INVERSIONS WITHIN RESTRICTED FILLINGS OF YOUNG TABLEAUX

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Abstract. In this paper we study inversions within restricted fillings of Young tableaux which describe geometric properties of Hessenberg varieties. We define the dimension of a filling in terms of certain inversions which occur. Most of the results in this paper answer or give partial answers to conjectures posed in [T]. We give an upper bound on the dimension of any allowed filling of a multitableau and show that this upper bound is achieved. In a special case which is an example of interest in numerical analysis, we give a smaller upper bound and show that it is in fact achieved. We also give a lower bound on the number of inversions given an extra condition and conjecture that when this condition holds it gives the maximum possible dimension. We describe all the zero-dimensional fillings and give the number of connected components of a Hessenberg function in terms of permutations of a multiset which have a particular descent set.

1. Introduction

In this paper we study inversions within restricted fillings of Young tableaux. These restricted fillings are of interest because they describe geometric properties of Hessenberg varieties, which have been studied in [DPS], [DS], [F], and [T]. We give answers and partial answers to some conjectures posed in [T]. In particular we find the number of components of Hessenberg varieties, give an upper bound on the dimension of a Hessenberg variety, and give an exact expression for the dimension in some special cases. The proofs given are all combinatorial.

The Young tableaux we use are both left-aligned and top-aligned. A multitableau λ_X is a collection of tableaux which are arranged vertically. We will refer to a step function $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots n\}$, so it is nondecreasing with $h(i) \geq i$ for all i as a Hessenberg function. The multitableau we study are filled with the numbers $\{1, 2, ..., n\}$, without repetition, with the restriction that $\begin{array}{|c|c|}\hline \text{j} & \text{k} \\ \hline\end{array}$ is allowed in the tableau only if $j \leq h(k)$.

For a linear operator X and a Hessenberg function h, the Hessenberg variety $\mathcal{H}(X, h)$ is a subvariety of the full flag variety and is defined in [T] by

$$\mathcal{H}(X, h) = \{\text{Flags } V_1 \subseteq \cdots \subseteq V_n : XV_i \subseteq V_{h(i)} \text{ for each } i\}.$$

A multitableau λ_X with $d_{i,j}$ boxes in the j^{th} row of the i^{th} tableau corresponds to a linear operator X if the Jordan blocks of X are of sizes $d_{i,1}, d_{i,2}, ..., d_{i,k_i}$ corresponding to each

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eigenvalue c_i. The way in which the restricted fillings of Young tableau that we study relate to Hessenberg varieties is also described in [T] as follows.

Theorem 1. (Tymoczko [T]). Fix any X and h. The Hessenberg variety $\mathcal{H}(X,h)$ is paved by affines by the intersection of the Hessenberg variety with a particular Bruhat decomposition of the flag variety. The nonempty cells are naturally bijective with the fillings of λ_X which contain the configuration $\boxed{j}\boxed{k}$ only if $j \leq h(k)$. The dimension of a nonempty cell is the sum of:

(1) the number of pairs i,j in the corresponding filling of λ_X such that
  - i and j are in the same tableau,
  - the box filled with i is to the left of or directly below the box filled with j,
  - $j < i$, and
  - if k fills the box immediately to the right of j *then* $i \leq h(k)$.

(2) the number of pairs i,j in λ_X such that
  - i and j are in different tableaux,
  - the box filled with i is below j, and
  - $j < i \leq h(j)$.

We will use the previous expression to describe the dimension of a filling of a multitableau λ_X.

We will use the following definition, which relates to the Hessenberg function, in Sections 5 and 3.

Definition 2. *Each Hessenberg function* $h : \{1, 2, ..., n\} \rightarrow \{1, 2, ..., n\}$ *partitions* $\{1, 2, ..., n\}$ *into blocks according to the rule that* $i + 1$ *is in the same block as* i *if and only if* $h(i) > i$. *We call this the* h*-partition, and say that it has* h*-blocks of sizes* $n_1, n_2, ..., n_k$ *if the first block contains the first* n_1 *numbers, the second block contains the next* n_2 *numbers, and so on.*

In Section 3 we study the zero-dimensional fillings of a multitableau. The number of zero-dimensional fillings is the number of components of the corresponding Hessenberg variety. We describe all the possible zero-dimensional fillings in terms of permutations of multisets. In particular Theorem 15 proves that if the Hessenberg function has h-blocks of sizes $n_1, n_2, ..., n_k$, the number of zero-dimensional fillings of a multitableau λ_X, with s_i boxes in the i^{th} tableau, is equal to the number of permutations π of the multiset $\{1^{n_1}, 2^{n_2}, ..., k^{n_k}\}$ with descent set $D(\pi) \subseteq \{s_1, s_1+s_2, s_1+s_2+s_3, ..., s_1+s_2+...+s_{l-1}\}$. We also prove a kind of duality with an unexplained geometric interpretation in Proposition 17. In particular, the number of zero-dimensional fillings of a multitableau with s_i boxes in the i^{th} tableau when the Hessenberg function has h-blocks of sizes $n_1, n_2, ..., n_k$ is the same as the number of zero-dimensional fillings of a multitableau with n_i boxes in the i^{th} tableau when the Hessenberg function has h-blocks of sizes $s_1, s_2, ..., s_l$.

The following definitions will allow us to rephrase this expression for the dimension, which is done in Section 2 in terms of entries in a multitableau which have certain properties which we call P_i and D_i. The alternate expressions, given in Corollaries 10 and 11, are used throughout the paper for computing the dimensions of particular fillings and upper bounds of fillings.

Definition 3. *If an entry j has the property that $j < i \leq h(j)$ for some other entry i, then we say that j is P_i.*

Definition 4. *We say that an entry j is D_i if $j < i$ and either j fills the rightmost box of a row, or else if k is in the box immediately to the right of j, then $i \leq h(k)$.*

In Theorem 18 from Section 4, we will use our expressions for the dimension in terms of P_i and D_i entries to prove a sharp upper bound for the dimension of any filling. If the j^{th} row of the i^{th} tableau has length $d_{i,j}$, then we find that the dimension of a filling is at most

$$\sum_{i=1}^{n} h(i) - i + \sum_{i=1}^{l} \sum_{j=2}^{k_i} (j-1) d_{i,j}.$$

We then describe a special case when this upper bound is actually achieved in Section 5.

For example, when $n = 7$ and the Hessenberg function is $h(1) = 3$, $h(2) = 3$, $h(3) = 4$, $h(4) = 4$, $h(5) = 5$, $h(6) = 7$, $h(7) = 7$, the h-partition consists of the blocks $\{1, 2, 3, 4\}$ $\{5\}$ $\{6, 7\}$. These h-blocks have sizes $4, 1$, and 2.

