LINEAR PERTURBATIONS OF A CLASS OF BEREZIN-TOEPLITZ OPERATORS

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Abstract. We investigate the asymptotic distribution of the eigenvalues of a sequence of perturbed Berezin-Toeplitz operators $\{P^{(n)}(\epsilon) = A_n + \epsilon Q_n\}_{n=1}^{\infty}$ and prove by means of "the averaging method" that choosing $\epsilon = \frac{1}{Cn}$ for an appropriate constant C will keep the eigenvalues of $P^{(n)}(\epsilon)$ close to those of $A_n + \epsilon Q_n^{av}$ for a particular diagonal matrix Q_n^{av}. This theorem is related to the splitting phenomenon observed numerically for several specific sequences of perturbed operators with symbols defined on the sphere and the cylinder.

1. Background and Preliminaries

1.1. **Notation & Conventions.** Throughout, we will make use of the notation $\|M\|$ to denote the operator norm of M:

$$\|M\| = \max_{|v|=1} \frac{\|Mv\|}{|v|}$$

We will also use the symbol $\Lambda(A)$ to denote the set of eigenvalues of A.

1.2. **Berezin-Toeplitz Operators.** In both the classical theory of Toeplitz operators and the theory of Berezin-Toeplitz operators, there is seen to be a deep connection between the "symbol" of an operator and various properties of the operator itself (e.g., eigenvalue distribution information for classical Toeplitz operators can be gleaned from the Szegö limit theorem and the location of the pseudospectrum for Berezin-Toeplitz operators by Trefethen and Chapman's "twist condition" in [2]). The theory of Berezin-Toeplitz operators begins in this spirit, but examines more general symbols. To begin our examination, we will take a function (the "symbol") $f : X \to \mathbb{C}$ where X is either the 2-sphere or the cylinder $\{(p, \theta) | p \in [0, 1], \theta \in [0, 2\pi]\}$. We begin by examining the case of the cylinder, which may be viewed as a superset of the theory of classical Toeplitz operators. To view the classical case as a subset of our framework, remember that the classical symbol is taken to be defined on all of the real line and to be 2π periodic, so it may be thought of as a function on the cylinder of the angular variable θ whose value does not depend on p (the altitude on the cylinder). For a much more in-depth discussion of the topics of this section, the reader is advised to look to [4].

1.2.1. *The Cylinder.* On the cylinder, we consider the Hilbert space

$$\mathcal{H}_n = \left\{ \sum_{k=1}^{n} a_k e^{ik\theta} | a_k \in \mathbb{C} \right\}$$

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of dimension (over $\mathbb{C}$) $n+1$ with the inner product defined by extending the relation $< e^{ik\theta}, e^{il\theta} >= \delta_{kl}$. What we will now illustrate is how to take a symbol f on the cylinder and to define an operator $Op_n(f) : \mathcal{H}_n \to \mathcal{H}_n$ for all n. Let $\psi \in \mathcal{H}_n$. In the classical Toeplitz case, we view f as an element of $L^2(S^1)$ and determine the induced operator by letting $Op_n(f)(\psi)$ be the result of mapping $\psi \to f\psi$ (an element of $L^2(S^1)$) and then projecting its Fourier Series onto $\mathcal{H}_n$. To extend this, for $f(p) = p$, we let

$$Op(f)(\psi) = \frac{1}{iN}\frac{d\psi}{d\theta}$$

and impose that $f \mapsto Op_n(f)$ be linear. Now, we have yet to specify how to quantize something of the form $f(p,\theta) = f(p)e^{i\theta}$. One way to quantize this is to follow Weyl and to define the operator symmetrically:

$$Op_n(f(p)e^{i\theta}) = \frac{1}{2}\left(Op_n(f(p)) \circ Op_n(e^{i\theta}) + Op_n(e^{i\theta}) \circ Op_n(f(p))\right)$$

Another approach, which we will take, is similar to that of [2]. First, we stipulate that $Op_n(f(p)e^{il\theta}) = Op_n(e^{il\theta}) \circ Op_n(f(p))$. Noting that $Op_n(p)(e^{ik\theta}) = \frac{k}{N}e^{ik\theta}$, it is natural to define $Op_n(f(p))(e^{ik\theta}) = f(\frac{k}{n})e^{ik\theta}$. If we further state that if

$$f(p,\theta) = \sum_{l=-\infty}^{\infty} f_l(p)e^{il\theta}$$

then we will define $Op_n(f)$ by the natural formula:

$$f(p,\theta) = \sum_{l=-\infty}^{\infty} Op_n(f_l(p)e^{il\theta})$$

We will refer to the n^{th} such operator as $T_n(f)$ from this point forward.

1.2.2. *The Sphere & SU(2).* SU(2) is defined as the group of 2×2 complex unitary matrices with determinant 1. To determine its Lie Algebra, we note that a matrix $A \in Lie(SU(2))$ iff for all t, $e^{tA} \in SU(2)$. This determines the Lie Algebra by noting that we must have that $e^{tA}(e^{tA})^* = I$ (using * to denote the Hermitian transpose) and $\det(e^{tA}) = 1$. As $(e^{tA})^* = e^{-tA}$, the first conditions gives us that $A + A^* = 0$ and by remembering that $det(e^{tA}) = e^{trace(A)}$, we have that $tr(A) = 0$. The set of all traceless 2×2 skew-Hermitian matrices is called su(2). su(2) is generated by the Pauli matrices:

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sigma_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Now, consider the following Hilbert space of homogeneous polynomials of degree $2n$:

$$\mathcal{H}_n = \left\{ \psi(z_1, z_2) | \sum_{j=0}^{2n} a_j z_1^{2n-j} z_2^j : a_j \in \mathbb{C} \right\}$$

An orthonormal basis, with respect to the Hermitian inner product

$$< \psi_1, \psi_2 >= \int_{S^3} f_1 \bar{f}_2 dV_{S^3}$$

is given by $\left\{ |j,n> = \sqrt{\begin{pmatrix} n \\ j \end{pmatrix}} z_1^{n-j} z_2^j \right\}$. SU(2) acts on $\mathcal{H}_n$ by letting $g \cdot \psi = \psi(g^{-1}(z_1, z_2))$. The interesting thing about this action is that the irreducible representations of SU(2) are indexed by $J = 0, \frac{1}{2}, 1, \frac{3}{2}, 2 \ldots$ and are given (up to isomorphism) by this action on $\mathcal{H}_n$.

