{v_A^1 - p_A, 0\}$. If he does not win object B, his payoff is then $\pi_1 = \max\{v_A^1 - p_A, 0\}$. Thus the payoffs from bidding can be categorized as follows:

- A: $v_A^1 - p_A$.
- B: $\geq \max\{v_A^1 - p_A, 0\}$.
- Drop out: 0.

It follows that there does not exist a profitable deviation for player 1.

Assume player 2 bids in period n (as well as one other player). Assume player 2 deviates and bids on object A. We know one other player bids in period n , and by assumption he bids on object B. We know that in the following period either player 1 or 3 bids and face prices $p_A \geq 4$ and $p_B = 1$. If player 3 bids in period $n+1$, we assume he drops out as he bids straightforwardly. If player 1 bids in period $n+1$, he bids on object B. We know player 3 previously held object B, and thus in period $n+2$, player 3 drops out and the auction ends. Thus player 2 wins object A at price $p_A \geq 3$. This implies his payoff is negative. We know straightforward bidding always yields a non-negative payoff, thus this is not a profitable deviation.

Now assume player 3 bids in period n (as well as one other bidder). We assume the other bidder, either player 1 or 2 bids on object B in period n . Assume player 3 deviates and bids on object A. If player 3 is not outbid, he wins object A at some price $p_A \geq 3$. If player 3 is outbid we know that he drops out in the following period as $p_A \geq 3$ and $p_B \geq 1$. We know $p_B \geq 1$ as one of the other players bids on object B in period n . Thus his payoff is then 0. Thus deviating to bidding on object A never yields a positive payoff. We know straightforward bidding always yields a non-negative payoff, thus this is not a profitable deviation. \square

Claim 10. *For all periods n such that $p_A = 0$ and $p_B \geq 2$, straightforward bidding is a sub-game perfect equilibrium.*

Proof. A symmetric argument to the proof of the previous claim. \square

Claim 11. *For any period, n , such that $p_A = 1$ or $p_A = 2$ and $p_B = 0$ straightforward bidding is a sub-game perfect equilibrium.*

Proof. Suppose only two players are present in the auction. Then straightforward bidding is a sub-game perfect equilibrium by Proposition 1. We therefore assume in this proof that no player has dropped out of the auction.

First assume player 1 bids in period n (as well as one other player). Under straightforward bidding player 1 bids on object A. Under the one-deviation principle we can assume all players bid straightforwardly for the remainder of the auction. It follows by assumption that the other player bidding in period n bids on object B. If player 1 bids on object A, in the following period it will be the case that $p_A = 2$ or $p_A = 3$ and $p_B = 1$. If player 3 bids in period $n+1$, he then drops out and the auction ends. If player 2 bids in period $n+1$, he will then bid on object B. In period $n+2$, player 3 will then drop out and the auction will end. Thus player 1 wins object A at price p_A under straightforward bidding. If player 1 deviates from this strategy, he may win either objects A or B at prices greater than or equal to the current prices, or drop out. Since $v_A^1 - 2 > v_B^1 > 0$, we know there can not be a profitable deviation from straightforward bidding for player 1.

Now assume player 2 bids in period n . First assume that player 3 is currently the high bidder on object A, and thus player 1 bids in period n as well. By assumption, player 1 will bid on object A in period n . If player 2 follows straightforward bidding, he will bid on object B. In the following period player 3 will bid and face prices $p_A > 1$ and $p_B = 1$. By assumption, player 3 will drop out, and player 2 will win object B at price 0. If player 2 deviates from this strategy, he may either he may win either objects A or B at prices greater than or equal to the current prices, or drop out. Since $v_B^2 > v_A^2 - 1$ and $v_B^2 > 0$, there does not exist a profitable deviation from straightforward bidding for player 2 under these assumptions.

Now assume that player 1 is the current high bidder on object A. We assume in period n player 3 will bid on object B. If player 2 bids on object B a winner is randomly chosen between players 2 and 3. If player 2 is given the high bid on object B at $p_B = 0$, in the following period, player 3 drops

out as he faces prices $p_A \geq 1$ and $p_B = 1$. If player 3 is given the high bid on object B, in the following period player 2 bids again. If $p_A = 1$, player 2 will bid then bid on object A in period $n+1$. In period $n+2$, player 1 will bid on object A. In period $n+3$, player 2 will then bid on object B at price $p_B = 1$. In period $n+4$, player 3 drops out and the auction ends. If $p_A = 2$, player 2 will bid on object B in period $n+2$. In period $n+3$, player 3 will drop out and the auction will end. Thus player 2 will win object B under straightforward bidding. He pays $p_B = 1$ with a .5 probability, and $p_B = 0$ with a probability of .5.

