UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 21
MARCH 27, 2004

Instructions. Write on the front of your blue book your student ID number. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must **prove** that your answers are correct even when the question doesn't say "prove". There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun. No calculators are allowed.

Problem 1. Show that $n! + 2004$ is not a perfect square for any positive integer n.

Problem 2. Twenty-four delegates sit around a round table. Two delegates can speak to each other if at most 4 people sit between them. After a break each person sits down again, not necessarily in the same seat as before. Show that there exist two delegates who are able to speak to each other before and after the break.

Problem 3. For positive integers n, let $S(n) = \lfloor\sqrt{n}\rfloor$ denote the integer part of the positive square root of n. Call a non-empty set of positive integers X *rooted* if whenever $n, m \in X$, then $S(n) + S(m) \in X$ (including the case when $n = m$). Find, with proof, all rooted sets of positive integers that do not contain 4.

Problem 4. A tetrahedron has as base equilateral triangle ABC and a fourth vertex V not in the plane of $\triangle ABC$ such that $|VA| = |VB| = |VC|$. Thus, $\triangle VAB$, $\triangle VBC$ and $\triangle VCA$ are congruent isosceles triangles. Let α be the angle at vertex V in each of these triangles, and let β be the interior angle of the tetrahedron between the planes of any two of these triangles. Express $y = \cos(\beta)$ in terms of $x = \cos(\alpha)$.

Problem 5. Let a be a fixed real number strictly between -2 and 2 and let A_n be the $n \times n$ matrix which has a along the diagonal, 1's along the super- and sub-diagonals and 0's everywhere else. For example, we have

$$A_2 = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}, \quad A_3 = \begin{pmatrix} a & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & a \end{pmatrix}, \quad A_4 = \begin{pmatrix} a & 1 & 0 & 0 \\ 1 & a & 1 & 0 \\ 0 & 1 & a & 1 \\ 0 & 0 & 1 & a \end{pmatrix}.$$

Prove that $\det(A_n)$ is negative for infinitely many n.

Problem 6. Show that for every positive integer k there is an integer n_k whose decimal expansion uses only the digits 1 and 2, such that 2^k divides n_k. For example $2 \mid 2$, $4 \mid 12$ and $8 \mid 112$.

Problem 7. Show that for any polynomial $p(z) \in \mathbb{R}[z]$ there is a polynomial $q(z) \in \mathbb{R}[z]$ such that $p(z)q(z) = \sum_k c_k z^k$ has the property that if c_k is nonzero then k is prime. For example, if we are given $p(z) = 1 + 2z + 3z^2$, then we can take $q(z) = 2z^2 - 3z^3$, for then $p(z)q(z) = 2z^2 + z^3 - 9z^5$ has the required form.

Problem 8. Given a natural number n let $\mathcal{S}_n$ denote the set of all integers m such that $\{n/m\} \geq 1/2$, where $\{x\} = x - \lfloor x \rfloor$ denotes the "fractional part" of x. Prove that

$$\sum_{m \in \mathcal{S}_n} \phi(m) = n^2,$$

where $\phi(m)$ is Euler's ϕ-function (that is, the number of integers k in $\{1, 2, \ldots, m\}$ for which the greatest common divisor of k and m is equal to 1). For example $\mathcal{S}_6 = \{4, 7, 8, 9, 10, 11, 12\}$, and we see that $\phi(4) + \phi(7) + \phi(8) + \phi(9) + \phi(10) + \phi(11) + \phi(12) = 2 + 6 + 4 + 6 + 4 + 10 + 4 = 36 = 6^2$.

Problem 9. Suppose that you are playing the following game. First a random number x_0 is chosen from the interval $[0, 1]$ (with a uniform distribution). In round 1, a second number x_1 will be chosen randomly from the interval $[0, 1]$ — but before this, you have to guess if this number is going to be *higher* or *lower* than the previous number x_0. If you were wrong, then the game is over. Otherwise you will proceed to round 2. The other rounds proceed similarly: In round k you first guess "higher" or "lower". Then a random number x_k is chosen from the interval $[0, 1]$. If you said "higher" and $x_k > x_{k-1}$ or you said "lower" and $x_k < x_{k-1}$ then you proceed to round $k + 1$. Otherwise the game will end in round k. Assume that you are using a strategy that in each round maximizes the probability to proceed to the next round.

(a) What is the probability that the game will last 3 or more rounds? In other words, what is the probability that the first two guesses will be right?
(b) What is the expected number of rounds that will be played?

Problem 10. Suppose that $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ are real numbers such that

$$y_1 \geq y_2 \geq \cdots \geq y_n > 0$$

and

$$x_1 x_2 \cdots x_k \geq y_1 y_2 \cdots y_k, \qquad \text{for } k = 1, 2, \ldots, n.$$

Prove that

$$x_1 + x_2 + \cdots + x_n \geq y_1 + y_2 + \cdots + y_n.$$