UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 22
APRIL 9, 2005

Instructions. Write on the front of your blue book your student ID number. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must **prove** that your answers are correct even when the question doesn't say "prove". There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun. No calculators are allowed.

Problem 1. Let $\mathcal{A}$ be a finite set of integers. Let D_e denote the number of all pairs (a, b) with $a, b \in \mathcal{A}$ such that $a - b$ is even, and let D_o denote the number of such pairs for which $a - b$ is odd. Show that $D_e \geq D_o$.

Solution. Suppose that $\mathcal{A}$ has x even and y odd elements. $a - b$ is even if and only if a and b are both even or both odd. So we have $D_e = x^2 + y^2$. Similarly, $a - b$ is odd if a is even and b odd or a is odd and b is even. Therefore $D_o = xy + yx = 2xy$. Now $D_e - D_0 = x^2 + y^2 - 2xy = (x - y)^2 \geq 0$. □

Problem 2. Let $k > 0$ be an integer. There are $2k$ clubs whose members are chosen from a set of M people. Each club has at least $M/2$ members. Let p_k be the fraction of the M people who belong to at least k clubs. Show that $p_k \geq \frac{1}{k+1}$.

Solution. Consider the number of pairs consisting of a person and a club to which the person belongs. The number of such pairs is $\geq (2k)(M/2) = kM$. From the $(1 - p_k)M$ people who belong to at most $k - 1$ clubs, we get at most $(1 - p_k)M(k - 1)$ such pairs, while from the other $p_k M$ people we get at most $p_k M(2k)$ such pairs. Therefore $(1 - p_k)M(k - 1) + p_k M(2k) \geq kM$. Divide by M and rewrite this as $(k - 1) - p_k(k - 1) + p_k(2k) \geq k$ or $p_k(k + 1) \geq 1$, yielding $p_k \geq \frac{1}{k+1}$. □

Problem 3. Suppose that $a, b \in \mathbb{R}$ with $a^2 + b^2 \leq 1$. Prove that the area of the set

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \text{ and } (x - a)(y - b) \geq 0\}$$

is equal to $\frac{\pi}{2} + 2ab$.

Solution. Let us assume that $a, b \geq 0$. Let $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Define

$$A_{-1} = C \cap \{(x, y) \mid x \leq -a\},$$

$$A_0 = C \cap \{(x, y) \mid -a \leq x \leq a\},$$

$$A_1 = C \cap \{(x, y) \mid x \geq a\}.$$

Similarly, define

$$B_{-1} = C \cap \{(x,y) \mid y \leq -b\},$$
$$B_0 = C \cap \{(x,y) \mid -b \leq y \leq b\},$$
$$B_1 = C \cap \{(x,y) \mid y \geq b\}.$$

For a set U we denote its area by $[U]$. The union of all $A_i \cap B_j$ $-1 \leq i,j \leq 1$ is equal to the unit disc, so

$$\sum_{-1 \leq i,j \leq 1} [A_i \cap B_j] = \pi.$$

Using symmetry, we also have $[A_i \cap B_j] = [A_{|i|} \cap B_{|j|}]$. So we have

$$[A_0 \cap B_0] + 2[A_1 \cap B_0] + 2[A_0 \cap B_1] + 4[A_1 \cap B_1] = \pi.$$

On the other hand,

$$t := [\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \text{ and } (x-a)(y-b) \geq 0\}] =$$
$$= [A_0 \cap B_0] + [A_1 \cap B_0] + [A_0 \cap B_1] + 2[A_1 \cap B_1],$$

It follows that

$$2t - \pi = [A_0 \cap B_0] = 4ab$$

and $t = \frac{\pi}{2} + 2ab$. □

Problem 4. Does there exist a polynomial with integral coefficients, such that $P(0) = 1$, $P(2) = 3$ and $P(4) = 9$?

Solution. Suppose that $P(x)$ is such a polynomial. Write $P(x) = xQ(x) + 1$. Then $Q(x)$ has integral coefficients and $Q(2) = (3-1)/2 = 1$ and $Q(4) = (9-1)/4 = 2$. Write $Q(x) = 1 + (x-2)R(x)$. Then $R(x)$ is a polynomial with integral coefficients and $R(4) = (2-1)/2 = 1/2$. Contradiction! Hence, no such polynomial $P(x)$ exists. □

Problem 5. Let a_n be the integer resulting from stringing together the decimal expansions of $1, 2, 4, \ldots, 2^n$, i.e., of the powers of 2 up to 2^n. Thus $a_1 = 12$ and $a_7 = 1248163264128$. Find $\lim_{n\to\infty} \sqrt[n^2]{a_n}$.