Now assume player 2 deviates and bids on object A in period n . We know player 3 bids on object B in period n as well. In period $n+1$, player 1 bids on object A. Then in period $n+2$, player 2 bids on object B. Player 3 then drops out of the auction. Thus player 2 wins object B at price $p_B = 1$. Thus this is not a profitable deviation from straightforward bidding.

Now assume player 3 bids in period n . If all players follows straightforward bidding player 3 drops out and receives a payoff of 0. Does there exist a profitable deviation from straightforward bidding for player 3? Assume there exist a profitable deviation. This implies player 3 wins either object A or B at a price of 0. If player 3 wins one of the objects, either player 1 or 2 does not. Without loss of generality assume 1 does not win any object. In order for player 1 to drop out of the auction it must be the case that $p_A > v_A^1$ and $p_B > v_B^1$. However, if player 3 wins either object at price 0, the final price of bidding must have been 1 for the object. Yet $v_A^1 > 1$ and $v_B^1 > 1$. This contradicts our assumption, thus there does not exist a profitable deviation. \square

Claim 12. *In any sub-game where $p_A = 0$ and $p_B = 1$, straightforward bidding is a sub-game perfect equilibrium.*

Proof. Suppose only two players are present in the auction. Then straightforward bidding is a sub-game perfect equilibrium by Proposition 1. We therefore assume in this proof that no player has dropped out of the auction.

First assume player 1 bids in period n . Since we assume all other players bid straightforwardly for the remainder of the auction, we know that in period n , the other player bidding will bid on object A. Thus, if player 1 deviates and bids on object B, he will be given the high bid on object B. The new price of object B will become $p_B = 2$, and neither player 2 or 3 will ever bid on object

B again as $2 > v_B^2 > v_B^3$. Thus player 1 will win object B at price $p_B = 1$. We shall complete the proof by showing that, if all players bid straightforwardly, player 1 wins object A at some price $p_A \leq 3$. Assume contrary to this claim that player 1 wins object B. This implies player 1 made the final choice that $2.1 - p_B > 4.5 - p_A$, as he bids straightforwardly. However we know that $p_B \geq 1$. This implies $2.1 - 1 = 1.1 \geq 2.1 - p_B > 4.5 - p_A$, or $p_A > 3.4$. Since p_A can only increase by integer multiples, it must be the case that the final price asking of object A was $p_A \geq 4$. Thus either player 2 or 3 wins object A at a price $p_A \geq 3$. However $p_A \geq 3 > 2.3 > .9$. This implies either player 1 or 2 bids over his/her valuation. Yet we assume all players bid straightforwardly following period n. This contradicts our assumption, thus player 1 wins object A. If all players follow straightforward bidding for all periods following n, we know players 2 or 3 will never bid on object A at any price $p_A \geq 3$. Thus the price of object A can never rise above this point.

Now assume player 2 bids in period n. First assume player 1 holds object B. If player 2 bids straightforwardly he bids on object A. By assumption, player 3 bids on object A in period n. If player 2 is given the high bid on object A in period n+1, player 3 drops out and player 2 wins object A. If player 3 is given the high bid on object A, in period n+1 player 2 bids on object A. In period n+2, player 3 drops out and the auction ends. Thus player 2 wins object A at price $p_A = 0$ with a probability of .5 or wins object A at price $p_A = 1$ with a probability of .5. Now assume player 2 deviates and bids on object B. We know player 3 bids on object A in period n, thus the new price of object A is $p_A = 1$. It follows then that player 2 either wins object A, B or drops out. If player 2 eventually wins object A, we know it must be at some price $p_A \geq 1$. If player 2 wins object B, it must be at $p_B \geq 1$. This implies his end payoff is at most $v_A^2 - 1$. If player 2 followed straightforward bidding we know his payoff is at least $v_A^2 - 1$. Thus bidding on object B is not a profitable deviation.

Now assume player 2 bids in period n and player 3 holds object B. We know player 1 bids on object A in period n by assumption. If player 2 bids on object A as well, he eventually wins object B at a price $p_B = 1$. If player 2 deviates and bids on object B in period n, we know that player 3 drops out in the following period as $p_A \geq 1$ and $p_B \geq 1$. Thus player 2 wins object B and receives a payoff of $\pi_2 = v_B^2 - 1$. It follows that there does not exist a profitable deviation from straightforward bidding for player 2.