Now we are ready to describe how, given an element $A \in su(2)$, we obtain an operator $\mathcal{H}_n \to \mathcal{H}_n$ for every $n \in \frac{1}{2}\mathbb{Z}$.

Definition 1.1. $Op(A)(\psi) = \frac{d}{dt} e^{tA} \psi|_{t=0}$.

As examples, we give the operators obtained for each of the Pauli matrices:

$$Op(\sigma_1) = \frac{1}{2i}(z_2 \frac{\partial}{\partial z_1} + z_1 \frac{\partial}{\partial z_2})$$

$$Op(\sigma_2) = \frac{1}{2}(z_2 \frac{\partial}{\partial z_1} - z_1 \frac{\partial}{\partial z_2})$$

$$Op(\sigma_3) = \frac{1}{2i}(z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2})$$

As each $\mathcal{H}_n$ is finite-dimensional, we may find a matrix representation of each of these operators with respect to the orthonormal basis presented before. Let $M_j(\sigma_k)$ denote the j^{th}supradiagonal entry of the matrix of σ_k, $m_j(\sigma_k)$ the j^{th} infradiagonal entry, and $d_j(\sigma_k)$ the j^{th} diagonal entry. Then the following may be verified by routine calculation (all unspecified entries are 0):

$$M_j(\sigma_1) = \frac{1}{2i}\sqrt{j(n-j+1)},\ m_j(\sigma_1) = \frac{-1}{2i}\sqrt{j(n-j+1)}$$

$$M_j(\sigma_2) = \frac{-1}{2}\sqrt{j(n-j+1)},\ m_j(\sigma_2) = \frac{1}{2}\sqrt{j(n-j+1)}$$

$$d_j(\sigma_3) = N - 2j$$

Now we introduce an inner product on su(2) by

$$< A, B >= -\frac{1}{2} Tr(AB)$$

which will allow us to introduce coordinate functions on the sphere (which is viewed as a subset of su(2)). We define $x_j : \mathrm{su}(2) \to \mathbb{R}$ by $x_j(A) =< A, 2\sigma_j >$. It turns out that the 2-sphere is diffeomorphic to the set $\{A \in \mathrm{su}(2) | x_1(A)^2 + x_2(A)^2 + x_3(A)^2 = \frac{1}{2}\}$ (for the diffeomorphism, see [4]). Now, we may quantize a symbol on the sphere with quadratic terms by the following procedure (we will not consider the more general case, which is also covered in [4]). We make the following definition:

$$T_n(x_j) = \frac{1}{in} Op_n(\sigma_j)$$

and for products of x_j, we follow Weyl:

$$T_n(x_i x_j) = \frac{1}{in}\left(Op_n(\sigma_i) \circ Op_n(\sigma_j) + Op_n(\sigma_j) \circ Op_n(\sigma_i)\right)$$

2. The Averaging Method

Given an operator of the form $P_\epsilon = Z + \epsilon\, Q$ whose behavior we would like to study as $\epsilon \to 0$, we may take the approach of Kato and attempt to extract information from bounds on coefficients of the analytic functions that represent the perturbed eigenvalues in terms of ϵ and to study the error induced by truncating these expansions, but these bounds often contain terms with small divisors that behave poorly in the limit $N \to \infty$ (e.g., factors of $\frac{1}{d}$ where d represents the separation distance between two unperturbed eigenvalues). These methods give information, as we will see shortly, but the averaging method allows us to strengthen these results. The method of averaging takes a different approach (that behaves well in the limit of N) to giving an approximation to the eigenvalues by conjugating our perturbed operator to another perturbed operator $Z + \epsilon\, Q^{av} + O(\epsilon^2)$ where Q^{av} is diagonal. To apply the averaging method, we must first note the following beautiful formula which we state as a lemma ($[V, P]$ denotes the commutator of V and P):

Lemma 2.1. $e^{\epsilon V} P\, e^{-\epsilon V} = P + \epsilon[V, P] + \frac{\epsilon^2}{2!}[V, [V, P]] + \frac{\epsilon^3}{3!}[V, [V, [V, P]]] + \dots$

Proof. The result follows by induction. □

Theorem 2.2. *(The Averaging Method)*
Given a perturbed operator $P_\epsilon = Z + \epsilon\, Q$ *where* Z *and* Q *are* $k \times k$ *matrices with* Z *diagonal having rational entries* $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_k}{b_k}$, *let* $n = lcm\{b_1, \dots, b_k\}$. *Then there exists an operator* V *such that* $e^{-\epsilon V} P_\epsilon\, e^{\epsilon V} = Z + \epsilon\, Q^{av} + O(\epsilon^2)$ *and* $[V, Z] = Q^{av} - Q$, *where* Q^{av} *is such that* $[Q^{av}, Z] = 0$.

