A GENERALIZATION OF CONWAY NUMBER GAMES TO MULTIPLE PLAYERS

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1. Introduction

There are many different mathematical meanings of the word 'game'. Regardless of the kind of games we consider, people agree that games of n players are much more difficult to understand for $n > 2$ than for $n \leq 2$. In this paper, we consider deterministic 'combinatorial' games, i.e. games where each player in each position has a well defined set of moves, which, in a fixed way, change the position to another position (in fact, it is clear that there is no point in distinguishing between positions and games, so we can substitute the word 'game' for the word 'position' everywhere). For some recent work on combinatorial games, see [4]. The main result of this paper is to analyze a certain, very special, class of combinatorial games for multiple players.

The definition given above, of course, describes only the 'static' aspect of the rules of a game. The 'dynamic' aspects refer to how the game is actually played. A play by play sequence of moves in a game will be called a 'match'. The dynamical rules of matches which we will consider specify a certain order of the set of players; the players shall move repeatedly in the same order of play until a certain player cannot move, at which point the match shall end. The player who cannot move shall then be declared the loser of the match. We shall consider only games where there is no possibility of infinite matches.

Even for such deterministic games, however, it is difficult to make any conclusions about the course of matches for $n > 2$. The reason is that unlike the case of $n = 2$, there is no natural order of preference of the outcomes of the game from the point of view of the i'th player. While the i'th player obviously prefers not to lose, there is no natural reason why he should a priori prefer one particular other player to lose. Yet, such preferences will determine strategies, and ultimately the outcome of a match. Preferences can even change throughout the course of a match. Thus, it is usually said that few strategic conclusions about deterministic games of $n > 2$ players can be made without introducing non-deterministic concepts, perhaps even non-mathematical concepts (e.g. psychology).

The purpose of this paper is to look at a certain very special class of deterministic games of n players, for which certain strategical conclusions can be made in a rigorous mathematical setting, without introducing outside concepts. The motivation for introducing our particular class of games is that they generalize the

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'number games' for 2 players from J.H.Conway's famous book [1]. For this reason, we shall call this class of games *number games for n players*.

Conway number games for 2 players are games to which one can assign a value which is a 'number'. Here the word number means element of a certain ordered field, known as the Conway field $\mathbb{C}$ (also known as the surreal numbers [1, 2, 3]). The Conway field contains, among other things, all ordinal numbers, as well as any other ordered field: it is a foundational-level object of set theory (in fact, Conway introduces his own approach to formal set theory based on number games in [1]).

However, this is not the aspect of number games which we will be most interested in. Rather, the main point of number games is that 'no player can possibly improve his own position by making a move'. This, of course, needs precise definition, (in particular in reference to the word 'improve'). Definitions will be provided later. We shall, however, remark here that one could adopt the point of view that for this reason, number games are generally strategically uninteresting, since the very meaning of strategy is being able to take advantage of one's own move in the best possible way. In number games, one always makes one's own position worse by moving. One can also take another point of view; in some sense, number games are *pure consumer models*: we can think of moving in a number game as using up one's assets (resources). The player who uses up his resources first dies.

With this in mind, it becomes interesting to try to define analogues of Conway number games for n players, and analyze what, if any, strategic conclusions one can make for such games. In this paper, we do give one possible definition of such number games for n players. The set of (equivalence classes of) such games is an $n-1$-dimensional vector space over the Conway field. It is somewhat surprising the set of number games of n players has many of the formal properties of number games of 2 players. Also, we shall be able to make certain strategic conclusions for number games: in particular, each match will have a well defined 'loser' who can always be defeated if all the other players act 'in concert'.

This paper is organized as follows: In the next section, we shall present basic definitions and facts about games and matches which do not involve numbers. In Section 3, we shall introduce number games, and prove what we can say about their strategic analysis. Section 4 contains, in some sense, our hardest result, namely constructing number games of n players with any given value. The paper has two appendices. In Appendix 1 (section 5), we draw some diagrams visualizing our concepts for games of three players. This may be helpful to the reader in understanding what we mean. In Appendix 2 (section 6), we show why our definition of number game cannot be simplified in one obvious way.

2. Games and matches

In this paper, a *game with set T of players* is defined recursively as follows:

(1) The empty set $\emptyset$ is a game (also called 0).

(2) If G_i are sets of games for all $i \in T$, then the tuple $(G_i)_{i\in T}$ is a game.

(3) Every game can be obtained by 1, 2 in a possibly transfinite number of steps.

Obviously, only the cardinality of the set T matters. We shall mostly consider the case $T = \{1, ..., n\}$ (in which case we shall simply speak of *games of n players*), but it is useful to allow other T's, notably $T \subset \{1, ..., n\}$.

We shall now introduce our main strategic concept for games of n players. It is important to notice that this concept does not involve dynamic aspects of games, i.e. matches.

Specifically, we shall inductively define

$$G <_S 0$$

for a non-empty set $S \subseteq T$ if the following conditions hold:

(1) If $i \in S$ then for all $H \in G_i$, $H <_{\{i\}} 0$.

(2) If $i \in S$ and $j \notin S$ then there exists an $H \in G_j$ and a set U with $i \in U \subseteq S \cup \{j\}$, such that $H <_U 0$.

We shall write

$$G \sim 0$$

if $G <_T 0$. The set S will be called the *set of possible losers of the game G*. We shall justify this terminology at the end of this section.

When working with games of n players, we shall use a notational convention analogous to that established in Conway's book [1], and denote a "general member" of the set G_i by G^i. Thus, for example, instead of referring to something that is true for *all* $H \in G_i$, we instead refer to something that is true for *all* G^i.

The members of G_i, or using the new convention, *the* G^i, are referred to as *i's options* in the game G.

In this notation, the above definition reads as follows:
Inductively define $G <_S 0$ for a non-empty set $S \subseteq T$ if the following conditions hold:

(1) If $i \in S$ then all $G^i <_{\{i\}} 0$.

(2) If $i \in S$ and $j \notin S$ then there exists a G^j and a set U with $i \in U \subseteq S \cup \{j\}$, such that $G^j <_U 0$.

Remark: When we are considering games of two players, we are in the context of Conway [1]. There, players are denoted by L and R, so $T = \{L, R\}$. Conway notes that it is easy to prove that all games of 2 players are of one of the following types: $0, L, R, F$. For 0, the first player loses, in F, the first player wins, in L (resp. R) the player L (resp. R) wins no matter whose move it is. In the above notation, $S = \{L, R\}$ for type 0, $S = R$ for type L, $S = L$ for type R and S does not exist for type F. The proof is left to the reader as an exercise.

Lemma 1. *For each game G, there exists at most one S such that $G <_S 0$.*

Proof: Induction: Note that

$$0 <_S 0 \text{ if and only if } S = T.$$

Assume the statement true for all G^i for all $i \in T$. Then if

$$G <_S 0,$$

note that $i \notin S$ if and only if there exists a G^i such that it is not true that $G^i <_{\{i\}} 0$, which uniquely characterizes S. □

We now explain the dynamical significance of $G <_S 0$. To this end, we must define matches. Assume now that T is finite, and that we have a bijection

$$\sigma : \{1, ..., n\} \to T.$$

Such bijection will be called an *order of play.* A match according to the order of play σ is a sequence of games

$$(G(j))_{j=1,...,N}$$

where $G(1) = G$,

$$G(j+1) \in G(j)_{k(j)} \text{ where } k(j) = \sigma(j'),\ j' \equiv j \mod n,$$

$$G(N)_{k(N)} = \emptyset.$$

Then $k(N)$ is called *the loser of the match.* Note that

$$(G(j))_{j=2,...,N}$$

is a match according to the order of play σ' where

$$\sigma'(j) = \sigma(j+1) \text{ for } j < n,$$

$$\sigma'(n) = \sigma(1).$$

Note also that by part 3 of our definition of game, it is impossible to have an infinite match, i.e. an infinite sequence satisfying the properties of a match without the N. (Proof: induction.)