Now assume player 3 bids in period n. Either player 1 or 2 bids in

period n , and by assumption bids on object A. Assume player 3 deviates from straightforward bidding and wins either object A or B. This implies either players 1 or 2 does not win an object. Without loss of generality assume player 1 does not win an object. This implies the final prices of the objects must be such that $p_A > 4.5$ and $p_B > 2.1$. If player 3 wins an object, we then know he pays $p_A > 3.5$ or $p_B > 1.1$. Thus the deviation yields a negative payoff. If player 3 deviates from straightforward bidding, and ends up dropping out, this can not be a profitable deviation as his payoff is 0. Since straightforward bidding yields a non-negative payoff, there does not exist a profitable deviation. \square

Claim 13. *Players 1 and 3 bidding for object A and player 2 bidding for object B is a Nash equilibrium of period $n = 0$ if in all subsequent periods all players bid straightforwardly.*

Proof. First analyze the game from the perspective of player 1. If player 1 follows the set strategy, in the initial period he is given the high bid on object A with a .5 probability. If player 1 is given the high bid on object A, in period 1 player 3 drops out, as $p_A = 1$ and $p_B = 1$. Player 1 wins object A at a price $p_A = 0$. If player 3 is given the high bid on object A, player 1 bids on object A in period 1. In period 2 player 3 drops out and the auction ends. Player 1 wins object A at price $p_A = 1$.

If player 1 deviates and bids on object B, we then know player 3 is given the high bid on object A in the initial period. It follows that $p_A \geq 1$ for the remainder of the auction. Thus player 1 may eventually either win object A or B, or drop out. If player 1 wins object A, it then must be at some price $p_A \geq 1$. It follows that his payoff must be such that $\pi_1 \leq \max\{v_A^1 - 1, v_B^1\}$. Since $v_A^1 - 1 \geq \max\{v_A^1 - 1, v_B^1\}$, it follows that this is not a profitable deviation.

Now we analyze the game from the perspective of player 2. The argument in the proof of Proposition 4 proves that player 2 has no incentive to deviate.

Now we analyze the game from the perspective of player 3. Assume player 3 deviates from straightforward bidding and eventually wins either object A or B or drops out. If player 3 wins an object, this implies either players 1 or 2 does not win an object. Assume player 1 does not win an object. This implies the final prices of the objects must be such that $p_A > 4.5$ and $p_B > 2.1$. If player 3 wins an object, we then know he pays $p_A > 4.5$ or

$p_B > 2.1$. Thus the deviation yields a negative payoff. Since straightforward bidding yields a non-negative payoff, this can not be a profitable deviation. If player 2 does not win an object, it must be the case that in the final period he made the choice that $p_A > v_A^2 = 2.3$ and $p_B > v_B^2 = 1.6$, as we assume player 3 bids straightforwardly following period 0. Thus if player 3 wins an object, he must pay some price $p_A > 1.3$ or $p_B > .6$. Since prices are only in integer multiples, this implies he pays a price greater than or equal to 1, and receives a negative payoff. This can not be a profitable deviation as straightforward bidding yields non-negative payoffs. If player 3 eventually drops out following the deviation, his payoff is 0. Following straightforward bidding guarantees a payoff greater than or equal to zero, thus this is not a profitable deviation. \square

This completes the proof of Proposition 5. \square

4.2 Other Equilibria

We now consider a slightly different example. In this example we assume that the bidders' valuations of the two objects are as follows:

	A	B
1	3.5	2.2
2	2.3	1.6
3	1.1	0.7

We assume that N , the index of the last period, is at least: $N \geq 4$.

Proposition 6. *There exists a sub-game perfect equilibrium where player 1 wins object B and player 2 wins object A.*

Proof. Consider the following strategies. In the initial period, player 1 bids on object B and players 2 and 3 bid on object A. If player 2 is given the high bid on object A in the initial period, player 3 drops out in period 1. If player 3 is given the high bid on object A in the initial period, player 2 bids on object A in period 1. In period 2 player 3 drops out. If any player deviates, all players bid straightforwardly for the remainder of the auction. We prove in a sequence of steps that these strategies constitute a sub-game perfect equilibrium.

Claim 14. *In period N straightforward bidding is a sub-game perfect equilibrium.*

Proof. Analogous to the proof of Claim 7. The details are omitted. \square

Now we analyze the game in some period n , such that $n < N$.

Claim 15. *If in some period, n , $p_A \geq 2$ and $p_B \geq 1$, straightforward bidding is a sub-game perfect equilibrium.*

Proof. Analogous to the proof of Claim 8. The details are omitted. \square

Claim 16. *For all periods, n , such that $p_A \geq 3$ and $p_B = 0$, straightforward bidding is a sub-game perfect equilibrium.*

Proof. Analogous to the proof of Claim 9. The details are omitted. \square

Claim 17. *For all periods n such that $p_B \geq 2$ straightforward bidding is a sub-game perfect equilibrium.*

Proof. If only two players bid in period n , we know that straightforward bidding follows as a sub-game perfect equilibrium by Proposition 2. Now we assume 3 bidders remain in period n . We have already shown this claim holds when $p_A \geq 2$. First prove true when $p_A = 1$. Assume player 1 bids in period n . If player 1 bids on object B, we know that in all following periods, he will never be outbid on object B. If player 1 bids on object A and follows straightforward bidding for the remainder of the auction, we know, if he win object A, it must be at some price, p_A , such that $v_A^1 - p_A > v_B^1 - p_B$ (where we assume p_B to be unchanged from the initial period). If player 1 ever drops out under straightforward bidding it must be the case that $0 > v_B^1 - p_B$, where p_B remains unchanged from the initial period. If player 1 wins object B under straightforward bidding, his payoff is $v_B^1 - p_B$. Thus the payoffs from bidding for player 1 can be characterized as follows:

- A: $\geq \max\{v_B^1 - p_B, 0\}$.
- B: $v_B^1 - p_B$.
- Drop out: 0.