Proof. First, we let V be given by the following:

$$V = \frac{n}{2\pi} \int_0^{2\pi} \int_0^t e^{-insZ} Q\, e^{insZ} ds\, dt$$

and we define Q^{av} by the formula

$$Q^{av} = \int_0^{2\pi} e^{-intZ} Q\, e^{intZ} dt$$

Then, we may proceed by utilizing the previous lemma as follows. Note that

$$e^{-\epsilon V} P_\epsilon\, e^{\epsilon V} = Z + \epsilon[V, Z] + \epsilon\, Q + O(\epsilon^2)$$

by the previous lemma. Now we verify that define Q^{av} and verify the condition stated with respect to $[V, Z]$. As we have that

$$[V, Z] = [\frac{n}{2\pi} \int_0^{2\pi} \int_0^t e^{-insZ} Q\, e^{insZ} ds\, dt,\ Z] = \frac{n}{2\pi} \int_0^{2\pi} \int_0^t [e^{-insZ} Q\, e^{insZ},\ Z]\, ds\, dt$$

and by calculation (letting $* = e^{-insZ} Q\, e^{in+1sZ}$), we see that $\frac{d}{ds}(*) = -inZ(*) + (*)inZ = in[*, Z]$, which is the integrand of the above integral. So,

$$[V, Z] = \frac{-1}{in2\pi} in \int_0^{2\pi} (e^{-intZ} Q\, e^{intZ} - Q)\, dt = Q - Q^{av}$$

Now, to prove that $[Q^{az}, Z] = 0$, we merely calculate once more:

$$[Q^{av}, Z] = \int_0^{2\pi} [e^{-intZ} Q\, e^{intZ}, Z]\, dt$$

$$= \frac{1}{in}\int_0^{2\pi} \frac{d}{dt}\left(e^{-intZ} Q\, e^{intZ}\right)$$

and by our definition of n, we have that $e^{in2\pi Z} = I$, the identity matrix. So, we have that

$$= \frac{1}{in}(Q - Q) = 0$$

and so the averaging method is justified. ☐

Remark. It is interesting to note that we also have that the j^{th} diagonal entry of Q^{av} is $< Qe_j, e_j >$ (where $<,>$ is the usual Hermitian product on $\mathbb{C}^n$). This may be seen by calculation once again, as

$$< Q^{av} e_j, e_j > = \frac{1}{2\pi}\int_0^{2\pi} < e^{-intZ} Q\, e^{intZ} e_j, e_j > dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} < Qe^{intZ}\, e_j, e^{intZ} e_j > dt$$

and by the properties of the Hermitian product, we have that this is

$$= \frac{1}{2\pi}\int_0^{2\pi} < Qe_j, e_j > dt = < Qe_j, e_j >$$

3. Main Results

First we state a well-known theorem which we will use shortly, and then we give a chain of lemmas leading to our main theorem.

Theorem 3.1. *(Bauer-Fike)*
Given two $n \times n$ matrices A and E such that there exists an $n \times n$ matrix V with $A = VDV^{-1}$ where D is diagonal, we have that for any $\hat{\lambda} \in \Lambda(A + E)$, there exists $\lambda \in \Lambda(A)$ such that $|\lambda - \hat{\lambda}| < \|V\|\,\|V^{-1}\|\,\|E\|$

Proof. Consult any book on numerical linear algebra. For example, Demmel's *Applied Numerical Linear Algebra* has a proof. ☐

Lemma 3.2. *Let $P^{(n)}(\epsilon) = A_n + \epsilon Q_n$ and pick some $\lambda \in \Lambda(A_n + \epsilon Q_n)$. If $\|\epsilon Q_n\| < \frac{d}{2}$ where d is the smallest distance to the other eigenvalues of A_n, then the perturbation series for $\lambda(\epsilon)$ converges.*

Proof. See [1]. ☐

Lemma 3.3. *For any symbol f, the associated B-T operator $T_n(f)$ has $\|T_n(f)\| \leq K_1$ for some constant K_1 and any entry of $T_n(f)$ for some n is bounded by some constant (independent of n).*

Proof. This is part of the symbolic calculus of B-T operators. See [3]. ☐

Lemma 3.4. *For the perturbed operator $P^{(n)}(\epsilon)$ as in the averaging method, we have that $\|V_n\| \leq K_2 n$ for some constant K_2.*

Proof. This may be seen by utilizing Egorov's theorem for B-T operators (also found in [3]) and the symbol calculus of B-T operators. Heuristically, the sum of two B-T operators is a B-T operator asymptotically, and this implies that the same holds for integrals of B-T operators. □

Now, with these lemmas in hand, we have a way of selecting ϵ in terms of n that will tell us that the perturbed eigenvalues of our special case (eigenvalues of $P^{(n)}(\epsilon) = T_n(x_3) + i\epsilon T_n(x_3x_1)$, which we shall discuss in Section 4) are close to the unperturbed eigenvalues. First, we show that the averaging method gives good asymptotic results. Note that there is no need for Q_n to be self-adjoint for the purpose of this theorem.

Theorem 3.5. *Let* $P^{(n)}(\epsilon) = A_n + \epsilon\, Q_n$ *with* A_n *a diagonal B-T operator as in the averaging method and* Q_n *a B-T operator. Let* $\epsilon = \frac{1}{2\hat{C}n}$*, where* $\hat{C}$ *is a constant. Then, for any eigenvalue* λ *of* $A_n + \epsilon Q_n^{av}$*, there exists* λ_j *an eigenvalue of* $P^{(n)}(\epsilon)$ *such that* $|\lambda - \lambda_j| < C^* \frac{1}{n^2}$ *for a fixed constant* C^* *and all* j*.*

Proof. (Note that we use the notation $P^{(n)}_{conj}(\epsilon)$ to denote $e^{\epsilon V_n} P^{(n)}(\epsilon) e^{-\epsilon V_n}$ and will switch between the two expressions at will).

By the averaging method, we may conjugate $P^{(n)}(\epsilon)$ to $P^{(n)}_{conj}(\epsilon) = e^{\epsilon V_n} P^{(n)}(\epsilon) e^{-\epsilon V_n} = A_n + \epsilon\, Q_n^{av} + \ldots$ and using the uniqueness of Taylor expansions (viewing this as a function from $\mathbb{R} \to \mathbb{C}^{n^2}$ and using the method for vector-valued functions of a real variable) we find that the remainder term is the following:

$$R_n(\epsilon) = P^{(n)}(\epsilon) - (A_n + \epsilon\, Q_n^{av}) \le \epsilon^2 \frac{1}{2!} \sup_{\tau\in[0,\epsilon]} \left\| \left(e^{-\tau V_n} P^{(n)}(\tau) e^{\tau V_n} \right)'' \right\|$$