In Section 3, we prove that if s_i is the number of boxes in the i^{th} tableau, if λ_X has l tableaux, and for all m and t with $0 \leq m \leq l$ and $1 \leq t \leq k_m$

$$h(\sum_{i=0}^{m} s_i + \sum_{j=1}^{t} d_{i+1,j}) = \sum_{i=0}^{m} s_i + \sum_{j=1}^{t} d_{i+1,j}$$

then the upper bound is actually achieved. Thus the largest dimension of a any filling in this case is

$$\sum_{i=1}^{n} h(i) - i + \sum_{i=1}^{l} \sum_{j=2}^{k_i} (j-1) d_{i,j}.$$

We also prove a smaller upper bound when the Hessenberg function is $h(i) = i + 1$. In this case, we assume that the bottom tableau of λ_X has k_l rows, and this is the largest number of rows of all the tableaux. Then we show that the dimension of any filling is at most

$$n - k_l + \sum_{i=1}^{l} \sum_{j=2}^{k_l} (j-1) d_{i,j}$$

and that this upper bound is achieved with a certain filling.

We also study the dimension of one other particular filling. This filling, when allowed, corresponds to the intersection of the big cell in the full flag variety with the Hessenberg variety. We find the dimension of the filling in this case and conjecture that it is in fact the largest possible filling when allowed.

2. AN ALTERNATE EXPRESSION OF THE DIMENSION

In this section, we describe properties about these entries which are P_i and D_i. For example for each fixed i, the entries which are P_i and D_i alternate in each row. We will also express the dimension of a filling in terms of these entries which are P_i and D_i, which will be useful throughout this paper.

Lemma 5. ***Slide Right Lemma.*** *Fix i. If $j_1, j_2, ..., j_k$ appear in a row as below with $i \neq j_k$ and $j_k < i \leq h(j_1)$, then at least one of $j_1, j_2, ..., j_k$ is P_i.*

$$\begin{array}{|c|c|c|c|}\hline j_1 & j_2 & \cdots & j_k \\ \hline\end{array}$$

Proof. We know that $i \leq h(j_1)$ so if $i > j_1$ then j_1 is P_i. Otherwise $i \leq j_1 \leq h(j_2)$ since all tableaux we are considering have $\begin{array}{|c|c|}\hline j_1 & j_2 \\ \hline\end{array}$ occurring only if $j_1 \leq h(j_2)$, so slide to the right and decide if j_2 is P_i. Continuing this "sliding right" procedure, either one of $j_1, j_2, ..., j_{k-1}$ is P_i or else $i \leq j_{k-1} \leq h(j_k)$ and since $i > j_k$ we can see that j_k is P_i. ☐

Corollary 6. *Fix i. If $j_0, j_1, j_2, ..., j_k$ appear in a row, with $i \neq j_0$ and $i \neq j_k$, where j_0 and j_k are D_i, then one of $j_1, j_2, ..., j_k$ is P_i.*

Proof. Since j_0 and j_k are D_i, we know that $j_k < i \neq h(j_1)$. Our claim then follows from the Slide Right Lemma. ☐

Lemma 7. ***Slide Left Lemma.*** *Fix i. If $j_1, j_2, ..., j_k$ are in some row as below, $i > j_1$, and if either*

- *j_k is at the end of the row or*
- *the box directly to the right of j_k is filled with j_{k+1} and $i \leq h(j_{k+1})$,*

then at least one of $j_1, j_2, ..., j_k$ is D_i.

$$\begin{array}{|c|c|c|c|c|}\hline j_1 & j_2 & \cdots & j_k & j_{k+1} \\ \hline\end{array}$$

Proof. If $i > j_k$ then since either j_k is at the end of a row or else $i \leq h(j_{k+1})$, we see that j_k is D_i. Otherwise $i \leq j_k \leq h(j_k)$ so slide to the left and decide if j_{k-1} is D_i. Continuing this "sliding left" procedure, either one of $j_2, ..., j_k$ is D_i or else $i \leq j_2 \leq h(j_2)$. Then because $i > j_1$ we see that j_1 is D_i. ☐

Corollary 8. *If $j_1, j_2, ..., j_k$ appear in a row where j_1 is P_i and either*

- *j_k is at the end of the row or*
- *j_{k+1} is directly to the right of j_k and j_{k+1} is P_i*

for some i, then one of $j_1, j_2, ..., j_k$ is D_i.

Proof. Since j_1 is P_i we have $i > j_1$ and either

- j_k is at the end of the row or
- j_{k+1} is directly to the right of j_k and is P_i, meaning $i \leq h(j_{k+1})$.

The claim then follows from the Slide Left Lemma. ☐

By Corollaries 6 and 8 the entries which are D_i and P_i alternate. For example, if we have two entries which are P_i, as in Figure 1, there will be an entry which is D_i in the specified region. The similar property for D_i entries is also shown.

Hence if there are two P_i entries with no other P_i entries between them, there is exactly one D_i entry in the specified region, and likewise for D_i.

Observe the following:

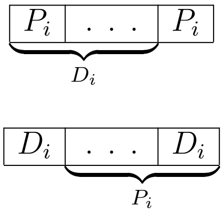


FIGURE 1. P_i and D_i entries

- After the last P_i entry, there is exactly one D_i entry, which is at the P_i entry or strictly after it, by the Slide Left Lemma or Corollary 8.
- Before the first P_i entry, there are either zero or one D_i entries by Corollary 6.
- There are no D_i entries to the right of i and before the first P_i entry which is in the same row as and to the right of i, by the Slide Right Lemma, since if k is immediately to the right of i, we have $i \leq h(k)$.

Definition 9. *We will say that an entry j_k which is D_i has no left P_i if*

- *i is directly below or to the left of j_k, j_1 is in the same row as j_k and directly above i, as in Figure 2, and*
- *j_1 and j_k are not P_i and for any entry k which is between j_1 and j_k is not P_i.*

Note that if i is directly below j_k, $k = 1$.

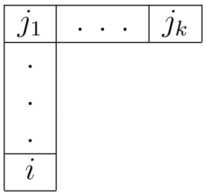


FIGURE 2. j_k is D_i with no left P_i if j_1 and j_k are not P_i, and there is no P_i entry between j_1 and j_k.

Note that it is possible for some entry k to the left of j_1 to be P_i, but we are not considering that.

Corollary 10. *The dimension of a nonempty cell is the sum of:*

1. *the number of pairs i,j in the filling of λ_X such that*
   - *i and j are in the same tableau,*
   - *the box filled with i is to the left of or directly below the box filled with j,*
   - *j is D_i; and*
2. *the number of pairs i,j in λ_X such that*
   - *i and j are in different tableaux,*
   - *the box filled with i is below j, and*
   - *j is P_i.*

Corollary 11. *The dimension of a filling is equal to the sum of*

(1) *the number of pairs* i,j *in the filling of* λ_X *such that*
   - i *and* j *are in the same tableau,*
   - *the box filled with* i *is to the left of or directly below the box filled with* j*,*
   - j *is* P_i*.*

(2) *the number of pairs* i,j *in the filling of* λ_X *such that*
   - i *and* j *are in the same tableau,*
   - *the box filled with* i *is to the left of or directly below the box filled with* j*,*
   - j *is* D_i *with no left* P_i*.*

(3) *the number of pairs* i,j *in* λ_X *such that*
   - i *and* j *are in different tableaux,*
   - *the box filled with* i *is below* j*, and*
   - j *is* P_i*.*

Proof. We first consider from Theorem 1, the number of pairs i,j in λ_X such that

- i and j are in the same tableau,
- the box filled with i is to the left of or directly below the box filled with j
- $j < i$, and
- if k is immediately to the right of j, then $i \leq h(k)$.