Thus there is not a profitable deviation from straightforward bidding for player 1.

Now assume player 2 bids in period n . If player 2 deviates from straightforward bidding, and bids on object B in period n , in the following period prices are such that $p_A = 1$ and $p_B \geq 3$. If player 1 bids in period $n+1$, player 1 bids on object A. Player 3 then bids in period $n+2$, but drops out as $p_A = 2$ and $p_B \geq 3$. If player 3 bids in period $n+1$, player 3 bids on object A. In period $n+2$, player 1 bids on object A. In period $n+3$, player 3 drops out as $p_A = 3$ and $p_B \geq 3$. Thus player 2 wins object B at some price $p_B \geq 2$ and receives a negative payoff. This can never be a profitable deviation from straightforward bidding.

Now assume player 3 bids in period n . If player 3 deviates and bids on object B, in the following period, the new bidder will always bid on object A. Thus the new prices facing bidders are $p_A = 2$ and $p_B \geq 3$. If player 2 is never outbid, he will then win object B with a negative payoff. If he is outbid, we know he will then drop out, as prices now exceed his valuations. Thus he can never receive a positive payoff from the deviation. This implies there can never be a profitable deviation from straightforward bidding.

If we now assume $p_A = 0$ and $p_B \geq 2$, a symmetric proof to claim 10 follows that there does not exist a profitable deviation from straightforward bidding. \square

Claim 18. *For any period, n , such that $p_A = 2$ and $p_B = 0$ straightforward bidding is a sub-game perfect equilibrium.*

Proof. Suppose only two players are present in the auction. Then straightforward bidding is a sub-game perfect equilibrium by Proposition 1. We therefore assume in this proof that no player has dropped out of the auction.

First assume player 1 bids in period n and player 3 holds the high bid on object A. Under straightforward bidding, all players bidding in period n bid on object B. If player 1 bids on object B and is given the high bid on object B, we know that in the following period, player 2 bids on object B. In period $n+2$, player 1 bids on object A. Player 3 drops out in period $n+3$ and the auction ends. If player 2 is given the high bid on object B, in period $n+1$ player 1 bids on object A. Player 3 then drops out as $p_A = 3$ and $p_B = 1$. Thus player 1 wins object A at price $p_A = 2$.

Now assume player 1 bids in period n and player 2 holds the high bid on object A. We assume player 3 bids on object B in period n . If player 1 bids on object B in period n , either player 1 or 3 is given the high bid at $p_B = 0$. If player 1 is given the high bid on object B, in period $n+1$, player 3 drops out as $p_A = 2$ and $p_B = 1$. Thus player 1 wins object B at price $p_B = 0$ and receives a payoff $\pi_1 = v_B^1 > v_A^1 - 2$. If player 3 is given the high bid on object B, in period $n+1$, player 1 bids on object A. The new asking price on object A is $p_A = 3$. Since $p_A > v_A^2 > v_A^3$, we may assume no other player outbids player 1. He then wins object A at price $p_A = 2$.

Now assume player 1 deviates and bids on object A in period n . By assumption the other bidder in period n bids on object B. Thus the new prices facing bidder in period $n+1$ are $p_A = 3$ and $p_B = 1$. With $p_A = 3$, we know that neither players 2 or 3 will bid on object A, as $p_A > v_A^2 > v_A^3$. Thus player 1 wins object A at price $p_A = 2$. This is not a profitable deviation.

Now assume player 2 bids in period n and player 3 holds the high bid on object A. We assume player 1 bids on object B in period n . If player 2 follows straightforward bidding, and bids on object B, either player 1 or 2 is given the high bid on object B. If player 2 is given the high bid on object B, in period $n+1$ player 1 bids on object A. Player 3 then drops out in period $n+2$ as $p_A = 3$ and $p_B = 1$. Player 2 wins object B at price $p_B = 0$. If player 1 is given the high bid on object B, in period $n+1$, player 2 bids on object B. Player 1 bids on object A in period $n+2$. In period $n+3$ player 3 drops out as $p_A = 3$ and $p_B = 2$. Now player 2 wins object B at price $p_B = 1$.