Evaluating the derivatives of the function above gives us that

$$R_n(\epsilon) \le \epsilon^2 \frac{1}{2} \sup_{\tau\in[0,\epsilon]} \left\| [V_n, [V_n, e^{-\tau V_n} P^{(n)}(\tau) e^{\tau V_n}] + e^{-\tau V_n} Q_n e^{\tau V_n}] + [V_n, e^{-\tau V_n} Q_n e^{\tau V_n}] \right\|$$

Now, note that $\left\| e^{-\tau V_n} \right\| \le e^{\|\tau V_n\|}$, so $e^{-\tau V_n}$ is an operator of order 0 (is bounded by a constant) by our earlier remark concerning V_n. By the symbolic calculus (see [3]), we have that the commutator of an operator of order 1 (V_n in this case) and an operator of order 0 (every other operator in this expansion) is a B-T operator of order 0, so we have that

$$R_n(\epsilon) \le \frac{1}{(\hat{C}n)^2} \frac{1}{2} K$$

Now we will make (what should now seem as) a mysterious choice of $\hat{C}$ to make the next leg of our proof work. By a previous lemma, the entries of $T_n(Q_n)$ are bounded for all n by some constant (call it M^*). If we let $M = \max\{1, M^*\}$ and $\hat{C} = \sqrt{2KM}$, then $R_n(\tau) \le \frac{1}{4Mn^2}$.

Now we show that $P^{(n)}(\epsilon^*)$ is diagonalizable for any $\epsilon^* < \epsilon$ so that we may utilize the Bauer-Fike theorem for the last leg of the proof. To show that $P^{(n)}(\epsilon)$ is diagonalizable, we show that $e^{\epsilon V} P^{(n)}(\epsilon) e^{-\epsilon V}$ is diagonalizable. To do this, we note that the averaging method, along with our bound on the remainder gives us that the eigenvalues of $P^{(n)}(\epsilon)$ (which are also the eigenvalues of $P^{(n)}_{conj}(\epsilon)$) may be found by investigating the eigenvalues of $A_n + \epsilon Q_n^{av}$ with the Bauer-Fike theorem. The above bound on the remainder term along with the Bauer-Fike theorem tells us that given an eigenvalue $\hat{\lambda}_i \in \Lambda(A + \epsilon Q_n^{av} + R_n(\epsilon))$, there exists an eigenvalue $\lambda_i \in \Lambda(A_n + \epsilon Q_n^{av})$ such that $|\hat{\lambda}_i - \lambda_i| < \frac{1}{4Mn^2}$ (note that this is half the distance between eigenvalues of A_n). Now, fix n and note that $\lambda_i = \lambda_i^* + \epsilon\lambda_i^{(1)}$ for some λ_i^* an eigenvalue of A_n and some $\lambda_i^{(1)} \in \mathbb{C}$ (by the fact that Q_n^{av} and A_n are both diagonal). We claim that for $\epsilon \in [0, \frac{1}{4Mn^2}]$, $\hat{\lambda}_i$ may not leave the ball of this form and enter another ball of this form around some λ_k. First note that by our choice of ϵ and by the form of the expansion for λ_i, we have that $|\lambda_i - \lambda_i^*| = |\epsilon\lambda_i^{(1)}| < \frac{1}{4n^2}$. This implies that $|\lambda_i^* - \hat{\lambda}_i| < \frac{1}{4Mn^2} + \frac{1}{4n^2} < \frac{1}{2n^2}$, so all each $\hat{\lambda}_i$ lies in one of the n disjoint balls of radius $\frac{1}{2n^2}$ around each unperturbed eigenvalue. Now, note that if $\hat{\lambda}_i$ left the ball around λ_i it could not be continuous, much less analytic, as all balls of this radius are disjoint. However, by an earlier lemma, we have that as $\|\epsilon Q_n\| < \frac{1}{2n^2}$, all eigenvalues of $\Lambda(A_n + \epsilon Q_n)$ are analytic functions of ϵ. So for ϵ in this range, we have that $\hat{\lambda}_i$ must stay in one single ball around an unperturbed eigenvalue. So, $\hat{\lambda}_i$ lies in the ball around λ_i^* for all $\epsilon \in [0, \frac{1}{4Mn^2}]$. Now, this means that for such ϵ, we have n distinct eigenvalues of $A_n + \epsilon Q_n^{av} + R_n(\epsilon)$. So, as no two $\hat{\lambda}$ coalesce for $\epsilon \in [0, \frac{1}{2Kn^2})$, and as A_n had n eigenvalues, it must be diagonalizable for any such ϵ.

Now, to prove the final bound we apply the Bauer-Fike theorem in the opposite manner. That is, we consider our "unperturbed" operator to be $P^{(n)}(\epsilon)$ and the perturbation to be $E = R_n(\epsilon)$. The Bauer-Fike theorem then tells us that given an eigenvalue $\lambda \in \Lambda(A_n + \epsilon Q_n^{av})$, there exists an eigenvalue $\lambda_j \in \Lambda(A_n + \epsilon Q_n^{av} + R_n(\epsilon))$ such that $|\lambda - \lambda_j| < \frac{1}{4Mn^2}$. So, we are done. □

4. RELATION TO NUMERICAL RESULTS

Before we discuss specific examples, we give a brief introduction to the concept of the ϵ- pseudospectrum.

4.1. **Pseudospectra.**

Definition 4.1. The ϵ-pseudospectrum of a matrix A, $\Lambda_\epsilon(A)$ is defined to be the set $\{z \in \mathbb{C} |\, ||(A - zI)^{-1}|| \geq \frac{1}{\epsilon}\}$.

There is quite a bit of activity in applications and theory of pseudospectral methods: applying pseudospectral methods to obtain information about eigenvalues, etc. One particularly interesting example is [2], which motivated the start of our own research. The interested reader should also peruse the Pseudospectrum Gateway website. We will neither use nor prove any theorems concerning pseudospectra, but as it has lingered in the back of our minds while examining this problem, we will state one theorem that will allow the reader unfamiliar with the basics of the theory to follow our later discussion of numerical evidence and examples.