By definition, the last two conditions are equivalent to j being D_i.

The second quantity is the number of pairs i,j in λ_X such that

- i and j are in different tableaux,
- the box filled with i is below j, and
- $j < i \leq h(j)$.

The last condition is equivalent to j being P_i.

Thus we get the expression from Corollary 10 for the dimension of a filling.

Now notice that since P_i and D_i entries alternate with possibly an initial D_i entry, the first condition from Corollary 10 is equivalent to the number of pairs i,j in the filling of λ_X such that

- i and j are in the same tableau,
- the box filled with i is to the left of or directly below the box filled with j, and
- either j is D_i with no left P_i, or else j is P_i.

From this, we get the expression for the dimension given in Corollary 11. ☐

3. Zero-Dimensional Fillings

The number of zero-dimensional fillings is of interest because it is the number of connected components of the corresponding Hessenberg variety. In this section, we prove that there is a bijection between the number of zero-dimensional fillings and permutations of a multiset with a fixed descent set. Recall from Definition 2, the Hessenberg function partitions $\{1, 2, ..., n\}$ into h-blocks according to the rule that $i + 1$ is in the same h-block as i if and only if $h(i) > i$. In this section we also prove some facts about how the number of zero-dimensional fillings relates to the sizes of h-blocks and give some closed formulas for the number of zero-dimensional fillings in special cases.

Recall that for a multitableau λ_X which has l tableau with s_i boxes in the i^{th} tableau, the base filling has the first s_l numbers in the bottom most tableau, the next s_{l-1} numbers in the next tableau up, the next s_{l-2} numbers in the next tableau, and so on. Each tableau is then filled according to the rule that the smallest number is in the bottom most entry of the left most column, and if r fills a box not in the top-most row of a tableau, $r + 1$ is directly above it. If r is in the top-most row of a tableau, $r + 1$ is in the bottom-most entry of the column to the left of the one containing r. An example of this filling is seen in Figure 3.

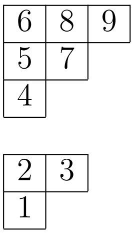


FIGURE 3. An example of the base filling.

Definition 12. *Suppose the i^{th} tableau of λ_X has k_i boxes. If the numbers $n_{i_1} < n_{i_2} < ... < n_{i_{k_i}}$ fill the i^{th} tableau according to the rule that n_{i_1} is in the bottom of the leftmost column, and if some n_{i_r} is in an entry not in the top-most row, $n_{i_{r+1}}$ is directly above it. If n_{i_r} is in the top-most row, then $n_{i_{r+1}}$ is in the bottom most entry of the column to the right of the one containing n_{i_r}. We call this filling a pseudo-base filling, as it is quite similar to the base filling.*

Note that in general, a pseudo-base filling on a multitableau λ_X depends on which entries are in each tableau, so it is not unique.

Lemma 13. *If a filling of λ_X is zero-dimensional, then it must be a pseudo-base filling.*

Proof. Suppose that some zero-dimensional filling is not a pseudo-base filling. Then for some i, the i^{th} tableau is not filled in this way. Then there will be some box filled with k to the left of or directly below a box filled with j_1 where $k > j_1$, and $j_2, j_3, ..., j_l$ are to the left of j_1 in that order, with j_l and the end of the row. Then the box filled with k is directly below or to the left of all of these entries. By the Slide Left Lemma, one of $j_1, j_2, ..., j_l$ is D_k, and the box filled with k is directly below or to the left of whichever entry is D_k, so this pair will contribute to the dimension. Hence the dimension must be at least 1. □

Corollary 14. *For nilpotent operators N, λ_N has only one zero-dimensional filling, the base filling, and $\mathcal{H}(N, h)$ is thus connected.*

Proof. Since N is nilpotent λ_N consists of only one tableau, so this one tableau λ_N contains the entries $1, 2, ..., n$. Any zero-dimensional filling must be a pseudo-base filling, and since there is only one tableau, there is only one possible filling. □

Given any pseudo-base filling of a multitableau λ_X, any pair i, k in the same tableau with i in a box to the left of or directly below that filled with k has $k > i$ by definition. Thus any pair i, k that contributes to the dimension must have i and k in different tableaux.

It follows that the dimension of a pseudo-base filling is the number of pairs i, k such that i and k are in different tableaux, the box filled with i is below the box filled with k, and $k < i \leq h(k)$.

We express a permutation of the multiset $\{1^{n_1}, 2^{n_2}, ..., k^{n_k}\}$ as $\pi = \pi_1\pi_2...\pi_s$ where each $\pi_k \in \{1^{n_1}, 2^{n_2}, ..., k^{n_k}\}$. Then the descent set of a permutation is $D(\pi) = \{k : \pi_k > \pi_{k+1}\}$. For example if the multiset is $\{1^2, 2\}$, then the permutations are $\pi_1 = 112$, $\pi_2 = 121$, and $\pi_3 = 211$.

Theorem 15. *For a Hessenberg function with a h-blocks of sizes $n_1, n_2, ..., n_k$ and a multitableau λ_X where the i^{th} tableau has s_i entries in it, the number of zero-dimensional fillings is equal to the number of permutations π of the multiset $\{1^{n_1}, 2^{n_2}, ..., k^{n_k}\}$ whose descent set $D(\pi) \subseteq \{s_1, s_1 + s_2, s_1 + s_2 + s_3, ..., s_1 + s_2 + ... + s_{l-1}\}$.*

Proof. We construct a bijection which sends a permutation π of the multiset $\{1^{n_1}, 2^{n_2}, ..., k^{n_k}\}$ whose descent set $D(\pi) \subseteq \{s_1, s_1+s_2, s_1+s_2+s_3, ..., s_1+s_2+...+s_{l-1}\}$ to a zero-dimensional filling as follows: $\pi = \pi_1\pi_2...\pi_s$ where $s = s_1 + s_2 + ... + s_l$. Look at the first s_1 numbers: $\pi_1, \pi_2, ..., \pi_{s_1}$. There will be $c_{1,1}$ 1's, $c_{1,2}$ 2's, ..., $c_{1,k}$ k's. Place the largest $c_{1,i}$ numbers of the i^{th} h-block in the top-most tableau for each i, with the pseudo-base filling on these numbers. Since $c_{1,1} + c_{1,2} + ... + c_{1,k} = s_1$, these numbers will fill the top tableau. Next, look at the next s_2 numbers $\pi_{s_1+1}, \pi_{s_1+2}, ..., \pi_{s_1+s_2}$. There will be $c_{2,1}$ 1's, $c_{2,2}$ 2's, ..., $c_{2,k}$ k's. Put the largest unused $c_{2,i}$ numbers of the i^{th} h-block in the second tableau for each i, again with the pseudo-base filling on these numbers. Since $c_{2,1} + c_{2,2} + ... + c_{2,k} = s_2$, these numbers will fill this tableau. Continuing this process will fill the multitableau with $1, ..., n$ since there were n_i i's in the multiset. If i and j are in the same h-block and $i > j$, the filling places i either in the same tableau as or above j. Moreover, each tableau is filled with the pseudo-base filling, so this filling of λ_X is guaranteed to be an allowed zero-dimensional filling.