Now assume player 2 bids in period n and player 1 holds the high bid on object A. We assume player 3 bids on object B in period n . If player 2 follows straightforward bidding and bids on object B in period n , either players 2 or 3 is given the high bid on object B. If player 2 is given the high bid on object B, player 3 drops out in period $n+1$ as $p_A = 2$ and $p_B = 1$. If player 3 is given the high bid on object B, player 2 bids on object B in period $n+1$. In period $n+2$ player 3 drops out as $p_A = 2$ and $p_B = 2$. Thus player 2 wins object B at price $p_B = 0$ with probability of .5 and at price $p_B = 1$ with probability of .5.

Now assume player 2 deviates from straightforward bidding and bids on object A. We assume the other bidder in period n , either player 1 or 3, bids

on object B. Thus in period $n+1$, the prices facing bidders are $p_A = 3$ and $p_B = 1$. If player 3 bids in period n , player 3 drops out as prices exceed player 3 valuations. If player 1 bids in period $n+1$, player 1 bids on object A. Since we assume player 2 holds object A, it must be the case that player 3 previously held object B. Thus in period $n+2$ player 3 drops out as $p_A = 3$ and $p_B = 2$. In either case player 2 wins object A at price $p_A = 2$. Since $v_B^2 - 1 > v_A^2 - 2$, this can never be a profitable deviation.

Now assume player 3 bids in period n . We assume the other bidder in period n , either player 1 or 2, bids on object B. Thus the price of object B in period $n+1$ is $p_B = 1$. If player 3 deviates and bids on object A, he either wins object A at price $p_A = 2$, or is outbid at some later period in the auction. If player 3 is out bid, we assume he drops out, as he now faces prices $p_A = 3$ and $p_B \geq 1$. Thus the deviation can never yield a positive payoff. Since straightforward bidding guarantees a non-negative payoff, this can not be a profitable deviation. \square

Claim 19. *In any period, n , where $p_A = 1$ and $p_B = 1$, there does not exist a profitable deviation from straightforward bidding.*

Proof. First assume player 1 bids in period n . If player 1 bids straightforwardly, he bids on object A in period n . If player 1 bids on object A in the following period either player 2 or 3 bids. If player 3 bids, he drops out and the auction ends. If player 2 bids, player 2 bids on object B. Player 3 then drops out and the auction ends. It follows that player 1 wins object A at price $p_A = 1$. If player 1 deviates from straightforward bidding he may either win objects A or B, at prices greater than or equal to current prices, or drop out of the auction. Thus the payoff from deviating is $\leq v_A^1 - 1$. It follows that there does not exist a profitable deviation from straightforward bidding.

Now assume player 2 bids in period n . Assume player 1 holds the high bid on object A and player 3 holds the high bid on object B. If player 2 bids on object A, in period $n+1$, player 1 bids on object A. In period $n+2$, player 2 bids on object B. In period $n+3$, player 3 drops out as $p_A = 3$ and $p_B = 2$. Player 2 wins object B at price $p_B = 1$. If player 2 deviates and bids on object B, in period $n+1$ player 3 bids on object A. In period $n+2$, player 1 bids on object A. In period $n+3$, player 3 drops out. Thus player 2 wins object B at price $p_B = 1$. This is not a profitable deviation.

Now assume player 3 holds object A and player 1 holds object B in period n . If player 2 bids on object A in period n , in period $n+1$ player 3 drops out as $p_A = 2$ and $p_B = 1$. Player 2 then wins object A at price $p_A = 1$. If player 2 deviates he either wins object A or B at prices greater than or equal to the current prices, or drops out. Thus his payoff is $\leq v_A^2 - 1$. This can not be a profitable deviation.

Now assume player 3 bids in period n . If player 3 deviates and bids on object B, in the following period either player 1 or 2 bids on object A as $p_A = 1$ and $p_B = 2$. The new price of object A is now $p_A = 2$. If player 3 is never outbid on object B, he then wins object B with a negative payoff. If player 3 is outbid on object B, we assume he drops out, as we know he faces prices $p_a \geq 2$ and $p_b \geq 2$. It follows that deviating never yields a positive payoff. Recall straightforward bidding always yields a non-negative payoff, thus this is not a profitable deviation. \square

Claim 20. *For any period, n , such that $p_A = 1$ and $p_B = 0$ straightforward bidding is a sub-game perfect equilibrium.*

Proof. Suppose only two players are present in the auction. Then straightforward bidding is a sub-game perfect equilibrium by Proposition 1. We therefore assume in this proof that no player has dropped out of the auction.