In some sense, the pseudospectrum measures the departure from normality of a matrix. The greater the departure from normality, the less the pseudospectrum mirrors the eigenvalues for small ϵ. The following theorem, the proof of which is a direct result of the Bauer-Fike theorem stated in section 4 and the spectral theorem, tells us that the pseudospectrum of a Hermitian operator is not terribly interesting (this generalizes to normal operators as well).

Theorem 4.2. *If A is Hermitian, $A = VDV^{-1}$ with D a real diagonal matrix and $\Lambda_\epsilon(A) \subset \Lambda(A) + \Delta_\epsilon$ where Δ_ϵ denotes the closed ball in $\mathbb{C}$ about 0.*

For examples of pseudospectra of a non-normal operators, the interested reader can consult [2] or the Pseudospectrum Gateway. Now we will examine a few specific examples to which our main theorem applies.

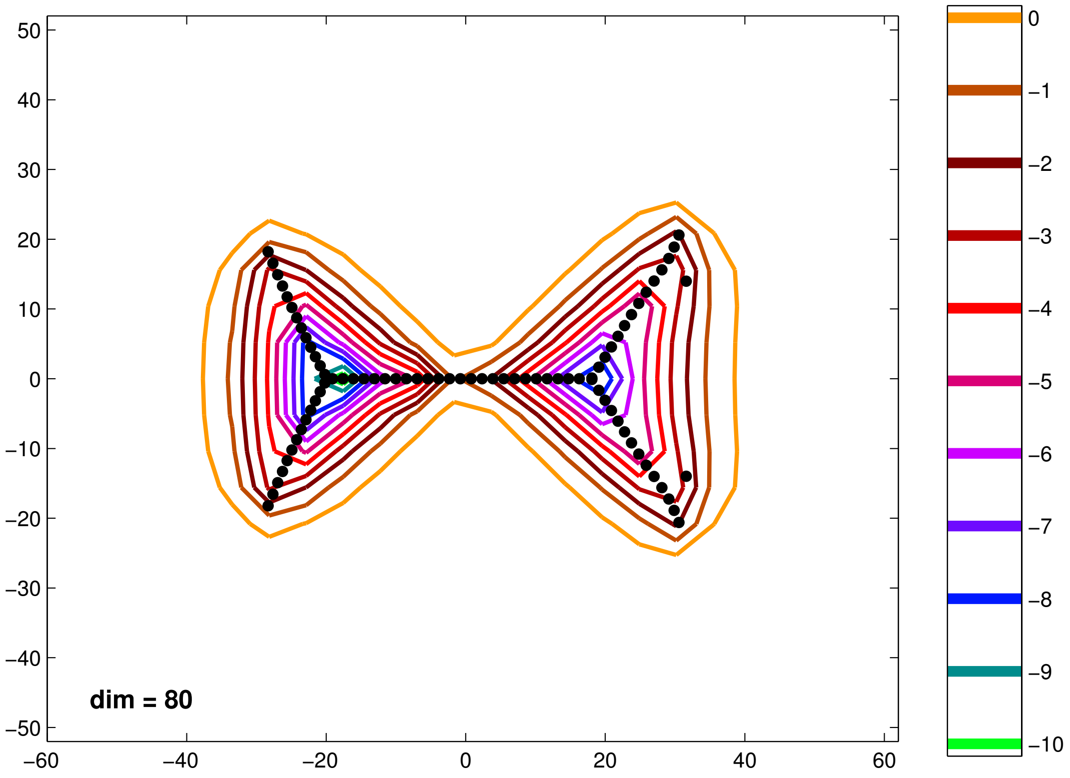

Figure 1:
$T_{80}(x_3 + 0.06x_1x_3)$

4.2. **Examples.** As remarked before, the study of the operators resulting from the symbol $f_\epsilon(x_1, x_3) = x_3 + i\epsilon x_1x_3$ has motivated nearly all of this work. Now we will discuss why this particular example is of interest. The pseudospectral portrait below for the symbol $f_1(x_1, x_3)$ shows the characteristic "winged" spectrum and lemniscate pseudospectrum for fixed $n = 80$ and different values of ϵ. Now, it is important to note that in this case we have $Q_n^{av} = 0$ for all n, which may be seen by a simple computation of Q_n and our earlier remark on determining the entries of the averaged operator. So, by the theorem that we have just proved, the spectrum lies very close to the real line for small values of ϵ. However, still fixing n, as ϵ increases from 0, the spectrum "unzips" to form a figure that is generally similar to the one below. It is unclear where the splitting points lie in general, as it is unclear at which value of ϵ the spectrum begins to unzip. It is possible that the perturbative approach could be extended to cover the case of $\epsilon > \frac{1}{2\hat{C}n}$, but it seems quite likely that a new approach would be required to explain the splitting phenomenon observed in this case. Any techniques used to explain this spectral splitting would likely apply to our next example on the cylinder as well.

There are also symbols on the cylinder that exhibit this sort of splitting behavior. For example, the Figure 2 was obtained from the symbol $f_\epsilon(p, \theta) = p + i\epsilon(p^2 - p)$. By our previous discussion of quantization of symbols on the cylinder, we may see that this symbol satisfies the hypotheses of our main theorem. As in the previous case, $Q_n^{av} = 0$ for this symbol.