Given two different permutations of the multiset π and π' whose descent sets are contained in $\{s_1, s_1+s_2, ..., s_1+s_2+...+s_{k-1}\}$, if $\pi \neq \pi'$ then the first s_1 numbers of π, or the s_2 after those, or the next s_3, or so on, will have a different number of i's than the same group of s_j numbers in π', for some i. Thus, the filling that π is sent to has a different number of entries from the i^{th} h-block in the j^{th} tableau than the filling corresponding to π'. This means the map is injective.

The map is also surjective. Given an allowed filling with $c_{j,i}$ numbers from the i^{th} h-block in the j^{th} tableau, in order for the filling to be zero-dimensional, if i and $i+1$ are in the same h-block, i is either in the same tableau as $i+1$ or in a tableau below it. So if we let π be the permutation which has $c_{1,1}$ 1's, $c_{1,2}$ 2's, ..., $c_{1,k}$ k's in the first group of s_1 numbers, then $c_{2,1}$ 1's, $c_{2,2}$ 2's, ..., $c_{2,k}$ k's in the next group of s_2 numbers, and so on, then π will correspond to this filling. For example, if we have the filling seen in Figure 3, where the h-blocks are $\{1, 2, 3, 4\}$, $\{5\}$, and $\{6, 7\}$, since the top tableau contains two entries, 3 and 4, from the first h-block, one entry from the second h-block, and one entry from the third h-block, π must start with 1123. Then the next tableau down has two entries from the first h-block and one entry from the third h-block, so the next three numbers of π are 113. Thus $\pi = 1123113$. ☐

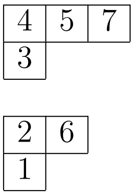


FIGURE 4. The filling corresponding to $\pi = 1123113$, with h-blocks $\{1, 2, 3, 4\}$, $\{5\}$, and $\{6, 7\}$.

Recall that the total number of permutations of a multiset $\{1^{n_1}, 2^{n_2}, ..., k^{n_k}\}$ where $n = n_1 + n_2 + ... + n_k$ is $\binom{n}{n_1, n_2, ..., n_k}$ ([S], Section 1.3). The following corollary gives special cases of the previous theorem.

Corollary 16. *The number of zero-dimensional fillings have a nice formula in the following cases:*

1. *if $s_1 = n$, this corresponds to the linear operator X for the Hessenberg variety being nilpotent, meaning there is only one tableau. Then there is only one zero-dimensional filling, so the Hessenberg variety is connected.*
2. *if $n_1 = n$, this corresponds to the Hessenberg function satisfying $h(i) > i$ for all i, which means there is only one h-block. Then there is only one zero-dimensional filling, so the Hessenberg variety is connected.*
3. *if $s_i = 1$ for all i, this corresponds to the linear operator X for the Hessenberg variety having n distinct eigenvalues, so the multitableau consists of n tableau, each containing only one box. Then there are $\binom{n}{n_1, n_2, ..., n_k}$ zero-dimensional fillings.*
4. *if $n_i = 1$ for all i, this means that $h(i) = i$ for all i. Then there are $\binom{n}{s_1, s_2, ..., s_l}$ zero-dimensional fillings.*
5. *if $n_i = 1$ for all i and $s_j = 1$ for all j, meaning X has n distinct eigenvalues, and $h(i) = i$ for all i. Then there are $n!$ zero-dimensional fillings.*

The following proposition proves a kind of duality between Hessenberg varieties with h-blocks of sizes $n_1, ..., n_k$ and multitableau λ_X with tableaux of sizes $s_1, ..., s_l$ Hessenberg varieties with h-blocks of sizes $s_1, ..., s_l$ and multitableau with tableaux of sizes $n_1, ..., n_k$. It is unknown if there is a geometric interpretation for this duality.

Proposition 17. *The number of zero-dimensional fillings corresponding to a Hessenberg function with h-blocks of sizes $n_1, ..., n_k$ and multitableau λ_X with tableaux of sizes $s_1, ..., s_l$ is equal to the number of zero-dimensional fillings corresponding to a Hessenberg function with h-blocks of sizes $s_1, ..., s_l$ and multitableau with tableaux of sizes $n_1, ..., n_k$*

Proof. We construct a bijection between permutations π of the multiset $\{1^{n_1}, 2^{n_2}, ..., k^{n_k}\}$ and permutations σ of the multiset $\{1^{s_1}, 2^{s_2}, ..., l^{s_l}\}$, with descent sets $D(\pi) \subseteq \{s_1, s_1 + s_2, ..., s_1 + s_2 + ... + s_{l-1}\}$ and $D(\sigma) \subseteq \{n_1, n_1 + n_2, ..., n_1 + n_2 + ... + n_{k-1}\}$. Given π, group the numbers so that the first group contains the first s_1 numbers of π, the second contains the next s_2 numbers, the third contains the next s_3, and so on. If the j^{th} group contains $c_{i,j}$ copies of i (for $j = 1, ..., l$ and $i = 1, ..., k$) then put $c_{i,j}$ copies of j in the i^{th} group of σ, and then rearrange each group of σ so that the numbers in the group are increasing. The first group of σ contains the first n_1 numbers, the second contains the next n_2 numbers, and so on, until the last group contains the last n_k numbers. This regrouping ensures that $D(\sigma) \subseteq \{n_1, n_1+n_2, ..., n_1+n_2+...+n_{k-1}\}$, and since the j^{th} group of π contained s_j numbers for each j, the permutation σ will have s_j copies of j in it. This means it is in fact a permutation of $\{1^{s_1}, 2^{s_2}, ..., l^{s_l}\}$.

This construction is well-defined and is symmetric in σ and π so it is in fact bijective. ☐

An example of this bijection is when $n_1 = 5$, $n_2 = 3$, $n_3 = 2$, and $s_1 = 3$, $s_2 = 3$, $s_3 = 2$, $s_4 = 2$. If $\pi = 1121331122$, then there are two 1's in the first group, one 1 in the second group, and two 1's in the third group, so the first group of σ is 11233. We end up with $\sigma = 1123314422$.