Solution. The number of decimal digits of r is $[\log_{10} r] + 1$. Thus, the number of decimal digits of 2^k is $[k \log_{10} 2] + 1$, which differs from $k \log_{10} 2$ by at most 1. It follows that the number of decimal digits of a_n differs from $(1 + 2 + \cdots + n)\log_{10} 2$ by at most n. Thus, $\log_{10} a_n$ differs from $\frac{1}{2}n(n+1)\log_{10} 2$ by at most $n+1$. Now, $\log_{10} a_n^{1/n^2} = \frac{1}{n^2}\log_{10} a_n$ and differs from $\frac{n^2+n}{2n^2}\log_{10} 2$ by at most $\frac{(n+1)}{n^2}$. Since the latter term $\to 0$ as $n \to \infty$, $\lim_{n\to\infty} \log_{10} a_n = \frac{1}{2}\log_{10} 2 = \log_{10}\sqrt{2}$. Since 10^x is a continuous function of x, $a_n \to \sqrt{2}$. □

Problem 6. The Fibonacci numbers are defined by $F_0 = F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for all positive integers n. Show that there exists a nonzero polynomial $P(x,y)$ such that $P(F_n, F_{n-1}) = 0$ for all positive integers n.

Solution. Take $Q(x,y) = x^2 - xy - y^2$. We have

$$Q(F_{n+1}, F_n) = Q(F_n + F_{n-1}, F_n) = (F_n + F_{n-1})^2 - (F_n + F_{n-1})F_n - F_n^2 =$$
$$= F_{n-1}^2 + F_{n-1}F_n - F_n^2 = -Q(F_n, F_{n-1}).$$

So $Q(F_n, F_{n-1})^2$ is constant and equal to $Q(F_1, F_0)^2 = Q(1,1)^2 = 1$. Hence we may take $P(x,y) = Q(x,y)^2 - 1 = (x^2 - xy - y^2)^2 - 1$. □

Problem 7. Let $\mathcal{A}$ be a collection of n vectors in the plane. Show that there is a subset $\mathcal{B} \subseteq \mathcal{A}$ such that

$$\left|\sum_{v \in \mathcal{B}} v\right| \geq \frac{1}{\pi} \sum_{v \in \mathcal{A}} |v|.$$

Solution. Define $p : \mathbb{R} \to \mathbb{R}$ by $p(x) = 1$ if $x > 0$ and $p(x) = 0$ if $x \leq 0$. Let $w(t) = (\cos(t), \sin(t))$, and let $\langle \cdot, \cdot \rangle$ be the usual bilinear form on $\mathbb{R}^2$. If $v = (1,0)$ then

$$\int_0^{2\pi} \langle v, w(t) \rangle p(\langle v, w(t) \rangle\ )dt = \int_0^{2\pi} \cos(t) p(\cos(t))\, dt = \int_{-\pi/2}^{\pi/2} \cos(t)\, dt = 2.$$

If we rotate v then the value of this integral will not change. If we multiply v with a scalar $\lambda \in \mathbb{R}$, then the integral will change with a factor $|\lambda|$ So for an arbitrary vector v we get

$$\int_0^{2\pi} \langle v, w(t) \rangle p(\langle v, w(t) \rangle)\, dt = 2|v|.$$

We have

$$\int_0^{2\pi} \sum_{v \in \mathcal{A}} \langle v, w(t) \rangle p(\langle v, w(t) \rangle)\, dt = \sum_{v \in \mathcal{A}} \int_0^{2\pi} \langle v, w(t) \rangle p(\langle v, w(t) \rangle)\, dt = 2 \sum_{v \in \mathcal{A}} |v|.$$

By the mean value theorem, we have

$$\sum_{v \in \mathcal{A}} \langle v, w(t) \rangle p(\langle v, w(t) \rangle) = \frac{1}{\pi} \sum_{v \in \mathcal{A}} |v|.$$

for some t. Let $\mathcal{B}$ be the set of all v for which $\langle v, w(t) \rangle > 0$. and let $u = \sum_{v \in \mathcal{B}} v$. Then we have

$$|u| \geq \langle u, w(t) \rangle = \sum_{v \in \mathcal{B}} \langle v, w(t) \rangle = \sum_{v \in \mathcal{A}} \langle v, w(t) \rangle p(\langle v, w(t) \rangle) = \frac{1}{\pi} \sum_{v \in \mathcal{A}} |v|.$$

□

Problem 8. Show that if $0 < j \leq k < n$ then

$$\gcd\left(\binom{n}{j}, \binom{n}{k}\right) > 1$$

where gcd denotes the *greatest common divisor*.