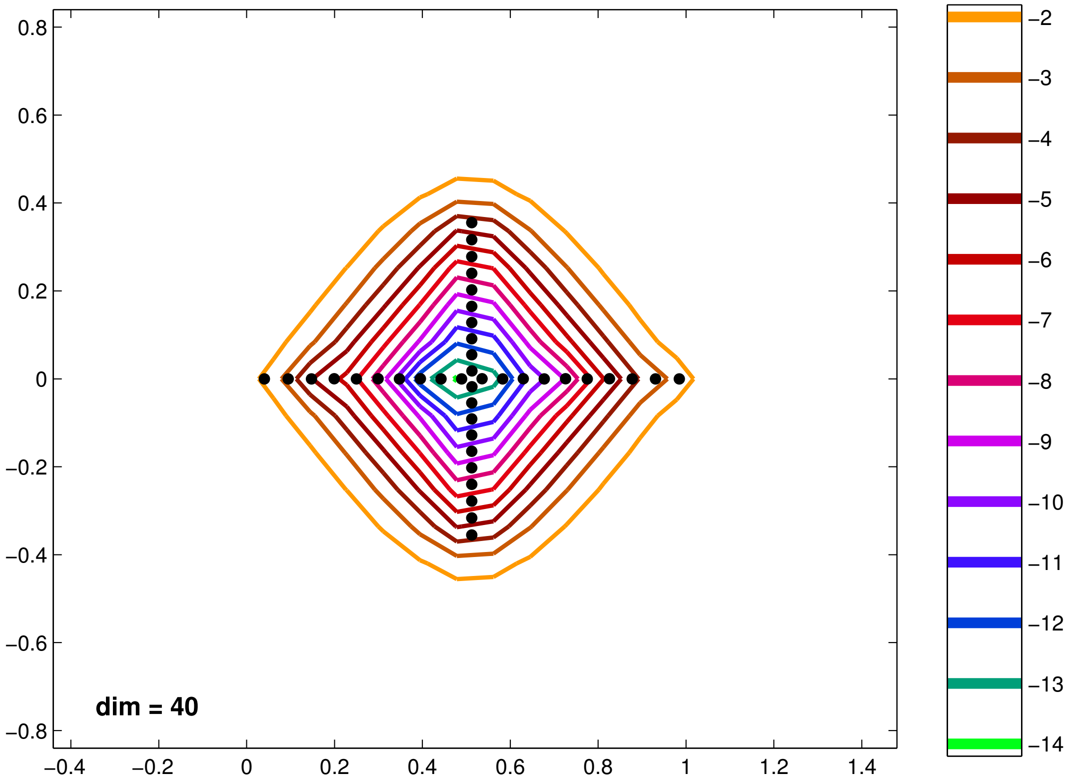

Figure 2:
$T_{40}(p) + \epsilon T_{40}(i(p^2 - p))$ for large ϵ ($\epsilon \approx 2$)

In Figure 3, we see that for small ϵ (here ϵ is relatively large with respect to the choice made in our main theorem) the averaged operator mimics the spectrum fairly well, although this breaks down quickly ($\epsilon \approx 0.2$). Of course, as $Q_n^{av} = 0$ for all n in this case, the averaged spectrum does not exhibit splitting behavior (or move at all, for that matter). However, our next example provides us with operators such that Q_n^{av} is not trivial.

In Figure 4, we examine the operators associated with the symbol $f(p, \theta) = p + i(p^2 - p)\cos^2(\theta)$. Note that in this case, $Q_n^{av} \neq 0$, which may be seen by remembering the formula $\cos^2(\theta) = \frac{1}{2} + \frac{\cos(2\theta)}{2}$ (so $Q_n^{av} = \frac{1}{2}T_n(p^2 - p)$). Here we see that the averaged operator provides a good approximation until approximately $\epsilon \approx \frac{1}{2}$. This behavior, along with that observed in the previous examples, suggests that the bounds obtained in our main theorem should be able to be improved dramatically.

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5. SOURCE CODE

cylindermatrixfirstpart.m

```
function output = cylindermatrixfirstpart( input )
% First part of a B-T operator with symbol defined on the cylinder.
% The full operator's symbol is f(p,theta)=p+i(p^2-p)cos(theta)
%
% This function returns the (input)x(input) B-T operator on the cylinder
% with symbol f(p,theta)=p
%
% Exhibits the same sort of splitting of the spectrum as
% our symbol on the sphere of f(x1,x3)=x3+i*eps*x3*x1 if we take
% cylindermatrixfirstpart(n) + eps*cylindermatrixsecondpart(n)
N=input;
for j=1:N
for k=1:N
if j==k
A(j,k)=(j/N);
else
A(j,k)=0;
end
end
end
output=A;
```

cylindermatrixsecondpart.m

```
function output = cylindermatrixsecondpart( input )
% Second part of a B-T operator with symbol defined on the cylinder.
% The full operator's symbol is f(p,theta)=p+i(p^2-p)cos(theta)
%
% This function returns the (input)x(input) B-T operator on the cylinder
% with symbol f(p,theta)=i(p^2-p)cos(theta)
%
% Exhibits the same sort of splitting of the spectrum as
% our symbol on the sphere of f(x1,x3)=x3+i*eps*x3*x1 if we take
% the nxn operator
% cylindermatrixfirstpart(n) + eps*cylindermatrixsecondpart(n)
N=input;
for j=1:N
for k=1:N
if j==k+1
```

```
B(j,k)=i*((k/N)^2-(k/N));
elseif j==k-1
B(j,k)=i*((j/N)^2-(j/N));
else
B(j,k)=0;
end
end
end
output=B;
```

cylindermatrixfirstpartQavnot0.m

```
function output = cylindermatrixfirstpartQavnot0( input )
% This is the first part of a B-T operator on the sphere with Q^(av)
% non-zero.
% This function returns the (input)x(input) B-T operator on the cylinder
% with symbol f(p,theta)=p
%
% To obtain the full nxn operator with symbol
% f(p,theta)=p+i*eps*(p^2-p)cos^2(theta), run
% cylindermatrixfirstpartQavnot0(n) + i*eps*cylindermatrixsecondpartQavnot0(n)
N=input;
for j=1:N
for k=1:N
if j==k
A(j,k)=(j/N);
else
A(j,k)=0;
end
end
end
output=A;
```