4. An Upper Bound for the Dimension

We will use the description of the dimension in Corollary 11 to find a sharp upper bound for the dimension of a filling of λ_X. This upper bound is achieved in a special case which is described in Section 5. The expression for the upper bound is made up of two terms. The first, $\sum_{i=1}^{n} h(i) - i$, is a parameter associated to ad-nilpotent ideals and to regular nilpotent Hessenberg varieties [ST]. The second term, $\sum_{i=1}^{l}\sum_{j=2}^{k_i}(j-1)d_{i,j}$, is the number of semistandard fillings and the rank of the irreducible representation if X is nilpotent.

Note that if k_i is the number of rows in the i^{th} tableau, and $d_{i,j}$ is the number of entries in the j^{th} row if the i^{th} tableau, then $\sum_{j=2}^{k_i}(j-1)d_{i,j}$ is the sum of the number of rows in the i^{th} tableau which are above the box filled with c for each entry c in the i^{th} tableau of λ_X. This sum can also be expressed as

$$\sum_{\text{each column}} 1 + 2 + ... + (\text{ length of the column } - 1)$$

$$= \sum_{\text{each column}} \binom{\text{length}}{2}$$

.

Theorem 18. *Let λ_X be a multitableau with l tableaux, such that the i^{th} tableau has k_i rows and the length of the j^{th} row of the i^{th} tableau is $d_{i,j}$. Then the dimension of any filling of λ_X is at most*

$$\sum_{i=1}^{n} h(i) - i + \sum_{i=1}^{l}\sum_{j=2}^{k_i}(j-1)d_{i,j}$$

.

Proof. First, note that if j is P_i, then $j < i \leq h(j)$, so j can be P_i only for $i = j+1, j+2, ..., h(j)$, a total of $h(j) - j$ possibilities. Thus there are $\sum_{j=1}^{n} h(j) - j$ pairs i,j where j is P_i. Also note that for any pair i,j where j is P_i, this pair can contribute to either part (1) or part (2) of the expression for the dimension in Corollary 11, but not both. So parts (1) and (2) can contribute at most $\sum_{j=1}^{n} h(j) - j$ to the dimension.

We show that the number of pairs satisfying the condition in part (3) is at most $\sum_{i=1}^{l}\sum_{j=2}^{k_i}(j-1)d_{i,j}$.

Recall that given a fixed i, there can be at most one D_i entry with no left P_i. Now given any two entries i_1 and j_1 which are both in the same column, as in Figure 4, we will show that at most one of the following can occur:

- there is an entry j_k in the same row as j_1 which is D_{i_1}, the pair i_1,j_k contributes to the dimension, and j_k has no left P_{i_1} or
- there is an entry i_l in the same row as i_1 which is D_{j_1}, the pair j_1,i_l contributes to the dimension, and i_l has no left P_{j_1}.

Assume that there is both a j_k and an i_l satisfying the given conditions. Note that there may in fact be no such j_k or i_l. In order for i_1,j_k to contribute to the dimension, i_1 is to the left of or directly below j_k, so $k \geq 1$. For j_1,i_l to contribute to the dimension, j_1 must be to the left of i_l (since i_1 is below j_1), so $l > 1$.

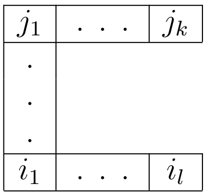


Figure 5

Suppose that $j_1 < i_1$. Then $j_1 < i_1 \leq h(i_2)$, and $j_1 > i_l$ since i_l is D_{j_1}. By the Slide Right Lemma, one of $i_2, ..., i_l$ is P_{j_1} and thus i_l has a left P_{i_1}. Otherwise, if $i_1 < j_1$, we have $i_1 < j_1 \leq h(j_1)$, and $i_1 > j_k$ since j_k is D_{i_1}. Again by the Slide Right Lemma, one of $j_1, ..., j_k$ is P_{i_1} and thus j_k has a left P_{j_1}.

Thus we can see that for any two entries in a column, at most one can have an entry in the same row as the other entry which is D_i with no left P_i. Since there are $\sum_{i=1}^{l}\sum_{j=2}^{k_i}(j-1)d_{i,j}$ pairs of such entries, this is also the maximum number of pairs satisfying the condition in part (3) of Corollary 11. □

5. THE DIMENSION IN A SPECIAL CASE

In this section we demonstrate that the upper bound from Section 4 can be achieved. Let s_i be the number of boxes in the i^{th} tableau, and let l be the number of tableaux in λ_X. Now suppose it is the case that for $0 \leq m \leq l$ and $1 \leq t \leq k_m$

$$h\left(\sum_{i=0}^{m} s_i + \sum_{j=1}^{t} d_{i+1,j}\right) = \sum_{i=0}^{m} s_i + \sum_{j=1}^{t} d_{i+1,j}.$$

This just means that the h is partitioned so that if the first $d_{1,1}$ numbers are put in the first row, and the next $d_{1,2}$ numbers are put in the second row, and so on, the h-blocks are not broken up between rows. Also, note that we are not completely specifying the Hessenberg function. One particular example that always satisfies this condition is $h(i) = i + 1$.

For example if $d_{1,1} = 4$, $d_{1,2} = 2$, $d_{1,3} = 1$, $d_{2,1} = 2$, and $d_{2,2} = 1$, then we must have $h(4) = 4$, $h(6) = 6$, $h(7) = 7$, $h(9) = 9$, and $h(10) = 10$. The multitableau for this example can be seen in Figure 5.

Recall from Definition 2 that we partition $\{1, 2, ..., n\}$ into h-blocks according to the rule that $i + 1$ is in the same h-block as i if and only if $h(i) > i$. Now consider the following filling. First, take the numbers in the first h-block, $1, ..., n_1$, and put them at the beginning of the first row in decreasing order

n_1	...	2	1

Then place $n_1+1, ..., n_2$, the numbers in the second h-block, in reverse order immediately after the first n_1 numbers in the first row.

n_1	...	2	1	n_2	...	$n_1 + 2$	$n_1 + 1$

Then put the next h-block in reverse order after the previous one, continuing until the first row is filled. Each row that is filled in this way will end exactly after some h-block is used. After the first row is full, go to the second row and put the next h-block in reverse order, then the next h-block in reverse order after it, skipping to the top row of a the next tableau after each one is filled, and continue this process until every row of every tableau is filled. We will call this the h-block filling.

An example of this filling is when $d_{1,1} = 4$, $d_{1,2} = 2$, $d_{1,3} = 1$, $d_{2,1} = 2$, $d_{2,2} = 1$, and the Hessenberg function is given by:

$h(1) = 2$ $\quad h(2) = 2$ $\quad h(3) = 4$ $\quad h(4) = 4$
$h(5) = 5$ $\quad h(6) = 6$
$h(7) = 7$
$h(8) = 9$ $\quad h(9) = 9$
$h(10) = 10$

This example gives the filling seen in Figure 5.

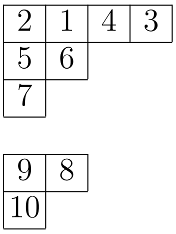


Figure 6. Example of the h-block filling.