We now define inductively our main dynamic strategic concept. A player i is called the *loser of a game G according to the order of play σ* if

(1) If $\sigma(1) = i$ then i is the loser of all its options G^i according the order of play σ'.
(2) If $\sigma(1) \neq i$, then there exists a $G^{\sigma(1)}$ such that i is the loser of $G^{\sigma(1)}$ according to the order of play σ'.

Intuitively speaking, this means that i will lose any match according to the order of play σ, provided that all the other players act "in concert".

Proposition 2. *Suppose $G <_S 0$ and suppose that σ is any order of play. Let j be minimal such that $\sigma(j) \in S$. Then $\sigma(j)$ is the loser of the game G according to the order of play σ.*

Proof: Induction. If $\sigma(1) = i$, then always $G^i <_{\{i\}} 0$, so the induction hypothesis applies. If $\sigma(1) \neq i$, then there exists a $G^{\sigma(1)}$ such that $G^{\sigma(1)} <_U 0$ for some $i \in U \subseteq S \cup \{\sigma(1)\}$. Note that i satisfies the induction hypothesis with G replaced by $G^{\sigma(1)}$, and σ replaced by σ'. □

With this new dynamic significance applied to our previous definitions, the definitions can be formulated in a more intuitive way. If we find a set $S \subseteq T$ with $G <_S 0$, then the set S is the set of players who, for some order of play σ, would definitely lose the game if the others acted in concert. This is the reason the set S can be thought of as the set of possible losers of the game G. This can yield intuitive versions of 1 and 2 of the previous definition. $G <_S 0$ means:

(1) If $i \in S$, then player i is the *only* possible loser of each of i's options.
(2) If $i \in S$ but $j \notin S$, then player j has an option of which i is a possible loser. In addition, this option must not add any new possible losers, except possibly player j himself.

For the purposes of the next section, we shall now define the *sum of games*: Define inductively

$$G + H$$

by

$$(G + H)_i = \{G + H^i\} \cup \{G^i + H\}.$$

The sum of games is understood as follows: playing $G+H$ is the same as playing the games G and H side by side, so that i's options in the game $G+H$ should be to either "move in G" or "move in H." If player i chooses to move in G, he chooses an option G^i of the game G, and the game progresses to the position G^i+H. Similarly, moving in H moves the game to some position $G+H^i$. Thus, the set of i's options is defined to be the set $\{G + H^i\} \cup \{G^i + H\}$.

3. Number games

We continue to assume that T is finite, of cardinality n. We shall work with T-tuples of real numbers (or more generally T-tuples of elements of any ordered field F)

$$g = (g_i)_{i \in T} \tag{1}$$

which satisfy

$$\sum_{i \in T} g_i = 0.$$

Obviously, the set of all such T-tuples is an $n-1$-dimensional vector space over F, which we shall denote by F_T. For $S \subseteq T$, and for the T-tuple (1), we now write

$$g <_S 0 \tag{2}$$

for the unique set S of all $i \in T$ with

$$g_i = \min_{k \in T} g_k.$$

Note that S is always non-empty. We also write

$$g \leq_S 0$$

if $g <_U 0$ for some $U \supseteq S$. Note that $g <_T 0$ is equivalent to $g \leq_T 0$ which is equivalent to $g = 0$. We shall write

$$g <_S h$$

if $g - h <_S 0$, and similarly for $\leq_S$. By abuse of notation, we write $<_i$ instead of $<_{\{i\}}$.

Lemma 3. *If $g <_i h$, then $g_i - g_j < h_i - h_j$ for all $j \neq i$.*

Proof: $g <_i h$ means $g - h <_i 0$, i.e. $g_i - h_i < g_j - h_j$ for all $j \neq i$. □

Below, we shall need the following construction. For $i \in T$, consider the function

$$p_i : F_T \to F_{T-\{i\}}$$

given by

$$p_i(g) = \left(g_j + \frac{g_i}{n-1} \right)_{j \in T-\{i\}}.$$

The function p_i takes n-tuples in F_T and creates $n-1$-tuples in $F_{T-\{i\}}$ in the most natural way: it evenly divides up the strength of the i^{th} element among all the others.

We now proceed to number games. We begin by recalling briefly Conway number games of 2 players [1]. The main point is that to each pair of subsets

$$\langle A|B \rangle$$

of the Conway field $\mathbb{C}$, such that for all $x \in A$, $y \in B$ we have

$$x < y,$$

there is assigned an element

$$v\langle A|B \rangle \in \mathbb{C}$$

such that, for all $x \in A$, $y \in B$,

$$x < v\langle A|B \rangle < y.$$

We refer the reader to [1] for details, but the following two properties are crucial for our purposes:

(1) $v(\emptyset) = 0$ and $v(G + H) = v(G) + v(H)$ where
$$\langle A|B \rangle + \langle C|D \rangle = \langle v\langle A|B \rangle + C | v\langle C|D \rangle + B \rangle.$$
(2) If $C \supseteq A$ and $D \supseteq B$ and for each $x \in C$ (resp. $y \in D$) $x < v\langle A|B \rangle$ (resp. $v\langle A|B \rangle < y$) then
$$v\langle C|D \rangle = v\langle A|B \rangle.$$

Using this, we define inductively a *number game of T players* as a game G for which there exists an n-tuple

$$v(G) \in \mathbb{C}_T$$

such that

(1) For all $i \in T$, all G^i are number games, and $v(G^i) <_i v(G)$.
(2) For all $i \neq j \in T$,
$$v_i(G) - v_j(G) = v\langle \{v_i(G^i) - v_j(G^i) : p_i v(G^i) \leq_j p_i v(G)\} | \\ \{v_i(G^j) - v_j(G^j) : p_j v(G^j) \leq_i p_j v(G)\} \rangle$$

On first glance, the conditions given in the definition of a number game do not seem natural, but intuitive meaning can be given to them. First, the n-tuple $v(G) = (v_1, v_2, \ldots, v_n)$ gives the strengths of the positions of each player. Larger, positive values of v_i indicate better positions for player i; smaller, negative values indicate worse positions.

Thus, the first statement, that $v(G^i) <_i v(G)$, can be understood as follows: Player i's move from G to G^i not only hurts player i's position; it hurts player i's position more than anyone else's position. This seems natural, as moving in a number game should never "improve" one's position compared to any other player.

The second statement defines the quantity $v_i - v_j$ for each i and j, which is understood to be *i's advantage over j in the game G.* This advantage is defined as the number $\langle A|B\rangle$, where A is a set of possible advantages i could have after moving, and B is the set of possible advantages j could have after moving. This means that i's advantage in G is more than any advantage he would have after choosing one of his own options G^i, but less than the advantage he would gain were his opponent to move to any G^j.

This would completely explain the definition, however, the sets A and B have an additional restriction on them. Take, for example, the definition of the set A, which contains a condition further restricting its members:

$$A = \{v_i(G^i) - v_j(G^i) : p_i v(G^i) \leq_j p_i v(G)\}.$$

The condition $v(G^i) \leq_j v(G)$ (disregarding the p_i's) would mean that the move from G to G^i must hurt player j's position the most. The $\leq$ allows for this to be nonstrict, namely, that the move may hurt other players just as much. However, in a number game, i's moves from G to G^i must hurt player i the most, so the condition would never be true without considering the p_i.