Solution.

$$\gcd\left(\binom{n}{j},\binom{n}{k}\right) = \gcd\left(\frac{n!}{j!(n-j)!},\frac{n!}{k!(n-k)!}\right) = \frac{1}{k!(n-j)!}\gcd\left(\frac{n!k!}{j!},\frac{n!(n-j)!}{(n-k)!}\right) =$$

$$= \frac{n!}{k!(n-j)!}\gcd\left(\frac{k!}{j!},\frac{(n-j)!}{(n-k)!}\right) = \frac{n!(k-j)!}{k!(n-j)!}\gcd\left(\binom{k}{k-j},\binom{n-j}{k-j}\right) =$$

$$= \frac{n}{k}\cdot\frac{n-1}{k-1}\cdot\dots\cdot\frac{n-j+1}{k-j+1}\gcd\left(\binom{k}{k-j},\binom{n-j}{k-j}\right) > 1$$

because $n > k$. □

Problem 9. Of a 2005-gon in the Euclidean plane, all the sides have equal length. Prove that one of the vertices has at least one coordinate which is irrational.

Proof. Suppose that there exists a 2005-gon with all sides of equal length such that all its vertices have rational coordinates. Let (x_i, y_i), $i = 0, 1, \dots, 2004$ be the vertices of this polygon. We also define $x_{2005+i} = x_i$ and $y_{2005+i} = y_i$. After a translation we may assume that $(x_0, y_0) = (0, 0)$. After scaling we may assume that $x_i, y_i \in \mathbb{Z}$ for all i, and that they are not all even. Let D be the sidelength of the 2005-gon. If D^2 is divisible by 4, then we have

$$(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2 = D^2$$

for all i. It follows that $x_i - x_{i-1}$ and $y_i - y_{i-1}$ are even for all i. By induction x_i and y_i are even for all i. Contradiction! So D^2 is not divisible by 4. Since D^2 is a sum of two squares it must be either congruent 1 or 2 modulo 4. If D^2 is congruent 2 modulo 4, then it follows that $x_i - x_{i-1}$ and $y_i - y_{i-1}$ are odd for all i. By induction, x_i is even if and only if i is even. But then $x_{2005} = 0$ is odd. Contradiction! If D^2 is congruent 1 modulo 4, then $(x_i + y_i) - (x_{i-1} + y_{i-1})$ is odd for all i. By induction $x_i + y_i$ is even if and only if i is even. But then $x_{2005} + y_{2005} = 0 + 0$ is odd. Contradiction! □

Problem 10. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function with a continuous third derivative such that $f(x) > 0$, $f'(x) > 0$, $f''(x) > 0$, $f'''(x) > 0$ and $f'''(x) \le f(x)$ for all $x \in \mathbb{R}$. Prove that $f'(x) < 2f(x)$ for all $x \in \mathbb{R}$.

Solution. Using translations it is clear that it suffices to show that $f'(0) < 2f(0)$. Using Taylor's theorem, we can write for $x \le 0$:

$$0 < f(x) = f(0) + f'(0)x + \frac{f''(y)}{2}x^2$$

with $x \le y \le 0$. Since $f''' > 0$ we have $f''(y) \le f''(0)$. Hence

$$0 < g(x)$$

for $x < 0$, where

$$g(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2.$$

for all $x \leq 0$. The minimum of $g(x)$ lies at $-f'(0)/2f''(0) < 0$, hence $g(x) < 0$ for all $x \in \mathbb{R}$. Therefore, the discriminant of $g(x)$ is negative:

$$f'(0)^2 - 2f(0)f''(0) < 0.$$

We repeat a similar argument for $f'(x)$. If $x \leq 0$ then we can write

$$0 < f'(x) = f'(0) + f''(0)x + \frac{f'''(y)}{2}x^2$$

with $x \leq y \leq 0$. Now $f'''(y) \leq f(y) \leq f(0)$ because $f' > 0$. So we have

$$0 < h(x)$$

for all $x \leq 0$ where

$$h(x) = f'(0) + f''(0)x + \frac{f(0)}{2}x^2.$$

Again, it follows that $h(x) > 0$ for all $x \in \mathbb{R}$ and the discriminant of $h(x)$ must be negative:

$$f''(0)^2 - 2f(0)f'(0) < 0.$$

Combining both results yields

$$(f'(0))^4 = (f'(0)^2)^2 < (2f(0)f''(0))^2 = 4f(0)^2(f''(0))^2 < 4f(0)^2(2f(0)f'(0)) = 8f(0)^3f'(0).$$

Dividing by $f'(0)$ and taking the third root gives us

$$f'(0) < 2f(0).$$

□