First assume player 1 bids in period n (as well as one other player). Under straightforward bidding player 1 bids on object A. Under the one-deviation principle we can assume all players bid straightforwardly for the remainder of the auction. It follows by assumption that the other player bidding in period n bids on object B. If player 1 bids on object A, in the following period it will be the case that $p_A = 2$ and $p_B = 1$. If player 3 bids in period $n+1$, he then drops out and the auction ends. If player 2 bids in period $n+1$, he will then bid on object B. In period $n+2$, player 3 will then drop out and the auction will end. Thus player 1 wins object A at price $p_A = 1$ under straightforward bidding. If player 1 deviates from this strategy, he may win either objects A or B at prices greater than or equal to the current prices, or drop out. Since $v_A^1 - 1 > v_B^1 > 0$, we know there can not be a profitable deviation from straightforward bidding for player 1.

Now assume player 2 bids in period n . First assume that player 3 is currently the high bidder on object A, and thus player 1 bids in period n as well. By assumption, player 1 will bid on object A in period n . If player 2 follows straightforward bidding, he will bid on object B. In the following period player 3 will bid and face prices $p_A = 2$ and $p_B = 1$. By assumption, player 3 will drop out, and player 2 will win object B at price 0. If player 2 deviates from this strategy, he may either he may win either objects A or B at prices greater than or equal to the current prices, or drop out. Since $v_B^2 > v_A^2 - 1$ and $v_B^2 > 0$, there does not exist a profitable deviation from straightforward bidding for player 2 under these assumptions.

Now assume that player 1 is the current high bidder on object A. We assume in period n player 3 will bid on object B. If player 2 bids on object B a winner is randomly chosen between players 2 and 3. If player 2 is given the high bid on object B at $p_B = 0$, in the following period, player 3 bids on object A. In period $n+2$ player 1 bids on object A. In period $n+3$, player 3 drops out and the auction ends. Thus player 2 wins object B at price $p_B = 0$. If player 3 is given the high bid on object B, in the following period player 2 bids again. Player 2 will bid then bid on object A in period $n+1$. In period $n+2$, player 1 will bid on object A. In period $n+3$, player 2 will then bid on object B at price $p_B = 1$. In period $n+4$, player 3 drops out and the auction ends. Thus player 2 will win object B under straightforward bidding. He pays $p_B = 1$ with a .5 probability, and $p_B = 0$ with a probability of .5.

Now assume player 2 deviates and bids on object A in period n . We know player 3 bids on object B in period n as well. In period $n+1$, player 1 bids on object A. Then in period $n+2$, player 2 bids on object B. Player 3 then drops out of the auction. Thus player 2 wins object B at price $p_B = 1$. Thus this is not a profitable deviation from straightforward bidding.

Now assume player 3 bids in period n . If all players follows straightforward bidding player 3 drops out and receives a payoff of 0. Does there exist a profitable deviation from straightforward bidding for player 3? Assume there exists a profitable deviation. This implies player 3 wins either object A or B at a price of 0. If player 3 wins one of the objects, either player 1 or 2 does not. Without loss of generality assume 1 does not win any object. In order for player 1 to drop out of the auction it must be the case that $p_A > v_A^1$ and $p_B > v_B^1$. However, if player 3 wins either object at price

0, the final price of bidding must have been 1 for the object. Yet $v_A^1 > 2$ and $v_B^1 > 1$. This contradicts our assumption, thus there does not exist a profitable deviation. \square

Claim 21. *For any period n , such that $p_A = 0$ and $p_B = 1$, straightforward bidding is a sub-game perfect equilibrium.*

Proof. Suppose only two players are present in the auction. Then straightforward bidding is a sub-game perfect equilibrium by Proposition 1. We therefore assume in this proof that no player has dropped out of the auction.

First assume player 1 bids in period n . Since we assume all other players straightforward bidding for the remainder of the auction, we know that in period n , the other player bidding will bid on object A. If player 1 follows straightforward bidding he bids on object A. If player 1 bids on object A, there is a .5 chance he is given the high bid on object A. In period $n+1$ we know prices are $p_A = 1$ and $p_B = 1$. If player 1 is given the high bid on object A in period n , by assumption, we know the bidder in period $n+1$ bids on object A. Player 1 now faces prices $p_A = 2$ and $p_B = 1$. Player 1 bids on object A. The new price of object A is $p_A = 3$. Since $3 > v_A^2 > v_A^3$, we know player 1 is never outbid. He then wins object A at price $p_A = 2$. If player 1 is not given the high bid on object A in period n , by assumption, he bids on object A in period $n+1$. The new bidder faces prices $p_A = 2$ and $p_B = 1$. If player 3 bids in period $n+2$, he drops out and the auction ends. If player 2 bids in period $n+2$, he bids on object A. Player 3 then drops out in period $n+3$. Thus player 1 wins object A at price $p_A = 1$.

Thus, if player 1 deviates and bids on object B, he will be given the high bid on object B. The new price of object B will become $p_B = 2$, and neither player 2 or 3 will ever bid on object B again as $2 > v_B^2 > v_B^3$. Thus player 1 will win object B at price $p_B = 1$. Since $v_A^1 - 1 > v_A^1 - 2 > v_B^1 - 1$, this is not a profitable deviation.