Recall that the function p_i takes n-tuples and creates $n-1$-tuples, with player i's strength equally distributed among all other players. So, the restriction $p_i v(G^i) \leq_j p_i v(G)$ means that the inequality is true once player i is no longer considered, namely, that the move from G to G^i must hurt player j at least as much as everyone else, with player i himself excluded. Such moves G^i are called *i's anti-j options*, since they do as much damage to player j as possible.

So, number games can be understood as games G for which each player has a well-defined strength of position, given by the n-tuple $v(G)$. G having a *well-defined* strength means that:

(1) In the game G, each player's options must be number games, and a player's move must always damage his own position the most.
(2) In the game G, *i's advantage over j* is the Conway field element $\langle A|B\rangle$, where A is the set of all advantages player i could have if he chose an *anti-j move*, and B is the set of all advantages player i could have if his opponent chose an *anti-i move*. Thus, player i's advantage over j only depends on the i- and j-options that are primarily directed against one another.

Lemma 4. *The T-tuple $v(G)$, if it exists, is uniquely determined.*

Proof: By (1) of the definition of number game and Lemma 1, for all G^i we have

$$v_i(G^i) - v_j(G^i) < v_i(G) - v_j(G)$$

while for all G^j we have

$$v_i(G^j) - v_j(G^j) > v_i(G) - v_j(G).$$

By property (2) of Conway games, (2) of the definition of number games then implies

$$v_i(G) - v_j(G) = v\langle v_i(G^i) - v_j(G^i) | v_i(G^j) - v_j(G^j)\rangle \tag{3}$$

which recursively determines $v(G)$. □

Lemma 5. *A sum of number games is a number game.*

Proof: By induction, both conditions (1), (2) are obviously additive. In particular, in (2), the right hand side for a sum of games contains the Conway sum of the right hand sides of (2) of the individual games, so we can use properties (1) and (2) of Conway games. □

Corollary 6. *(of (3))*

$$v(G + H) = v(G) + v(H).$$

□

Proposition 7. *If G is a number game and $v(G) <_S 0$, then $G <_S 0$.*

Recall that $v(G) <_S 0$ means that for each $i \in S$, $v_i = \min_{k \in T} v_k$. So, this will show that those players with the least strength of position are exactly those players who will lose a match of this game for some order of play, if all others act in concert.

Proof: Induction. By the induction hypothesis, condition (1) in the definition of number games implies condition (1) for $G <_S 0$.

Suppose condition (2) for number games is valid for G. Choose $i \notin S$, $j \in S$. Then, by definition of $v(G) <_S 0$,

$$v_i(G) > v_j(G).$$

By (2) for number games and properties of Conway games, there is an option G^i such that

$$v_i(G^i) \geq v_j(G^i),\ p_i v(G^i) \leq_j p_i v(G). \tag{4}$$

The second condition implies that

$$v_j(G^i) - v_k(G^i) \leq v_j(G) - v_k(G) \leq 0 \text{ for all } k \neq i, j, \tag{5}$$

so together with (4) this implies that

$$v_j(G^i) = min\{v_p(H) | p \in T\}.$$

On the other hand, if $k \notin S$, (5) implies that

$$v_j(G^i) < v_k(G^i).$$

Thus, $G^i <_T 0$ for some $j \in T \subseteq S \cup \{i\}$, as required in condition 2 for $G <_S 0$. □

Remark: Since for $g \in \mathbb{C}_T$, there is always a unique $S \subseteq T$, $S \neq \emptyset$ with $g <_S 0$, the converse of the Proposition is also true.

Corollary 8. *If G is a number game and $v(G) = 0$ then $G \sim 0$.*

□

Corollary 9. *A number game G has an inverse, i.e. a game H such that $G+H \sim 0$.*

Proof: The symmetric group Σ_T obviously acts on number games by permuting players. Now we obviously have

$$v\left(\sum_{\sigma\in\Sigma_T} \sigma G\right) = \sum_{\sigma\in\Sigma_T} \sigma v(G) = 0,$$

so

$$\sum_{\sigma\in\Sigma_T} \sigma G \sim 0$$

by the previous Corollary. □

4. The Existence theorem.

In this section, we prove that number games of arbitrary values exist.

Theorem 10. *(Existence theorem) For every $g \in \mathbb{C}_T$ there exists a number game G with*

$$v(G) = g.$$

Proof: We begin by constructing games that are "worth one move" to each player, then from there games that are "worth x moves" to each player for any $x > 0 \in \mathbb{C}$. Sums and inverses of these games will then be enough to construct a game with $v(G) = g$ for any $g \in \mathbb{C}_T$.

First, we construct the game ${}^{(\mathbf{i})}\mathbf{1}$ for each $i \in T$, the *game worth one move to player i*. It is constructed by

$$ {}^{(\mathbf{i})}\mathbf{1}_i = \{0\},$$
$$ {}^{(\mathbf{i})}\mathbf{1}_j = \emptyset \text{ for } j \neq i.$$

In this game, player i has only one option: the move to the zero game. No other players have any options. This is a number game with

$$v({}^{(\mathbf{i})}\mathbf{1}) = {}^{(i)}v \in \mathbb{C}_T$$

where

$${}^{(i)}v_i = \frac{n-1}{n}, \qquad {}^{(i)}v_j = -\frac{1}{n}.$$

For example, for $T = \{1, 2, 3\}$, this construction yields three games:

$$\begin{aligned} {}^{(1)}\mathbf{1}: &\quad v({}^{(1)}\mathbf{1}) = (\ \ 2/3, -1/3, -1/3) \\ {}^{(2)}\mathbf{1}: &\quad v({}^{(2)}\mathbf{1}) = (-1/3, \ \ 2/3, -1/3) \\ {}^{(3)}\mathbf{1}: &\quad v({}^{(3)}\mathbf{1}) = (-1/3, -1/3, \ \ 2/3) \end{aligned}$$

To demonstrate how to check that a game is a number game, we will check that indeed the games ${}^{(i)}\mathbf{1}$ above are number games, with $v(G) = ((n-1)/n, -1/n, ..., -1/n)$. Obviously, it suffices to consider $i = 1$.

To prove ${}^{(1)}\mathbf{1}$ is a number game, we need to check the two conditions. First, we must check that all the options of G are number games, which they are, and additionally, that $v(G^i) <_i v(G)$ for all i and all G^i.

There is only one option to check, namely $G^i = \mathbf{0}$ and $i = 1$. Indeed, $\mathbf{0}$ is a number game, with $v(\mathbf{0}) = (0,0,...,0)$. So, to check that $v(\mathbf{0}) <_1 v({}^{(1)}\mathbf{1})$, we need only that

$$\begin{aligned} v(\mathbf{0}) - v({}^{(1)}\mathbf{1}) &<_1 0 \\ (0, 0, ..., 0) - ((n-1)/n, -1/n, ..., -1/n) &<_1 0 \\ (-(n-1)/n, 1/n, ..., 1/n) &<_1 0 \end{aligned}$$

which is true. So, condition (1) for number games is satisfied here.

Now, to check condition (2), we need to make sure that the definition's $v_i - v_j$ match up with what we claimed they were by setting $v(G) = ((n-1)/n, -1/n, ..., -1/n)$.

