

UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 22
APRIL 9, 2005

Instructions. Write on the front of your blue book your student ID number. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must **prove** that your answers are correct even when the question doesn't say "prove". There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun. No calculators are allowed.

Problem 1. Let \mathcal{A} be a finite set of integers. Let D_e denote the number of all pairs (a, b) with $a, b \in \mathcal{A}$ such that $a - b$ is even, and let D_o denote the number of such pairs for which $a - b$ is odd. Show that $D_e \geq D_o$.

Problem 2. Let $k > 0$ be an integer. There are $2k$ clubs whose members are chosen from a set of M people. Each club has at least $M/2$ members. Let p_k be the fraction of the M people who belong to at least k clubs. Show that $p_k \geq \frac{1}{k+1}$.

Problem 3. Suppose that $a, b \in \mathbb{R}$ with $a \geq 0$, $b \geq 0$ and $a^2 + b^2 \leq 1$. Prove that the area of the set

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \text{ and } (x - a)(y - b) \geq 0\}$$

is equal to $\frac{\pi}{2} + 2ab$.

Problem 4. Does there exist a polynomial $P(x)$ with integral coefficients, such that $P(0) = 1$, $P(2) = 3$ and $P(4) = 9$?

Problem 5. Let a_n be the integer resulting from stringing together the decimal expansions of $1, 2, 4, \dots, 2^n$, i.e., of the powers of 2 up to 2^n . Thus $a_1 = 12$ and $a_7 = 1248163264128$. Find

$$\lim_{n \rightarrow \infty} a_n^{(1/n^2)}.$$

Problem 6. The Fibonacci numbers are defined by $F_0 = F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for all positive integers n . Show that there exists a nonzero polynomial $P(x, y)$ such that $P(F_n, F_{n-1}) = 0$ for all positive integers n .

Problem 7. Let \mathcal{A} be a collection of n vectors in the plane. Show that there is a subset $\mathcal{B} \subseteq \mathcal{A}$ such that

$$\left| \sum_{v \in \mathcal{B}} v \right| \geq \frac{1}{\pi} \sum_{v \in \mathcal{A}} |v|.$$

Problem 8. Show that if $0 < j \leq k < n$ are integers, then

$$\gcd\left(\binom{n}{j}, \binom{n}{k}\right) > 1$$

where gcd denotes the *greatest common divisor*.

Problem 9. Of a 2005-gon in the Euclidean plane, all the sides have equal length. Prove that one of the vertices has at least one coordinate which is irrational.

Problem 10. Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function with a continuous third derivative such that $f(x) > 0$, $f'(x) > 0$, $f''(x) > 0$, $f'''(x) > 0$ and $f'''(x) \leq f(x)$ for all $x \in \mathbb{R}$. Prove that $f'(x) < 2f(x)$ for all $x \in \mathbb{R}$.