Proposition 19. *Given the hypothesis on* h *and* λ_X*, the dimension of the large filling is exactly*

$$\sum_{i=1}^{l} \sum_{j=2}^{k_i} (j-1)d_{i,j} + \sum_{i=1}^{n} h(i) - i.$$

Proof. It suffices to show that there are at least $\sum_{i=1}^{l} \sum_{j=2}^{k_i} (j-1)d_{i,j} + \sum_{i=1}^{n} h(i) - i$ pairs i,j which contribute to the dimension.

First, for each tableau and each entry i in a row which is not the top-most row, we have $i > j$ whenever j at the end of a row which is above i and in the same tableau. So then the pair i, j contributes to the dimension. For each i, there is one of these j for each row above the row containing i. So for each tableau, there are

$$\sum_{i \in \text{ tableau}} \text{number of rows above the row containing } i$$

$$= \sum_{j=2}^{k_i} (j-1) d_{i,j} \text{ pairs in the } i^{th} \text{ tableau.}$$

Summing up over all tableaux, the number of pairs of this type contributing to the dimension is at least

$$\sum_{i=1}^{l} \sum_{j=2}^{k_i} (j-1) d_{i,j}.$$

Now for each entry i, count up the number of entries j to the right of i in the same row where j is P_i. If j is P_i, then $j < i \leq h(j)$ and for each j, there are $h(j) - j$ possible entries i for which j is P_i. Also note that whenever $j < i \leq h(j)$, we must have that $h(k) \geq h(j) \geq i$ for all $j < k$. Therefore $h(k) > k$ for all k with $j < k < i$. So for any such k, both k and $k+1$ are in the same h-block. Also since $h(j) > j$, we see that j and $j+1$ are in the same h-block. It follows that $j, j+1, ..., i$ are all in the same h-block. Since all of these i are to the left of j by the hypothesis, there are at least $\sum_{j=1}^{n} h(j) - j$ pairs of this type contributing to the dimension.

Adding these two parts, the dimension of the filling is at least

$$\sum_{i=1}^{n} h(i) - i + \sum_{i=1}^{l} \sum_{j=2}^{k_i} (j-1) d_{i,j}.$$

Since this is also the upper bound for the dimension, the dimension must equal to this sum. □

6. The Dimension when $h(i) = i + 1$

In this section, we require that a multitableau λ_X with the j^{th} row of the i^{th} tableau having length $d_{i,j}$ for $1 \leq i \leq l$ and $1 \leq j \leq k_i$, that $k_l > k_i$ for all i, meaning that the bottom tableau has the greatest number of rows out of all the tableaux. We describe a particular filling of a multitableau when $h(i) = i + 1$ and give its dimension. We then prove a smaller upper bound than the one in the previous section for the case when $h(i) = i + 1$. This upper bound is then used to show that the filling described actually gives the maximum possible dimension.

For a multitableau satisfying our condition that $k_l > k_i$ for all i, and the Hessenberg function $h(i) = i+1$, consider the following filling of λ_X. First, put the first $d_{1,1}$ numbers in the top-most row of the top tableau, in decreasing order. Next, put the next $d_{2,1}$ numbers in the top-most row of the next tableau, again in decreasing order. Continue this process, putting the numbers in decreasing order in the top-most row of each tableau, continuing downward to each tableau. After this is done, go back to the top tableau and put the next $d_{1,2}$ numbers in the second row of the top tableau, in decreasing order. Continue this same process going down the tableaux for the second row of each. Then continue for the third rows, fourth rows, and so on, skipping a tableau if all of its rows have been filled. Note that since the bottom tableau has the greatest number of rows, it will be filled last. We call this filling the large $i+1$ filling. An example of this filling is given in Figure 6.

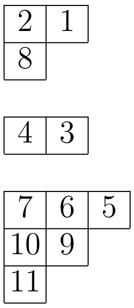


Figure 7. An example of large $i+1$ filling.

Proposition 20. *The dimension of the large $i+1$ filling is*

$$n - k_l + \sum_{i=1}^{l}\sum_{j=2}^{k_l}(j-1)d_{i,j}.$$

Proof. Note that if j is P_i, this means that $j < i \leq h(j) = j+1$. So if j is P_i, this means that $i = j+1$.

Using that fact to slightly modify the expression of the dimension from Corollary 11, the dimension of a filling as the sum of

1. the number of $j < n$ in the filling of λ_X such that
    - $j+1$ is in the same tableau as j, and
    - the box filled with $j+1$ is to the left of or directly below the box filled with j.
2. the number of pairs i,j in the filling of λ_X such that
    - i and j are in the same tableau,
    - the box filled with i is to the left of or directly below the box filled with j,
    - j is D_i with no left P_i, meaning $i-1$ is not anywhere directly to the left of j and above i.
3. the number $j < n$ in λ_X such that
    - $j+1$ and j are in different tableaux, and
    - the box filled with $j+1$ is below j.

Consider part (1) of this sum. By construction of the filling if the entry j is not at the beginning of a row, $j \neq n$ and $j+1$ is to the left of and in the same tableau as j. The number of these boxes is $n-(k_1+k_2+...+k_l)$. If j is at the beginning of a row, $j+1$ will be in a tableau below the one containing j. Thus part (1) contributes $n-(k_1+k_2+...+k_l)$ to the dimension.

Similarly, for part (3) of the sum, the number of entries j ($j \neq n$) such that $j+1$ is in a tableau below the one containing j is equal to the number of boxes at the beginning of a row not in the bottom-most tableau, which is $k_1+k_2+...+k_{l-1}$.

Note that if i and j are in different rows with j not at the end of a row, $j-1$ is immediately to the right of j. If $i > j$, then we also have $i > h(j-1) = j$. Thus j cannot be D_i. So, the number of pairs i,j satisfying the conditions in part (2) of the sum is equal to the number of pairs i,j where j is at the end of a row which does not contain i, i is to the left of or directly below j, and j is D_i, meaning $i > j$.

So for part (2) of the sum, we need to count the number of pairs i,j such that i and j are in the same tableau, j is at the end of a row, i is to the left of or directly below j, and $i > j$. Given some entry i, every entry j below i has $i < j$, by the rules of our filling. But any entry j above i in the same tableau has $i > j$, and if j is at the end of a row above i, then i is to the left of or directly below j. Thus the number of these types of pairs is equal to

$$\sum_i \text{the number of rows above } i \text{ in the same tableau}$$

$$= \sum_{i=1}^{l} \sum_{j=2}^{k_i} (j-1) d_{i,j}.$$

Adding up these three parts, we get that the dimension of this filling is equal to

$$n - (k_1 + k_2 + ... + k_l) + (k_1 + k_2 + ... + k_{l-1}) + \sum_{i=1}^{l} \sum_{j=2}^{k_i} (j-1) d_{i,j}$$

$$= n - k_l + \sum_{i=1}^{l} \sum_{j=2}^{k_i} (j-1) d_{i,j}.$$

□

Theorem 21. *For a multitableau λ_X with the j^{th} row of the i^{th} tableau having length $d_{i,j}$ for $1 \leq i \leq l$ and $1 \leq j \leq k_i$, and the Hessenberg function $h(i) = i + 1$, the dimension of any filling is at most*

$$n - k_1 + \sum_{i=1}^{l} \sum_{j=2}^{k_i} (j-1) d_{i,j}.$$

Proof. Recall from Corollary 10, the dimension of a nonempty cell is the sum of

(1) the number of pairs i,j in the filling of λ_X such that
  - i and j are in the same tableau,
  - the box filled with i is to the left of or directly below the box filled with j,
  - j is D_i; and

(2) the number of pairs i,j in λ_X such that
  - i and j are in different tableaux,
  - the box filled with i is below j, and
  - j is P_i.