First, check $v_1 - v_2$. We should get that $v_1 - v_2 = (n-1)/n + 1/n = 1$. Indeed,

$$\begin{aligned} v_1 - v_2 = v\langle &\{v_1(G^1) - v_2(G^1) : p_1 v(G^1) \leq_2 p_1 v({}^{(1)}\mathbf{1})\} \\ &|\{v_1(G^2) - v_2(G^2) : p_2 v(G^2) \leq_1 p_2 v({}^{(1)}\mathbf{1})\}\rangle \\ &= v\langle\{v_1(G^1) - v_2(G^1) : p_1 v(G^1) \leq_2 p_1 v({}^{(1)}\mathbf{1})\}|\emptyset\rangle \\ &= v\langle\{v_1(\mathbf{0}) - v_2(\mathbf{0}) : p_1 v(\mathbf{0}) \leq_2 p_1 v({}^{(1)}\mathbf{1})\}|\emptyset\rangle \\ &= v\langle 0 - 0|\emptyset\rangle \qquad \text{(we have } (0, ..., 0) = p_1 v(\mathbf{0}) \leq_2 p_1 v({}^{(1)}\mathbf{1}) = (0, ..., 0)) \\ &= v\langle 0|\emptyset\rangle \\ &= 1. \end{aligned}$$

Checking $v_1 - v_i$ is similar for other i. And, checking $v_i - v_j$ for $i, j \neq 1$ $(i \neq j)$ is easy, since we want $v_i - v_j = 0$, and indeed, it is

$$\begin{aligned} &v\langle\{v_i(G^i) - v_j(G^i) : p_i v(G^i) \leq_j p_i v({}^{(1)}\mathbf{1})\}|\{v_i(G^j) - v_j(G^j) : p_j v(G^j) \leq_i p_j v({}^{(1)}\mathbf{1})\}\rangle \\ =&v\langle\emptyset|\emptyset\rangle \\ =&0. \end{aligned}$$

Now, we construct games ${}^{(i)}\mathbf{x}$ for all numbers $x \in \mathbb{C}$. The game ${}^{(i)}\mathbf{x}$ is *the game worth x moves to player i.*

Lemma 11. *Given $x \geq 0 \in \mathbb{C}$ and $i \in T$, there exists a number game* $\overset{(\mathbf{i})}{}\mathbf{x}$ *such that*

$$v(\overset{(\mathbf{i})}{}\mathbf{x}) = x \cdot v(\overset{(\mathbf{i})}{}\mathbf{1}).$$

For $x = 0$, the zero game satisfies this condition, so we need only consider $x > 0$. Once this construction is complete, the proof will be nearly finished.

Given $x \in \mathbb{C}$, we know that it is constructed by

$$x = v\langle L|R\rangle$$

for some sets L, R of simpler numbers in $\mathbb{C}$. In addition, since $x > 0$, we have that all $x^R > 0$. We can assume without loss of generality that all $x^L \geq 0$ as well, so by induction, we may assume that we have already constructed the "simpler" number games $\overset{(\mathbf{j})}{}\mathbf{x}^{\mathbf{L}}$ and $\overset{(\mathbf{j})}{}\mathbf{x}^{\mathbf{R}}$ for all $x^L \in L$, $x^R \in R$, and $j \in T$.

Now define an inverse $\overset{(\mathbf{j})}{}-\mathbf{x}^{\mathbf{R}}$ of $\overset{(\mathbf{j})}{}\mathbf{x}^{\mathbf{R}}$ as the sum of previously constructed number games:

$$\overset{(\mathbf{j})}{}-\mathbf{x}^{\mathbf{R}} = \sum_{k \neq j} \overset{(\mathbf{k})}{}\mathbf{x}^{\mathbf{R}}.$$

To prove that $\overset{(\mathbf{j})}{}-\mathbf{x}^{\mathbf{R}}$ is an inverse of $\overset{(\mathbf{j})}{}\mathbf{x}^{\mathbf{R}}$, note that by Corollary 9, we have that every game G has an inverse given by the sum of all the games that are the result of permuting the players of G. So, it suffices to show that $\overset{(\mathbf{j})}{}-\mathbf{x}^{\mathbf{R}}$ is the sum of the $(n! - 1)$ number games which are permutations of the game G. These permutations are still number games, and they are all given by $\overset{(\mathbf{k})}{}\mathbf{x}^{\mathbf{R}}$ for some k:

$$\begin{aligned}
\overset{(\mathbf{j})}{}-\mathbf{x}^{\mathbf{R}} &= \left(\left((n-1)! - 1\right) \cdot \overset{(\mathbf{j})}{}\mathbf{x}^{\mathbf{R}}\right) + \left(\left((n-1)!\right) \cdot \sum_{k \neq j} \overset{(\mathbf{k})}{}\mathbf{x}^{\mathbf{R}}\right) \\
&= \left(\left((n-1)! - 1\right) \cdot \overset{(\mathbf{j})}{}\mathbf{x}^{\mathbf{R}}\right) + \left(\left((n-1)! - 1\right) \cdot \sum_{k \neq j} \overset{(\mathbf{k})}{}\mathbf{x}^{\mathbf{R}}\right) + \left(\sum_{k \neq j} \overset{(\mathbf{k})}{}\mathbf{x}^{\mathbf{R}}\right) \\
&= \left(\left((n-1)! - 1\right) \cdot \sum_{\text{all k}} \overset{(\mathbf{k})}{}\mathbf{x}^{\mathbf{R}}\right) + \left(\sum_{k \neq j} \overset{(\mathbf{k})}{}\mathbf{x}^{\mathbf{R}}\right) \\
&= \left(\left((n-1)! - 1\right) \cdot \mathbf{0}\right) + \left(\sum_{k \neq j} \overset{(\mathbf{k})}{}\mathbf{x}^{\mathbf{R}}\right) \\
&= \sum_{k \neq j} \overset{(\mathbf{k})}{}\mathbf{x}^{\mathbf{R}}.
\end{aligned}$$

Now consider the game G given by

$$\begin{aligned}
G_i &= \{\overset{(\mathbf{i})}{}\mathbf{x}^{\mathbf{L}}\}, \\
G_j &= \{\overset{(\mathbf{j})}{}-\mathbf{x}^{\mathbf{R}}\}, \text{ for } j \neq i.
\end{aligned}$$

Claim : G is the game $\overset{(\mathbf{i})}{}\mathbf{x}$ that satisfies the lemma, i.e. it is a number game with $v(G) = x \cdot v(\overset{(\mathbf{i})}{}\mathbf{1})$.

To show that G is a number game with the given tuple, we need to check the conditions (1) and (2) for a number game.

For condition (1), we first need that each option of G is a number game, which is true by induction. Then we must show that each option G^k has $v(G^k) <_k v(G)$.

For player i, then, we must show that

$$v({}^{(i)}\mathbf{x^L}) <_i x \cdot v({}^{(i)}\mathbf{1}).$$

But by induction, the left-hand side is given by

$$x^L \cdot v({}^{(i)}\mathbf{1}),$$

so it must be proven that

$$\begin{aligned} x^L \cdot v({}^{(i)}\mathbf{1}) &<_i x \cdot v({}^{(i)}\mathbf{1}) \\ (x^L - x) \cdot v({}^{(i)}\mathbf{1}) &<_i 0. \end{aligned}$$

This is true: since $x^L - x$ is negative, while the only positive entry of $v({}^{(i)}\mathbf{1})$ is in the i^{th} position, we have that the only negative value of the n-tuple is in the i^{th} position. So, it is $<_i 0$.