Now assume player 2 bids in period n . First assume player 1 holds the high bid on object B. If player 2 follows straightforward bidding, he bids on object A in period n . By assumption, player 3 bids on object A in period n . Either player 2 or 3 is given the high bid on object A. If player 2 is given the high bid on object A, in period n , player 3 bids on object A in period

$n+1$. Player 2 then bids on object B in period $n+2$. In period $n+3$, player 1 bids on object A. In period $n+4$, player 3 drops out as prices are $p_A = 3$ and $p_B = 2$. Player 2 wins object B at price $p_b = 1$. If player 3 is given the high bid on object A in period n , in period $n+1$, player 2 bids on object A. In period $n+2$, player 3 drops out as $p_A = 2$ and $p_B = 1$. Player 2 then wins object A at prices $p_A = 1$.

Now assume player 3 holds the high bid on object B. We know player 1 bids on object A. If player 2 bids on object A, in the following period prices are $p_A = 1$ and $p_B = 1$. By assumption, in period $n+1$, either player 1 or 2 bids on object A. In period $n+2$, the new prices are $p_A = 2$ and $p_B = 1$. If player 2 bids, he bids on object B, and player 3 drops out. If player 1 bids, he bids on object A. In period $n+3$, player 2 bids on object B, and in period $n+3$, player 3 drops out. Thus player 2 wins object B at $p_B = 1$.

If player 2 deviates and bids on object B in period n , we know that in the following period prices are $p_A = 1$ and $p_B = 2$. Player 3 bids on object A in period $n+1$. The new price of object A is $p_A = 2$. It follows that either player 2 wins object B at price $p_B = 1$, or is out bid. If outbid, player 2 wins either object A or B, or drops out. If player 2 wins object A or B, it must be at prices $p_A \geq 2$ or $p_B \geq 1$ as prices never fall. Thus his payoff must be $\leq \max\{v_B^2 - 1, v_A^2 - 2\} = v_B^2 - 1$. Recall the payoff of bidding straightforwardly is $\pi_2 \geq v_B^2 - 1$. Thus there is no profitable deviation from straightforward bidding.

Now assume player 3 bids in period n . Recall we assume the other player bidding in period n bids on object A. Assume player 3 deviates from straightforward bidding and bids on object B. The new prices facing either player 1 or 2 in period $n+1$ are then $p_A = 1$ and $p_B = 2$. We know that either player bids on object A. Thus in period $n+2$, the new prices facing bidders are $p_A = 2$ and $p_B = 2$. If player 3 is outbid in any later period, we then assume player 3 drops out, as prices exceed his valuation. If player 3 is never outbid, he wins object B, at price $p_B = 1 > v_B^3$. Thus he never receives a positive payoff from the deviation. Recall straightforward bidding guarantees a non-negative payoff. Thus this can not be a profitable deviation. \square

Claim 22. *No player has an incentive to deviate in period 0.*

Proof. If all players follow the set strategy starting in period 0, the expected

payoffs of bidding are $(\pi_1, \pi_2, \pi_3) = (v_B^1, v_A^2 - .5, 0) = (2.2, 1.8, 0)$.

We have already shown that in all possible sub-games following the initial period, straightforward bidding is a sub-game perfect equilibrium.

Does there exist a profitable deviation for player 1, assuming all other bidders play the set strategy? Assume player 1 bids on object A in the initial period. We know all players bid straightforwardly for all remaining periods. We know there is a $\frac{1}{3}$ chance player 1 is given the high bid on object A in the initial period. If player 1 is given the high bid on object A, in period 1, both player 2 and 3 bid on object B. There is a $\frac{1}{2}$ chance player 2 is given the high bid on object B in period 1. If player 2 is given the high bid on object B, player 3 bids on object A in period 2. In period 3, player 1 bids on object A. In period 4, player 3 drops out and the auction ends. Thus player 1 wins object A at price $p_A = 2$. If player 3 is given the high bid on object B in period 1, player 2 bids on object A in period 2. Player 1 bids on object A in period 3. Player 2 bids on object B in period 4. In period 5, player 3 drops out as $p_A = 3$ and $p_B = 2$. Player 1 wins object A at $p_A = 2$. Thus, there is a $\frac{1}{3}$ chance player 1 wins object A at $p_A = 2$.

If player 1 is not given the high bid on object A, we still know the auction will never end unless $p_A \geq 2$. Thus the winner of object A, must pay some price $p_A \geq 1$. Under such a case, it must be that $\pi_1 \leq \max\{v_A^1 - 1, v_B^1, 0\} = v_A^1 - 1$. Thus $\pi_1 \leq 1.5(\frac{1}{3}) + 2.5(\frac{2}{3}) = 2\frac{1}{6}$. Since $2\frac{1}{6} < 2.2$ player 1 has no incentive to deviate from the set strategy.