We will use this expression to give an upper bound for the dimension of the filling when $h(i) = i + 1$.

We begin by showing that given some entry i, for any row in the same tableau as i excluding the one containing i, there are at most two entries which are D_i in the row.

Suppose this is not the case. Then we have three entries j, l and q which are all D_i. Without loss of generality, assume the entries are arranged as follows possibly with k and l coinciding.

j	k	. . .	l	m	. . .	q

If k is immediately to the right of j, then $i \leq h(k) = k+1$ Since $i \neq k$, we have either $i < k$ or $i = k+1$. If $i < k$, then $l < i < k$, meaning that the entries in the row must decrease from k to l. But since $h(i) = i+1$, each entry i can be followed by either $i-1$ or some number larger than i. So for the row to decrease from k to l, we must have $k-1, k-2, ..., l+2, l+1$ in between k and l. But since $l < i < k$ and i is in a different row, this is not possible. Thus $i = k+1$. It is possible that the entry k is actually l, but the argument works either way. Now suppose that m is the entry immediately following l. We have $i \leq h(m) = m+1$, but since $i = k+1$ and $i \neq m$ and $k \neq m$, it must be the case that $i < m$. Now by the same reasoning as before, $q < i < m$, so in order for the entries in the row to decrease from m to q, i would have to be between them, but it is not. Thus we cannot have more than two entries which are D_i in a row not containing i.

Also note that when we have two entries which are D_i, j and l as above, whenever this occurs, we have some entry k which is in a different row than $i = k+1$ and $k+1$ is to the left of k. Recalling definition 9, this is equivalent to saying that l is D_i with a left P_i entry. Then with $h(i) = i+1$, if j, had a left P_i entry, this would imply that either $k = j$ or k is to the left of j, since $i = k+1$ and the only entry which is P_i is k. We know this is not the case, since k is to the right of j, so j is D_i with no left P_i.

Also now, recall from our observations in Section 2, that whenever j is D_i and in the same row as and to the right of i, j has a left P_i entry.

Now, notice that part (1) of the sum is equal to the sum of

- the number of pairs i, j where i and j are in the same row, j is to the right of i and j is D_i, and
- the number of pairs i, j where i and j are in the same tableau but different rows, j is to the right of i and j is D_i.

Or, we can rephrase this as the sum is equal to the sum of

- the number of pairs i, j where i and j are in the same row, j is to the right of i and j is P_i,
- the sum over each i of the number of rows which have at least one entry j which is to the right of or directly below above i, and j is D_i, and
- the sum over each i of the number of rows which have two entries j and l which are both to the right of or directly below above i, and j and l are both D_i.

Then the number of pairs i, j where i and j are in the same row, j is to the right of i and j is P_i is simply equal to the number of entries j for which $j+1$ is to the left of j in the same row. And the sum over each i of the number of rows which have two entries j and l which are both to the right of or directly below above i, and j and l are both D_i is equal to the number of entries k for which $k+1$ is to the left of k and in the same tableau, but in a different row.

Also, given any two entries i_1 and j_1 which are both in the same column with j_1 above i_1, we will show that we can have an entry j_k in the same row as j_1 which is D_{i_1} contributing to the dimension, or that we can have an entry i_l in the same row as i_1 which is D_{j_1} contributing to the dimension, but not both can occur at once. In order for j_k to be D_{i_1} and contributing to the dimension, it must be the case that $k \geq 1$, and for i_l to be D_{j_1} and contributing to the dimension, it must be the case that $l > 1$. (If there is no entry j_k which is D_{i_1} contributing to the dimension, or no i_l which is D_{j_1} contributing to the dimension, then we obviously have at most one of the above cases, so we assume that there are such j_k and i_l). Now, one of the following cases will occur:

- If $j_1 > i_1$, then suppose that j_k is D_{i_1}. We then have $j_1 > i_1 > j_k$, and in order for the row containing j_1 to decrease to j_k, we would need every number $j_1 - 1, j_1 - 2, ..., j_k + 1$ to be between the two. But we know that i_1 is not between the two, so this cannot be the case. Therefore in this case, there can be no entry j_k which is D_{i_1}.
- If $i_1 > j_1$, then suppose that i_l is D_{j_1}. We then have $i_1 > j_1 > i_l$, and in order for the row containing i_1 to decrease to i_l, we would need every number $i_1 - 1, i_1 - 2, ..., i_l + 1$ to be between the two. But we know that j_1 is not between the two, so this cannot be the case. Therefore in this case, there can be no entry i_l which is D_{j_1}.

Therefore, the sum over each i of the number of rows in the same tableau as i, but not the same as the row containing i for which there is at least one entry j where i is to the left of or directly below j and j is D_i is at most

$$\sum_{i=1}^{l} \sum_{j=2}^{k_i} (j-1) d_{i,j}$$

.

If we add this to the sum of the other parts of part (1) of the sum and part (2) of the sum, which is

- the number of entries j where $j+1$ is to the left of j and in the same row,
- the number of entries j where $j+1$ is in a different tableau below the one containing j, and
- the number of entries j where $j+1$ is in the same tableau but in a different row than j, and $j+1$ is to the left of j

we will get the maximum possible dimension.

Note that for any entry j, at most one of the above three situations can occur. Also, for any entry j which is not in the left most column of the bottom tableau, any one of the three could occur. If j is in the left most column of the bottom tableau, there can be nothing to the left of j and nothing in a tableau below the one containing j. Therefore we can have at most one of these situations occurring for each entry j which is not in the left most column of the bottom tableau. There are $n - k_l$ such possible entries j.

Now adding these two parts, the dimension of a filling for $h(i) = i + 1$ is at most

$$n - k_l + \sum_{i=1}^{l} \sum_{j=2}^{k_i} (j-1) d_{i,j}.$$

□

Corollary 22. *For a multitableau λ_X with the j^{th} row of the i^{th} tableau having length $d_{i,j}$ for $1 \leq i \leq l$ and $1 \leq j \leq k_i$, that $k_l > k_i$ for all i, and the Hessenberg function $h(i) = i+1$, the Hessenberg variety has dimension*

$$n - k_1 + \sum_{i=1}^{l} \sum_{j=2}^{k_i} (j-1) d_{i,j}.$$

7. The Dimension of Another Filling

We now consider the filling which corresponds to the intersection of the big cell in the flag variety with the Hessenberg variety. This is the filling where we put n in the bottom leftmost entry of the bottom tableau, put $n-1$ above it, and continue up the column with numbers decreasing. When a column is full, we begin filling the column to the right of it from the bottom to the top with decreasing numbers, moving up to the bottom leftmost entry of the next tableau if one is full. We will call this the big filling.