For other players j, we must show that

$$v({}^{(j)} - \mathbf{x^R}) <_j x \cdot v({}^{(i)}\mathbf{1}).$$

But by induction, we already know $v({}^{(j)} - \mathbf{x^R})$ is a number game, and since it is the inverse of ${}^{(j)}\mathbf{x^R}$, we know $v({}^{(j)} - \mathbf{x^R}) = -v({}^{(j)}\mathbf{x^R})$. So, we need to show

$$\begin{aligned} v({}^{(j)} - \mathbf{x^R}) &<_j x \cdot v({}^{(i)}\mathbf{1}) \\ -v({}^{(j)}\mathbf{x^R}) &<_j x \cdot v({}^{(i)}\mathbf{1}) \\ -x^R \cdot v({}^{(j)}\mathbf{1}) &<_j x \cdot v({}^{(i)}\mathbf{1}) \\ -x^R \cdot v({}^{(j)}\mathbf{1}) - x \cdot v({}^{(i)}\mathbf{1}) &<_j 0. \end{aligned}$$

The n-tuple on the left-hand side has positive entries for every index other than i and j, but negative entries for indices i and j. However, the j^{th} entry is most negative, since $x^R > x$, so the inequality holds.

This finishes the verification of the first condition for G to be a number game with $v(G)$ as desired.

To finish the proof that G is a number game, we must show condition (2), that for each $j, k \in T$, the difference $v_j(G) - v_k(G)$ is the same as what is required. We split this into two cases: one where both $j, k \neq i$, and one where one of them does equal i.

(1) Check condition (2) for $v_j - v_k$, where $j, k \neq i$.

We have claimed that G is a number game with $v(G) = x \cdot v({}^{(i)}\mathbf{1})$, so we claim that $v_j - v_k = (x \cdot v({}^{(i)}\mathbf{1}))_j - (x \cdot v({}^{(i)}\mathbf{1}))_k$. This equals zero since $v({}^{(i)}\mathbf{1})_j = v({}^{(i)}\mathbf{1})_k$ for $j, k \neq i$.

So, we must show that the right-hand side of the equation in condition (2) is $0 \in \mathbb{C}$. The right-hand side of the equation reads as follows:

$$v\langle\{v_j(G^j) - v_k(G^j) : p_j v(G^j) \leq_k p_j v(G)\}|$$
$$\{v_j(G^k) - v_k(G^k) : p_k v(G^k) \leq_j p_k v(G)\}\rangle.$$

Now, a number $y = v\langle L|R\rangle$ is $0 \in \mathbb{C}$ if and only if all $y^L < 0$ and all $y^R > 0$ (if they exist). We shall thus show that all the left options, if they exist, are less than zero, and all the right options, if they exist, are greater than zero:

$$v_j(G^j) - v_k(G^j) < 0 \text{ when } p_j v(G^j) \leq_k p_j v(G),$$
$$v_j(G^k) - v_k(G^k) > 0 \text{ when } p_k v(G^k) \leq_j p_k v(G),$$
which simplifies to
$$v_j(G^j) < v_k(G^j) \text{ when } p_j v(G^j) \leq_k p_j v(G),$$
$$v_j(G^k) > v_k(G^k) \text{ when } p_k v(G^k) \leq_j p_k v(G).$$

In fact, we shall show more strongly that $v_j(G^j) < v_k(G^j)$ and $v_j(G^k) > v_k(G^k)$ always, regardless of the restrictions. Additionally, these two statements are symmetric in j and k, so we need only show the first one.

Since all the G^j are of the form ${}^{(\mathbf{j})}\!-\mathbf{x}^{\mathbf{R}}$, we only need to show that

$$v_j({}^{(\mathbf{j})}\!-\mathbf{x}^{\mathbf{R}}) < v_k({}^{(\mathbf{j})}\!-\mathbf{x}^{\mathbf{R}}).$$

But this is true, since $x^R > 0$ tells us that

$$v_j({}^{(\mathbf{j})}\!-\mathbf{x}^{\mathbf{R}}) = (-x^R) \cdot (n-1)/n < 0,$$
$$v_k({}^{(\mathbf{j})}\!-\mathbf{x}^{\mathbf{R}}) = (-x^R) \cdot (-1/n) > 0.$$
$$\text{So } v_j({}^{(\mathbf{j})}\!-\mathbf{x}^{\mathbf{R}}) < 0 < v_k({}^{(\mathbf{j})}\!-\mathbf{x}^{\mathbf{R}}).$$

This shows that $v_j - v_k = 0$, as desired.

(2) Check condition (2) for $v_i - v_j$, where $j \neq i$.

We have claimed that G is a number game with $v(G) = x \cdot v({}^{(\mathbf{i})}\mathbf{1})$. So, we have claimed that $v_i - v_j$ is as follows:

$$\begin{aligned} v_i - v_j &= x \cdot (v_i({}^{(\mathbf{i})}\mathbf{1}) - v_j({}^{(\mathbf{i})}\mathbf{1})) \\ &= x \cdot (\frac{n-1}{n} - \frac{-1}{n}) \\ &= x. \end{aligned}$$

So, we must show that the right-hand side of condition (2) for number games really yields the number $x \in \mathbb{C}$. The right-hand side is here the number $v\langle A|B\rangle$, where

$$v\langle A|B\rangle = v\langle\{v_i(G^i) - v_j(G^i) : p_i v(G^i) \leq_j p_i v(G)\}|$$
$$\{v_i(G^j) - v_j(G^j) : p_j v(G^j) \leq_i p_j v(G)\}\rangle.$$

First, look at the set $A = \{v_i(G^i) - v_j(G^i) : p_i v(G^i) \leq_j p_i v(G)\}$. Using the fact that we know all the G^i and G^j yields

$$\begin{aligned} A &= \{v_i(G^i) - v_j(G^i) : p_i v(G^i) \leq_j p_i v(G)\} \\ &= \{v_i(\overset{(\mathbf{i})}{}\mathbf{x^L}) - v_j(\overset{(\mathbf{i})}{}\mathbf{x^L}) : p_i v(\overset{(\mathbf{i})}{}\mathbf{x^L}) \leq_j p_i v(G)\} \\ &= \{\big(x^L \cdot ((n-1)/n)\big) - \big(x^L \cdot (-1/n)\big) : p_i v(\overset{(\mathbf{i})}{}\mathbf{x^L}) \leq_j p_i v(G)\} \\ &= \{x^L : p_i v(\overset{(\mathbf{i})}{}\mathbf{x^L}) \leq_j p_i v(G)\} \end{aligned}$$

So, $A \subseteq L$. Deciphering the condition at right yields that

$$\begin{aligned} A &= \{x^L : p_i v(\overset{(\mathbf{i})}{}\mathbf{x^L}) \leq_j p_i v(G)\} \\ &= \{x^L : p_i(x^L \cdot \overset{(i)}{v}) \leq_j p_i(x \cdot \overset{(i)}{v})\}. \\ &= \{x^L : p_i(x^L \cdot \overset{(i)}{v}) - p_i(x \cdot \overset{(i)}{v}) \leq_j 0\}. \end{aligned}$$

We will show that A = L, by showing that the right-hand condition is actually true for all x^L.