cylindermatrixsecondpartQavnot0.m

```
function output = cylindermatrixsecondpartQavnot0( input )
% This is the first part of a B-T operator on the sphere with Q^(av)
% non-zero
% This function returns the (input)x(input) B-T operator on the
% cylinder with symbol f(p,theta)=i(p^2-p)cos^2(theta)
%
% To obtain the full nxn operator with symbol
% f(p,theta)=p+i*eps*(p^2-p)cos^2(theta), run
% cylindermatrixfirstpartQavnot0(n) + i*eps*cylindermatrixsecondpartQavnot0(n)
N=input;
a=input;
for j=1:N
for k=1:N
if j==k
B(j,k)=(1/2)*i*((j/N)^2-(j/N));
elseif j==k+2
B(j,k)=i*((k/N)^2-(k/N))/2;
```

```
elseif j==k-2
B(j,k)=i*((j/N)^2-(j/N))/2;
else
B(j,k)=0;
end
end
end
output=B;
```

firstorderapproxplot.m

```
function output = firstorderapproxplot(A,Q,eps)
% Plots the eigenvalues of A+eps*Q as red circles as well as the
% eigenvalues of the first-order averaged operator.
% Note, the 0.00000001*i in a few lines shouldn't have to be there,
% but if it's not there, matlab doesn't like to treat its entries
% as imaginary numbers and plots them as if they were purely
% imaginary.  I have no clue why.
[V,D1]=eig(A+eps*Q);
D=D1+0.00001*i;
s=size(D);
for j=1:s(1,1)
hold on;
Qav=firstorderaveraging(Q)+0.00000001*i*eye(s(1,1));
appr=A+eps*Q+0.00000001*i*eye(s(1,1));
plot(D(j,j),'ro');
plot(appr(j,j),'bx');
end
% Modify this for different axes.
% The syntax is axis([xmin xmax ymin ymax]);
axis([0 1 -0.25 0.25]);
```

firstorderaveraging.m

```
function output = firstorderaveraging(Q)
% Apply the method of averaging up to first order and return
% the averaged matrix.  To see why this works, see the remark
% about determining the entries of the averaged matrix.
s=size(Q);
n=s(1,1);
Id=eye(n);
for j=1:n
for k=1:n
if j==k
Qav(j,k)=dot(Q*Id(:,k),Id(:,k));
else Qav(j,k)=0;
end
end
end
output=Qav;
```

giveusthecommutatormatrix.m

```
function output = giveusthecommutatormatrix(A,Q)
```

```
% Computation of the matrix V_n from A_n and Q_n
% Takes as arguments A_n and Q_n
% Returns V_n s=size(Q);
syms x t;
n=s(1,1);
for j=1:n
for k=1:n
if j==k
matrix1(j,k)=exp(-i*(n+1)*A(j,k)*x);
else matrix1(j,k)=0;
end
end
end
for j=1:n
for k=1:n
if j==k
matrix2(j,k)=exp(i*(n+1)*A(j,k)*x);
else
matrix2(j,k)=0;
end
end
end
matrix=matrix1*Q*matrix2;
for j=1:n
for k=1:n
f=matrix(j,k);
commutatormatrix(j,k)=(((n+1)*i)/(2*pi))*double(int((int(f,x,0,t)),t,0,2*pi));

end
end
output=commutatormatrix;
```

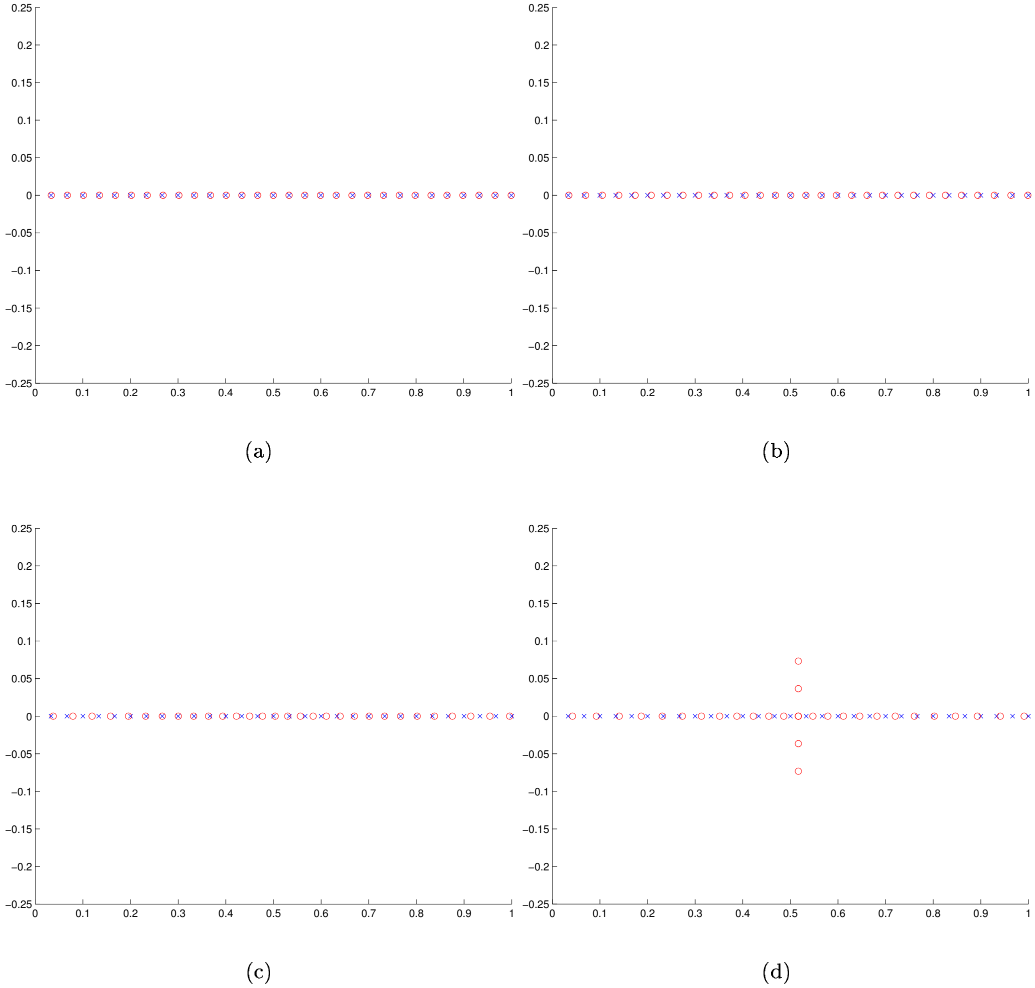
spherematrixfirstpart.m

```
function output = spherematrixfirstpart(input)
%This function returns the first part of a
%(2*input + 1)x(2*input + 1)
%B-T operator on the sphere with symbol f(x1,x3)=x3+i*eps*x3*x1
%That is, the B-T operator that this function returns has symbol
%f(x1,x3)=x3
%To obtain the nxn final operator, replace eps with the desired
%number and run this command:
%spherematrixfirstpart(n) + eps*spherematrixsecondpart(n)
N=2*input+1;
for i=1:N
for j=1:N
if i==j
Z(i,j)=(j-J-1)/N;
else
Z(i,j)=0;
```

```
end
end
end
output=Z;
```