An example of the big filling can be seen in Figure 7.

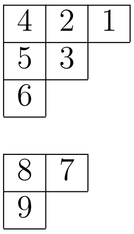


Figure 8. An example of the big filling.

Let s_j denote the number of entries in the j^{th} tableau, counted from the bottom tableau upward. Then $\sum_{i=1}^{m} s_i$ is the largest entry in the m^{th} tableau of the big filling.

Proposition 23. *When when the big filling is allowed, its dimension is*

$$\sum_{i=1}^{n} h(i) - i + \sum_{j \in \text{ leftmost column of the } m^{th} \text{ tableau}} \max\left\{ \left(\sum_{i=1}^{m} s_i \right) - h(j), 0 \right\}.$$

Proof. First, recall that from Corollary 11 in Section 2, the dimension of a filling is equal to the sum of

1. the number of pairs i,j in the corresponding filling of λ_X such that
   - i and j are in the same tableau,
   - the box filled with i is to the left of or directly below the box filled with j,
   - j is P_i.
2. the number of pairs i,j in the corresponding filling of λ_X such that
   - i and j are in the same tableau,
   - the box filled with i is to the left of or directly below the box filled with j,
   - j is D_i with no left P_i.
3. the number of pairs i,j in λ_X such that

- i and j are in different tableaux,
- the box filled with i is below j, and
- j is P_i.

For each i, any j such that $j < i \leq h(j)$ is either in a tableau above the one containing i, or else i and j are in the same tableau with the box filled with j directly above or to the right of the one filled with i, by the rules of the filling.

So the total number of P_i entries contributing to the dimension is $\sum_{j=1}^{n} h(j) - j$.

Thus the dimension is equal to the sum of

(1) $\sum_{j=i}^{n} h(j) - j$ and
(2) the number of pairs i, j where i is to the left of or directly below j and j is D_i with no left P_i.

If i is not in the leftmost column and j_m, $1 \leq m \leq 3$, is in one of the three relative positions seen in Figure 7, since i is directly below or to the left of j_m, we have $i > j_m$. Also, in the middle case, $i \leq h(j_2)$, since the filling is allowed, and in the bottom and top cases, $k_m > i$ by the rules for this filling, and $k_m \leq h(j_m)$, so it follows that $j_m < i \leq h(j_m)$ in any of the three cases. Thus j_m is P_i.

By the same argument, if i is in the leftmost column, j_2 and j_m will be P_i.

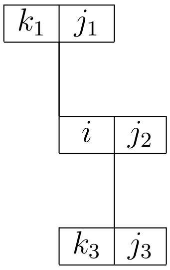


FIGURE 9

Therefore, the only possible way to have an entry j be D_i with no left P_i is if i is in the leftmost column and j is in a row above the one containing i, since anything else is assured to have a left P_i entry.

Hence the number of pairs i,j where i is to the left of or directly below j and j is D_i with no left P_i is equal to the number of pairs i,j where i is in the leftmost column and j is the rightmost entry in a row above i. Given any entry j that is the rightmost entry in a row above i and in the same tableau $i > j$ in this fulling, so j is D_i. We simply need to count up the number of rows above i in the same tableau for which the leftmost entry is not P_i, for each i in the leftmost column of each tableau. We can rephrase this as the sum over each entry k in the left most column of the number of entries i below k in the same column and same tableau for which k is not P_i.

If s_j is the total number of boxes in the j^{th} tableau, the largest entry in the m^{th} tableau (counted from the bottom) is $\sum_{j=1}^{m} s_j$. Any k in the m^{th} tableau k is P_i for $i = k+1, k+2, ..., h(k)$. So if $h(k) \geq \sum_{j=1}^{m} s_j$, then there are no entries i in the same tableau as k and below k for which k is P_i. But if $h(k) < \sum_{j=1}^{m} s_j$, we know k is not P_i for $i = h(k)+1, h(k)+2, ..., \sum_{j=1}^{m} s_j$, of which there are $\sum_{j=1}^{m} s_j - h(k)$ possibilities, and each of these occurs below k. k is P_i for all $i = k+1, ..., h(k)$ since these are the other entries occurring below k. Thus there are $\sum_{j \in \text{ leftmost column of the } m^{th} \text{ tableau}} \max\{(\sum_{i=1}^{m} s_i) - h(j), 0\}$ pairs of this type in the m^{th} tableau.

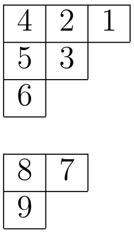
Adding this to $\sum_{i=1}^{n} h(i) - i$, gives the dimension of the filling. ☐

For example, in the case where

$h(1) = 3$	$h(2) = 5$	$h(3) = 5$
$h(4) = 5$	$h(5) = 5$	$h(6) = 7$
$h(7) = 8$	$h(8) = 9$	$h(9) = 9$

the dimension of the filling, seen in Figure 7, is

$$\sum_{i=1}^{9}(h(i) - i) \; + \sum_{j=4,5,6} \max\{6 - h(j), 0\} \; + \sum_{j=8,9} \max\{9 - h(j), 0\} = 11 + 2 + 0 = 13.$$



4	2	1
5	3	
6		

8	7
9	

FIGURE 10. An example of the big filling.

Corollary 24. *When the big filling is allowed for a multitableau* λ_X *and a Hessenberg function, the maximum possible dimension of any filling is at least*

$$\sum_{i=1}^{n} h(i) - i + \sum_{j \in \text{ leftmost column of the } m^{th} \text{ tableau}} \max\left\{\left(\sum_{i=1}^{m} s_i\right) - h(j), 0\right\}.$$

Conjecture 25. *We conjecture that whenever the big filling is an allowed filling, it gives the maximum dimension for any cell, and is thus the dimension of the Hessenberg variety.*

REFERENCES

[DPS] F. de Mari, C. Procesi, and M. A. Shayman, *Hessenberg varieties*, Trans. Amer. Math. Soc. **332** 1992, 529–534.

[DS] F. de Mari and M. A. Shayman, *Generalized Eulerian numbers and the topology of the Hessenberg variety of a matrix*, Acta Appl. Math. **12** 1988, 213–235.

[F] J. Fulman, *Descent identities, Hessenberg varities, and the Weil conjectures*, J. Combin. Theory Ser. A **87** 1999, 390–397.

[S] R. P. Stanley, *Enumerative Combinatorics, Vol. 1*, Cambridge Studies in Advanced Mathematics 49, Cambridge University Press, Cambridge, 1997.

[ST] E. Sommers and J. Tymoczko, *Exponents for B-stable ideals*, math.RT/0406047.

[T] J. Tymoczko, *Paving Hessenberg varieties by affines*, math.AG/0409118.

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