Consider the j^{th} element of the n-tuple in the condition:

$$\big(p_i(x^L \cdot \overset{(i)}{v}) - p_i(x \cdot \overset{(i)}{v})\big)_j.$$

$$\begin{aligned} \big(p_i(x^L \cdot \overset{(i)}{v}) - p_i(x \cdot \overset{(i)}{v})\big)_j &= \big(p_i(x^L \cdot \overset{(i)}{v})\big)_j - \big(p_i(x \cdot \overset{(i)}{v})\big)_j \\ &= x^L \cdot \big(\overset{(i)}{v}_j + \frac{1}{n-1}\overset{(i)}{v}_i\big) - x \cdot \big(\overset{(i)}{v}_j + \frac{1}{n-1}\overset{(i)}{v}_i\big) \\ &= (x^L - x) \cdot \big(\overset{(i)}{v}_j + \frac{1}{n-1}\overset{(i)}{v}_i\big) \\ &= (x^L - x) \cdot \big(-\frac{1}{n} + \frac{1}{n-1} \cdot \frac{n-1}{n}\big) \\ &= (x^L - x) \cdot 0 \\ &= 0. \end{aligned}$$

Thus $p_i(x^L \cdot \overset{(i)}{v}) - p_i(x \cdot \overset{(i)}{v}) = \mathbf{0}$. So, it is $\leq_j 0$ for all $j \neq i$. Thus $A = \{\text{all } x^L\} = L$.

Now consider the set $B = \{v_i(G^j) - v_j(G^j) : p_j v(G^j) \leq_i p_j v(G)\}$. Using the fact that we know all the G^i and G^j yields, analogously to the case of A,

$$\begin{aligned} B &= \{v_i(G^j) - v_j(G^j) : p_j v(G^j) \leq_i p_j v(G)\} \\ &= \{v_i(\overset{(\mathbf{j})}{}{-\mathbf{x^R}}) - v_j(\overset{(\mathbf{j})}{}{-\mathbf{x^R}}) : p_j v(\overset{(\mathbf{j})}{}{-\mathbf{x^R}}) \leq_i p_j v(G)\} \\ &= \{\big(-x^R \cdot (-1/n)\big) - \big(-x^R \cdot ((n-1)/n)\big) : p_j v(\overset{(\mathbf{j})}{}{-\mathbf{x^R}}) \leq_i p_j v(G)\} \\ &= \{x^R : p_j v(\overset{(\mathbf{j})}{}{-\mathbf{x^R}}) \leq_i p_j v(G)\} \end{aligned}$$

So, $B \subseteq R$. Similarly as before, we will show $B = R$. So, looking closely at the condition at right leads to

$$\begin{aligned} B &= \{x^R : p_j v(\overset{(\mathbf{j})}{-\mathbf{x}^\mathbf{R}}) \leq_i p_j v(G)\} \\ &= \{x^R : p_j(-x^R \cdot \overset{(j)}{v}) \leq_i p_j(x \cdot \overset{(j)}{v})\} \\ &= \{x^R : p_j(-x^R \cdot \overset{(j)}{v}) - p_j(x \cdot \overset{(j)}{v}) \leq_i 0\}. \end{aligned}$$

Again the final step is to show that $p_j(-x^R \cdot \overset{(j)}{v}) - p_j(x \cdot \overset{(j)}{v}) = \mathbf{0}$, so that it is in particular $\leq_i 0$.

So consider $\big(p_j(-x^R \cdot \overset{(j)}{v}) - p_j(x \cdot \overset{(j)}{v})\big)_k$. Again,

$$\begin{aligned} \big(p_j(-x^R \cdot \overset{(j)}{v}) - p_j(x \cdot \overset{(j)}{v})\big)_k &= \big(p_j(-x^R \cdot \overset{(j)}{v})\big)_k - \big(p_j(x \cdot \overset{(j)}{v})\big)_k \\ &= -x^R \cdot \big(\overset{(j)}{v}_k + \frac{1}{n-1}\overset{(j)}{v}_j\big) - x \cdot \big(\overset{(j)}{v}_k + \frac{1}{n-1}\overset{(j)}{v}_j\big) \\ &= (-x^R - x) \cdot \big(\overset{(j)}{v}_k + \frac{1}{n-1}\overset{(j)}{v}_j\big) \\ &= (-x^R - x) \cdot \big(-\frac{1}{n} + \frac{1}{n-1} \cdot \frac{n-1}{n}\big) \\ &= (-x^R - x) \cdot 0 \\ &= 0. \end{aligned}$$

Thus the condition was true for all x^R, and B = $\{$all $x^R\}$ = R.

So, we have A = L, B = R, which implies that, as desired,

$$x = v\langle L|R\rangle = v\langle A|B\rangle.$$

This conclude the proof of the lemma, that number games of the form $\overset{(\mathbf{i})}{\mathbf{x}}$ can be constructed for all $x \in \mathbb{C}$, $i \in T$.

Now that these games have been constructed, the proof of the theorem can be completed: given a T-tuple $(v_1, v_2, \ldots, v_n) \in \mathbb{C}_T$, it is a linear combination of the T-tuples $\overset{(1)}{v}, \overset{(2)}{v}, \ldots, \overset{(n-1)}{v}$, of the form

$$(v_1, v_2, \ldots, v_n) = \sum_{i=1}^{n-1} a_i \cdot \overset{(i)}{v}$$

for some "scalars" $a_i \in \mathbb{C}$.

Then a number game G with $v(G) = (v_1, v_2, \ldots, v_n)$ is constructed by

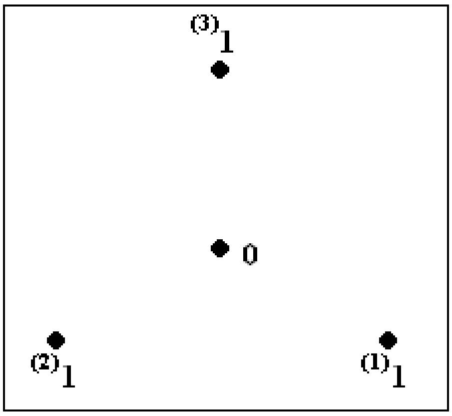
$$G = \sum_{i=1}^{n-1} \overset{(\mathbf{i})}{\mathbf{a_i}}.$$

□

5. Appendix 1: Examples of diagrams of three-player games

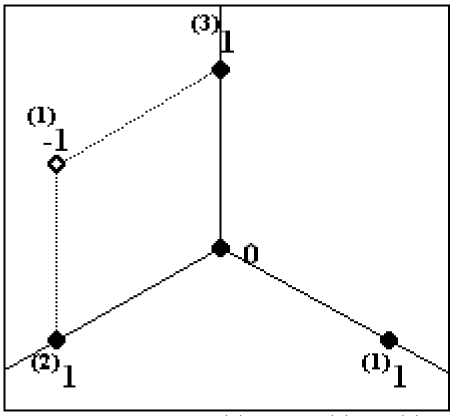
By the results of the previous section, we know that number games of any position strength can be constructed, and that for games of n players, the games form an $n-1$-dimensional vector space over the Conway field $\mathbb{C}$. In [1], one often uses the "number line" to visualize the position strengths of games of two players. When three player games are considered, we have a 2-dimensional vector space, i.e. a "number plane." To help understand the meaning of the results of this paper, this appendix is dedicated to the development of a similar visualization of the "number plane" of games of three players.

To begin, we place the zero game $\mathbf{0}$ at the origin. The next games to be constructed, ${}^{(1)}\mathbf{1}$, ${}^{(2)}\mathbf{1}$, and ${}^{(3)}\mathbf{1}$, are placed next. These are the games worth one move to each of players 1, 2, and 3 respectively, so we place these games on the plane one unit from the origin. Let us separate them by 120° angles, with ${}^{(1)}\mathbf{1}$ at the lower right, ${}^{(2)}\mathbf{1}$ at the lower left and ${}^{(3)}\mathbf{1}$ at the top as follows:



The beginning of the number plane.