Does there exist a profitable deviation for player 2, assuming all other bidders play the set strategy. If player 2 deviates in the initial period and bids on object B, we know that in period 1, the new prices are $p_A = 1$ and $p_B = 1$, as player 3 still bids on object B in the initial period. If player 2 is given the high bid on object B in the initial period, he may either win object B at price $p_B = 0$, or be outbid, and win either object A or B, at prices greater than or equal to current prices, or drop out. It follows that $\pi_2 \leq \max\{v_B^2, v_A^2 - 1\} = v_B^2$. If player 1 is given the high bid on object B in the initial period, player 2 bids on object A. In the following period, player 3 drops out, as $p_A = 2$ and $p_B = 1$. Thus player 2 wins object A at price $p_A = 1$. It follows that $\pi_2 \leq \max\{v_B^2, v_A^2 - 1\} = v_B^2$ under any possible deviation in the initial period. Since $v_B^2 = 1.6 < v_A^2 - .5 = 1.8$, player 2 does not have an incentive to deviate in the initial period.

Now assume player 2 bids in period 1, and bids on object B. If player 2

follows the set strategy his payoff is $\pi_2 = v_A^2 - 1$. If player 2 deviates and bids on object B, he may either win object A or B, at prices greater than or equal to current prices, or drop out. Thus the payoff from deviating is $\pi_2 \leq \max\{v_A^2 - 1, v_B^2 - 1\} = v_A^2 - 1$. Thus there does not exist a profitable deviation for player 2 in period 1.

Does there exist a profitable deviation for player 3 in the initial period. Assume player 3 deviates in the initial period and bids on object B. If player 3 is given the high bid on object B, in the following period, player 1 bids on object A. In period 2, player 2 bids on object B, and player 3 drops out of the auction as $p_A = 2$ and $p_B = 2$. If player 1 is given the high bid on object A in the initial period, player 3 bids on object A in period 1. In period 2, player 2 bids on object B. In period 3, player 1 bids on object A. In period 4, player 3 drops out as $p_A = 3$ and $p_B = 2$. Thus there does not exist a profitable deviation in the initial period.

Does there exist a profitable deviation for player 3 in periods 1 or 2. If player 3 bids in period 1 or 2, he drops out under the set strategy. Can player 3 strictly improve his payoff by deviating? Assume there exist a profitable deviation for player 3. We know that once player 3 deviates, all players bid straightforwardly. For there to exist a profitable deviation player 3 needs to win object A or B at some price $p_A > v_A^3$ or $p_B > v_B^3$. Yet in period 1 or 2, it must be the case that $p_B > v_B^3$, thus player 3 needs to win object A. However, this implies either player 1 or 2 drops out. However, players 1 or 2 drop out only if $p_A > v_A^2$. Thus the final price facing bidders on object A must be such that $p_A > v_A^2 = 2.2$. Thus $p_A \geq 3$, as prices only increase in integer multiples. This implies player 3 wins object A at some price $p_A \geq 2$. However, this yields a negative payoff. Thus there can not exist a profitable deviation from the set strategy for player 3. \square

This completes the proof of Proposition 6. \square

5 Conclusion

The results of the paper provide interesting findings on multiplicity of sub-game perfect equilibrium allocations. We find that in the 2 player 2 object case, there exist a multiplicity of sub-game perfect equilibrium bidding strategies. However, all of these strategies result in the same final allocation. A variety of equilibrium selections can be justified. If bidders are motivated

by harming their opponents, high price equilibria appear plausible. If bidders' primary motivation is to end the auction early, then low price equilibria seem plausible. The intermediate equilibrium in which bidders bid straightforwardly appears plausible when there is incomplete information. Indeed, we show that straightforward bidding is the only sub-game perfect equilibrium that passes the informational robustness test posed by the concept of ex-post equilibrium.

The 3 player 2 object case is a further interesting example. We have seen that straightforward bidding need not be a sub-game perfect equilibrium in this case. Moreover, we find that the introduction of a third bidder into the auction potentially causes a multiplicity of not only equilibrium prices but also equilibrium allocations. While the new third player may not actually win either of the two objects, under certain conditions, this player can still be used to ensure an allocation opposite of the 2 player 2 object case.

These findings lead to further questions about the simultaneous ascending auction. Under which general conditions is straightforward bidding a sub-game perfect equilibrium? Are there always strategies "close" to straightforward bidding that will be sub-game perfect equilibria? Is there a general sense in which straightforward bidding is informationally robust? It may also be worthwhile to formalize the equilibrium selection arguments that we did not analyze formally in this paper, but that we mentioned informally at the beginning of section 3.3. The analysis of this paper could be used as a starting point towards answering these questions.

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