spherematrixsecondpart.m

```
function output = spherematrixsecondpart(input)
%This function returns the first part of a
%(2*input + 1)x(2*input + 1)
%B-T operator on the sphere with symbol f(x1,x3)=x3+i*eps*x3*x1
%That is, the B-T operator that this function returns has symbol
%f(x1,x3)=i*x3*x1
%To obtain the nxn final operator, replace eps with the desired number
%and run this command:
%spherematrixfirstpart(n) + eps*spherematrixsecondpart(n)
N=2*input+1;
for i=1:N
for j=1:N
if i==j
Z(i,j)=(j-J-1)/N;
else Z(i,j)=0;
end
end
end
for i=1:N
for j=1:N
if i==j+1
X(i,j)=sqrt((i+j)*(2*N-i-j))/(2*N);
elseif j==i+1
X(i,j)=sqrt((i+j)*(2*N-i-j))/(2*N);
else
X(i,j)=0;
end
end
end
output=(Z*X+X*Z)/2;
```

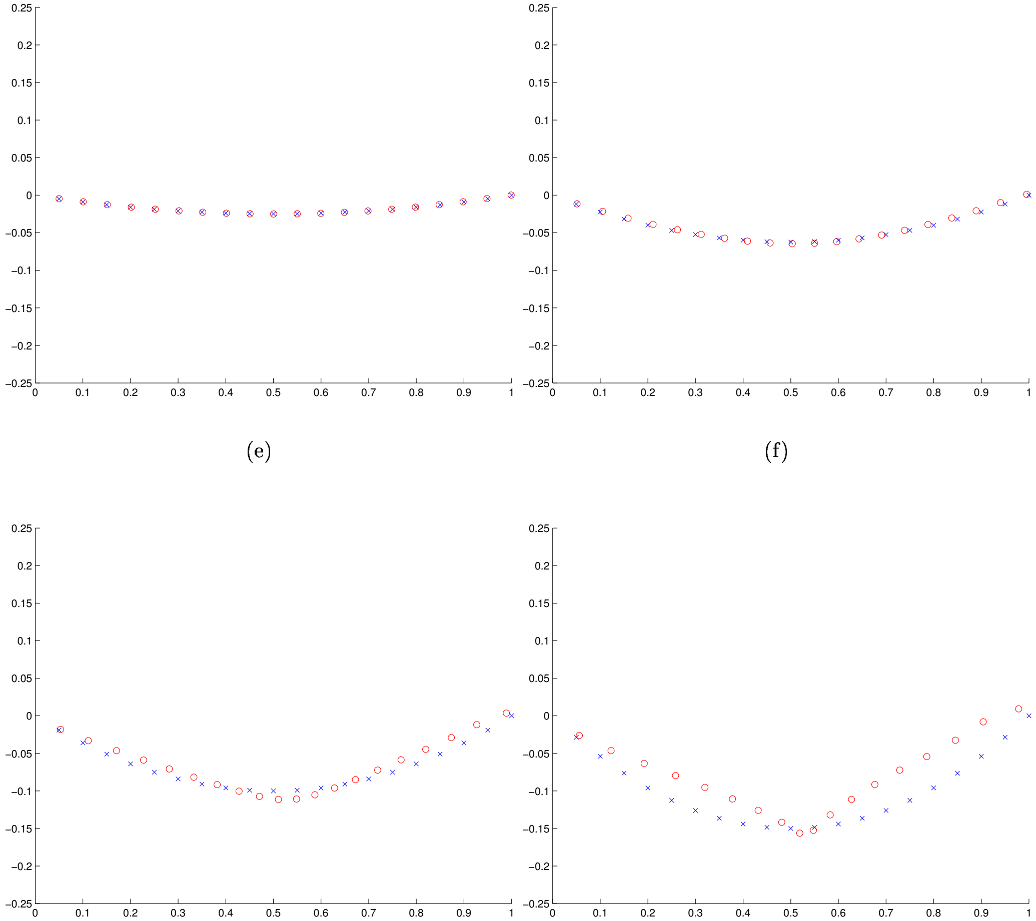


(a) (b)

(c) (d)

Figure 3: Comparison of eigenvalues of **(o)** $T_{30}(p) + \epsilon T_{30}(i(p^2 - p)\cos(\theta))$ with eigenvalues of **(x)** $T_{30}(p) + \epsilon T^{av}_{30}(i(p^2 - p)\cos(\theta))$

(a) $\epsilon = \frac{1}{10}$(b) $\epsilon = \frac{2}{10}$(c) $\epsilon = \frac{4}{10}$(d) $\epsilon = \frac{6}{10}$



(e) (f)

(g) (h)

Figure 4: Comparison of eigenvalues of **(o)** $T_{20}(p) + \epsilon T_{20}(i(p^2 - p)\cos^2(\theta))$ with eigenvalues of **(x)** $T_{20}(p) + \epsilon T_{20}^{av}(i(p^2 - p)\cos^2(\theta))$

(e) $\epsilon = \frac{2}{10}$ (f) $\epsilon = \frac{5}{10}$ (g) $\epsilon = \frac{8}{10}$ (h) $\epsilon = \frac{12}{10}$