University of Michigan
Undergraduate Mathematics Competition 15

April, 1998

Problem 1. Let $x_1, x_2, \ldots, x_n$ be positive real numbers satisfying $x_1 = 19$, $x_2 = 98$, and $x_k^3 \le x_{k-1}^2 x_{k+1}$ for $2 \le k \le n-1$. Determine the largest of $x_1, x_2, \ldots, x_n$.

Problem 2. The Fibonacci sequence is defined by $F_n = F_{n-1} + F_{n-2}$ for all $n \ge 2$, with $F_0 = 0$, $F_1 = 1$. Prove, for all n, that F_n and F_{n+2} have no common factor > 1.

Problem 3. Prove that $\cos\left(\frac{2\pi}{5}\right) = \frac{-1+\sqrt{5}}{4}$.

Problem 4. Prove that, if the coordinates of the vertices of a triangle are integers, then the area of the triangle is either an integer or an integer plus $\frac{1}{2}$.

Problem 5. A bird walks at a constant speed along the x-axis from $(0,0)$ to $(1,0)$. At the same time, a cat walks at the same speed along the y-axis from $(0,1)$ to $(0,0)$. Instead of watching where it is going, the cat always looks directly at the bird. The line of sight, from cat to bird, sweeps out a region bounded by the bird's path, the cat's path, and a curve. Find the equation of this curve. Also describe the shape of the curve, as accurately as you can, in synthetic terms (i.e., without coordinates).

Problem 6. Find all real solutions of $1998^x + 1999^x = 1997^x + 2000^x$. [Hint: Mean value theorem]

Problem 7. Let $f(x)$ denote the unique function defined on the interval $[1, 1+e^{\pi/2}]$, taking values in the interval $[0, \pi/2]$, and satisfying $\sin f(x) + e^{f(x)} = x$ for all x in its domain. Calculate

$$\int_1^{1+e^{\pi/2}} f(x)\,dx.$$

Problem 8. Suppose a, b, c, d are positive real numbers such that the sets $\{a+b, ab\}$ and $\{c+d, cd\}$ are equal.

(a) Give an example showing that $\{a, b\}$ can be different from $\{c, d\}$.

(b) Find an additional condition on a and b ensuring that $\{a, b\} = \{c, d\}$.

Problem 9. Suppose that f is a real-valued function on the positive real numbers such that, for every positive real x, $f(x) = f(x^2) + f(x^4)$. Assume that f is continuous at $x = 1$. Prove that there exists a real constant c such that $f(x^2) = cf(x)$ for all $x > 0$, and determine the value of c.

Problem 10. Let

$$F(x,y) = \sum_{k=0}^{\infty} (1+x)(1+2x)\ldots(1+(k-1)x) \cdot \frac{y^k}{k!}.$$

Prove that

$$F(-x,-y) = \frac{1}{F(x,y)}.$$