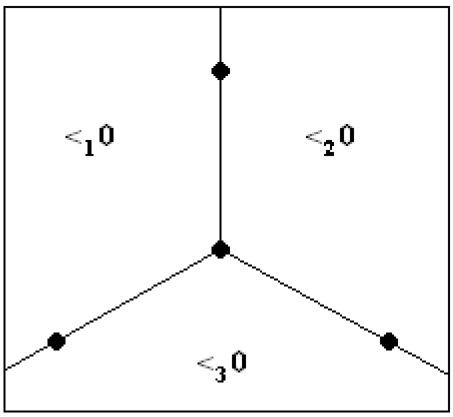
At this point, it becomes clear what must be the location of each of the number games. All games of the form ${}^{(1)}\mathbf{x}$ for $x > 0$ must lie on a ray through the zero game and ${}^{(1)}\mathbf{1}$; similarly for the other players. At that point, all other games are constructed by sums of those games, and are placed on the plane by vector addition.



The placement of ${}^{(1)}-\mathbf{1} = {}^{(2)}\mathbf{1} + {}^{(3)}\mathbf{1}$.

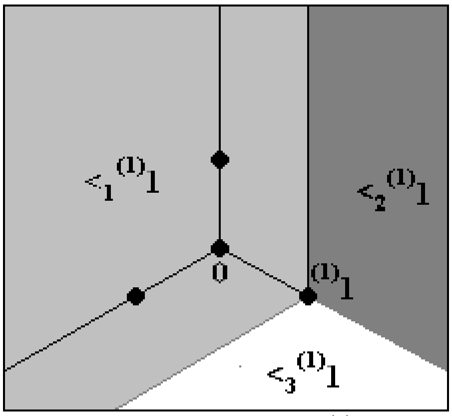
In this way, the plane is partitioned into three open regions, consisting of the games which are $<_1 0$, $<_2 0$, and $<_3 0$. These games will be lost by player 1, 2, and 3 respectively, if the others work in concert.

For example, all the games $<_1 0$ are in the region of the plane opposite the game ${}^{(1)}\mathbf{1}$, which is worth one move to player 1. The game ${}^{(1)}\mathbf{1}$ itself lies on the boundary between the regions $<_2 0$ and $<_3 0$, since it is itself $<_{\{2,3\}} 0$: the loser will depend on the order of play.



Partitioning the number plane by who loses the game.

While the preceding partition of the plane is very natural, we need not restrict ourselves to comparing number games to $\mathbf{0}$. For example, we may partition the plane by comparing the games to our first constructed game, ${}^{(1)}\mathbf{1}$. Thus, the region i consists of all the games "worse than ${}^{(1)}\mathbf{1}$ for player i."



Comparing games to ${}^{(1)}\mathbf{1}$.

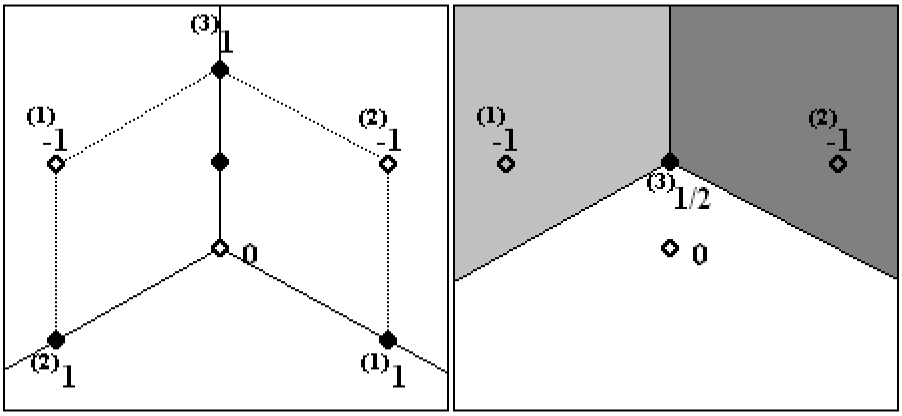
This visualization provides a means of testing the first condition for a game to be a number game. The condition was:

For all $i \in T$, all G^i are number games, and $v(G^i) <_i v(G)$.

For example, suppose that we were to test the claim that the game

$$G = ({}^{(1)} - \mathbf{1}, {}^{(2)} - \mathbf{1}, \mathbf{0}),$$

which has one option for each player, is really deserving of the name ${}^{(3)}(\mathbf{1/2})$, the game worth 1/2 move to the third player. Then we only need to plot the positions of the options, and the position of the "expected" value:



At left, the options are plotted on the plane.
At right, they are shown compared to the expected value ${}^{(3)}(\mathbf{1/2})$.

The figure above shows that player 1's option in G, which is the game ${}^{(1)} - \mathbf{1}$, is in the region $<_1 {}^{(3)}(\mathbf{1/2})$. Similarly, we have ${}^{(2)} - \mathbf{1} <_2 {}^{(3)}(\mathbf{1/2})$ and $\mathbf{0} <_3 {}^{(3)}(\mathbf{1/2})$. This visually checks the first condition for the game G to actually be the game ${}^{(3)}(\mathbf{1/2})$.

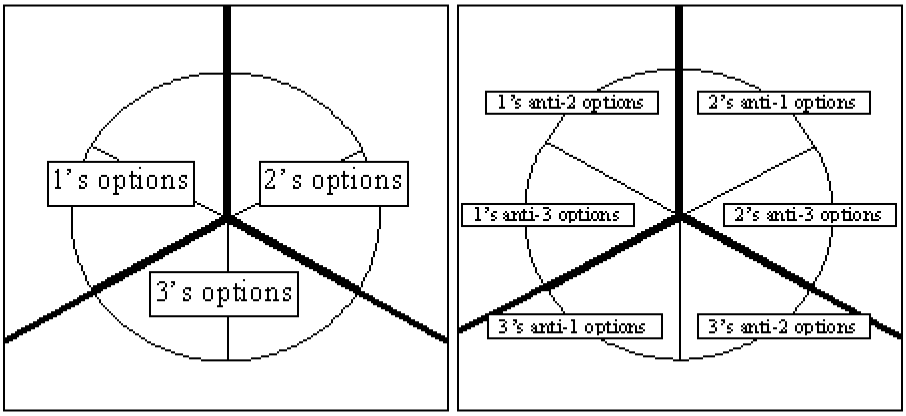
The number plane can also help in visualizing the second condition for a game to be a number game. In the second condition, we need that the difference $v_i - v_j$

in the tuple $v(G)$ is given by a particular number; that number is defined by the options of players i and j.

As previously discussed, the number $v_i - v_j$ must depend only on player i's *anti-j options*, and player j's *anti-i options*. These can be given graphical meaning as well.

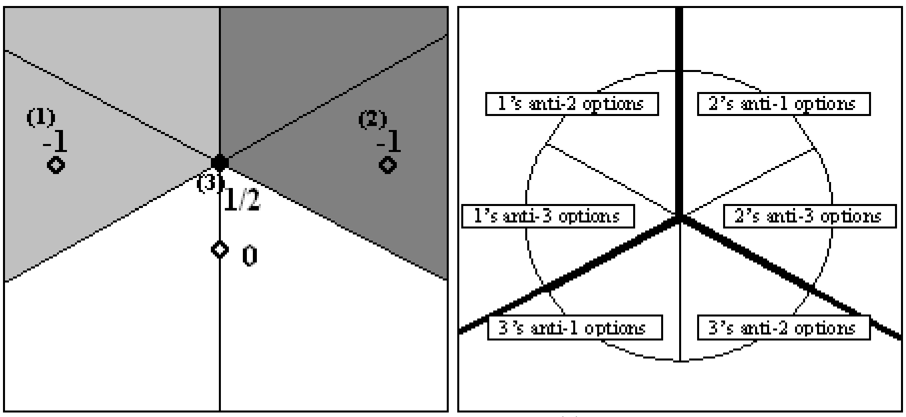
One of i's *anti-j options* G^i of G must first be one of i's options in G, so we first must have $G^i <_i G$. However, the additional restriction $p_i(G^i) \leq_j p_i(G)$ means that disregarding player i, player j must be hurt most by the move.

Thus, an *anti-j option* of G must lie in the region $<_i G$, but it must also be closer to the $<_j G$ region than any $<_k G$ region. The plane then looks like this:



Finding whose options are anti-who.

Returning to the case of the game $^{(3)}(\mathbf{1/2})$, we can use this reference to classify each option in the game.



Classifying the options of $^{(3)}(\mathbf{1/2})$.

So, player 1's option $^{(1)} - \mathbf{1}$ is an *anti-3* option,
Player 2's option $^{(2)} - \mathbf{1}$ is an *anti-3* option,

and player 3's option $\mathbf{0}$ is both *anti-1* and *anti-2*.

So, we calculate $v_1 - v_2$ to be 0, since player 1 has no anti-2 options, and player 2 has no $anti - 1$ options.

$v_1 - v_3$ must be given by

$$\begin{aligned} v_1 - v_3 &= \left\langle v_1\left({}^{(\mathbf{1})} - \mathbf{1}\right) - v_3\left({}^{(\mathbf{1})} - \mathbf{1}\right) \middle| v_1(\mathbf{0}) - v_3(\mathbf{0})\right\rangle \\ &= \langle (-2/3) - (1/3) | 0 - 0 \rangle \\ &= \langle -1 | 0 \rangle \\ &= -1/2, \text{ as desired. Player 3 has a 1/2-move advantage.} \end{aligned}$$

$v_2 - v_3$ would be calculated similarly.

6. Appendix 2: Why do we define number games in this way?

At first glance, our definition of a number game of multiple players may seem somewhat artificial. In particular, it is not obvious why we need to include in condition (2) the restriction on options which can be considered during evaluation of the $(n-1)$-tuple associated with the n-player game. To determine the advantage player i has over player j, we only use i's *anti-j options* and j's *anti-i options*. It seems that it would be more natural to include all the options on both sides.

However, it turns out that allowing all options to be considered would cause our theory of number games to break down. To be more precise, let us, for the moment, provisionally define a *quasi number game of T players* as a game G for which there exists an n-tuple

$$v(G) \in \mathbb{C}_T$$

such that

(1) For all $i \in T$, all G^i are quasi number games, and $v(G^i) <_i v(G)$.
(2) For all $i \neq j \in T$,

$$v_i(G) - v_j(G) = v\langle \{v_i(G^i) - v_j(G^i)\} | \{v_i(G^j) - v_j(G^j)\} \rangle$$

The problem with this definition is as follows.

Proposition 12. *There exists a quasi number game that does not possess a set of losers as defined in Section 2.*

Proof: : Consider the following game:

$$G = \left({}^{(\mathbf{3})}\mathbf{1}, {}^{(\mathbf{1})}\mathbf{1} + {}^{(\mathbf{3})}\mathbf{2}, \mathbf{0}\right)$$

Each player has one option, and all the options are number games. We shall show that G is a quasi number game. First, note that every number game is a quasi number game by the proof of Lemma 4. Next, every option of G is a number game, and hence a quasi number game. In fact, we find the 3-tuples associated with each of the options of G to be:

$$\begin{aligned} v({}^{(3)}\mathbf{1}) &= (-1/3, -1/3, 2/3) \\ v({}^{(1)}\mathbf{1} + {}^{(3)}\mathbf{2}) &= (0, -1, 1) \\ v(\mathbf{0}) &= (0, 0, 0). \end{aligned}$$

Now we must have for G:

$$\begin{aligned} v_1(G) - v_2(G) &= v\langle\{v_1({}^{(3)}\mathbf{1}) - v_2({}^{(3)}\mathbf{1})\}|\{v_1({}^{(1)}\mathbf{1} + {}^{(3)}\mathbf{2}) - v_2({}^{(1)}\mathbf{1} + {}^{(3)}\mathbf{2})\}\rangle \\ &= v\langle(-1/3) - (-1/3)|(0) - (-1)\rangle \\ &= v\langle 0|1\rangle \\ &= 1/2 \\ v_1(G) - v_3(G) &= v\langle\{v_1({}^{(3)}\mathbf{1}) - v_3({}^{(3)}\mathbf{1})\}|\{v_1(\mathbf{0}) - v_3(\mathbf{0})\}\rangle \\ &= v\langle(-1/3) - (2/3)|(0) - (0)\rangle \\ &= v\langle -1|0\rangle \\ &= -1/2 \end{aligned}$$

$$\begin{aligned} v_2(G) - v_3(G) &= v\langle\{v_2({}^{(1)}\mathbf{1} + {}^{(3)}\mathbf{2}) - v_3({}^{(1)}\mathbf{1} + {}^{(3)}\mathbf{2})\}|\{v_2(\mathbf{0}) - v_3(\mathbf{0})\}\rangle \\ &= v\langle(-1) - (1)|(0) - (0)\rangle \\ &= v\langle -2|0\rangle \\ &= -1 \end{aligned}$$

Then, $v_1(G), v_2(G)$, and $v_3(G)$ must satisfy the equations

$$\begin{aligned} v_1 - v_2 &= 1/2 \\ v_1 - v_3 &= -1/2 \\ v_2 - v_3 &= -1. \end{aligned} \tag{6}$$

The tuple $(0, -1/2, 1/2)$ satisfies these equations, so G is be a quasi number game under the above definition.

However, the game G has no set of possible losers S. To see this, suppose such set S existed.

First, player 1 cannot be a member of S, because under the order of play 1-2-3, player 1 would be the first member of S to move, and so it would have to be possible for player 1 to lose the game G. However, player 2 is the loser in this case.

Second, player 3 cannot be a member of S, because under the order of play 3-2-1, player 3 would be the first member of S to move, and so it would have to be possible for player 3 to lose the game G. However, player 2 is the loser in this case.

Finally, player 2 cannot be a member of S, because if that were true, we would have to have $S = \{\mathbf{2}\}$. However, under the order of play 3-1-2, player 2 is the first member of S to move, and player 1 is still the loser.

Thus, the game G has no set of possible losers S, but is a quasi number game, thus completing the proof of the Proposition. □

For completeness, let us see explicitly why the game G considered in the above proof is not an actual number game. It turns out that the only all of the calculations in the proof remain valid for the number game definition, except restricting to anti-2 and anti-3 options changes the last equation of (6) to

$$v_2(G) - v_3(G) = v\langle\emptyset|\emptyset\rangle = 0.$$

This change will render (6) inconsistent, and thus G is not a number game.

To explain this intuitively, there being no set of possible losers S means roughly that it is sometimes to one's advantage to move first in the game G. In a number game, it should never be to a player's advantage to move first, as a move should always imply a consumption of one's limited resources. The game G does not fit this model, so the relaxed definition must be rejected in favor of the one that we have proposed.

References

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