The search for Ultimate $L$

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The projective sets

**Definition**

A set \( A \subseteq \mathbb{R}^n \) is *projective* if it can be generated from the open subsets of \( \mathbb{R}^n \) in finitely many steps of taking complements and images by continuous functions, 

\[
f : \mathbb{R}^n \to \mathbb{R}^n.
\]

**Definition**

Suppose that \( A \subseteq \mathbb{R} \times \mathbb{R} \). A function \( f \) *uniformizes* \( A \) if for all \( x \in \mathbb{R} \):

- if there exists \( y \in \mathbb{R} \) such that \( (x, y) \in A \) then \( (x, f(x)) \in A \).
Two questions of Luzin

1. Suppose $A \subseteq \mathbb{R} \times \mathbb{R}$ is projective. Can $A$ be uniformized by a projective function?

2. Suppose $A \subseteq \mathbb{R}$ is projective. Is $A$ Lebesgue measurable and does $A$ have the property of Baire?

Luzin’s questions are questions about $\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, \cdot, +, \in \rangle$

Luzin conjectured that “we will never know the answer to the measure question for the projective sets”.

Both questions are unsolvable on the basis of the ZFC axioms
Determinacy and the answers to Luzin’s questions

Suppose $A \subseteq \mathbb{R}$. There is an associated infinite game involving two players.

- The players alternate choosing $\epsilon_i \in \{0, 1\}$.
- After infinitely many moves an infinite binary sequence $\langle \epsilon_i : i \in \mathbb{N} \rangle$ is defined.
- Player I wins this run of the game if

$$\sum_{i=1}^{\infty} \epsilon_i/2^i \in A$$

otherwise Player II wins.

**Definition**

The set $A$ is *determined* if there is a winning strategy for one of the players in the game associated to $A$. 
The Axiom of Determinacy (AD)

**Definition (Mycielski-Steinhaus)**

Axiom of Determinacy (AD): Every set $A \subseteq \mathbb{R}$ is determined.

**Lemma (Axiom of Choice)**

There is a set $A \subseteq \mathbb{R}$ such that $A$ is not determined.

**Corollary**

AD is false.
Definition

*Projective Determinacy (PD):* Every projective set $A \subseteq \mathbb{R}$ is determined.

Theorem

Assume every projective set is determined.

1. (Mycielski-Steinhaus) *Every projective set has the property of Baire.*

2. (Mycielski-Swierczkowski) *Every projective set is Lebesgue measurable.*

3. (Moschovakis) *Every projective set $A \subseteq \mathbb{R} \times \mathbb{R}$ can be uniformized by a projective function.*

Key questions

*Is PD even consistent and if consistent, is PD true?*
Basic notions: Logical definability from parameters

- A set $X$ is *transitive* if $A \subset X$ for all $A \in X$.
- A transitive set $X$ is an *ordinal* if $(X, \in)$ is a totally ordered set.
- $\omega$ is the least infinite ordinal, $(\omega, \in) \cong (\mathbb{N}, <)$.
- $\omega_1$ is the least uncountable ordinal.

**Definition**

Suppose $X$ is a transitive set. A subset $Y \subseteq X$ is logically definable in $(X, \in)$ from parameters if for some formula $\varphi[x_0, \ldots, x_n]$ and for some parameters $a_1, \ldots, a_n \in X$,

$$Y = \{ a \in X \mid (X, \in) \models \varphi[a, a_1, \ldots, a_n] \}$$

□
Basic notions: Elementary embeddings

**Definition**

Suppose $X$ and $Y$ are transitive sets. A function $j : X \rightarrow Y$ is an *elementary embedding* if for all logical formulas $\varphi[x_0, \ldots, x_n]$ and all $a_0, \ldots, a_n \in X$,

$$(X, \in) \models \varphi[a_0, \ldots, a_n] \text{ if and only if } (Y, \in) \models \varphi[j(a_0), \ldots, j(a_n)].$$

- Isomorphisms are elementary embeddings but the only isomorphisms of $(X, \in)$ and $(Y, \in)$ are trivial.

**Lemma**

Suppose that $j : X \rightarrow Y$ is an elementary embedding and that $X \models \text{ZFC}$. Then the following are equivalent.

1. $j$ is not the identity.
2. There is an ordinal $\beta \in X$ such that $j(\beta) \neq \beta$. 
Strong axioms of infinity: large cardinal axioms

Basic template for large cardinal axioms

A cardinal $\kappa$ is a large cardinal if there exists an elementary embedding,

$$j : V \rightarrow M$$

such that $M$ is a transitive class and $\kappa$ is the least ordinal such that $j(\alpha) \neq \alpha$.

- Requiring $M$ be close to $V$ yields a hierarchy of large cardinal axioms:
  - simplest case is where $\kappa$ is a measurable cardinal.
  - $M = V$ contradicts the Axiom of Choice.

The hierarchy of large cardinal axioms has emerged as the fundamental core of Set Theory.

- It is (empirically) a wellordered hierarchy and provides a calibration of the unsolvability of problems in Set Theory.
The validation of Projective Determinacy

**Theorem (Martin-Steel)**

Assume there are infinitely many Woodin cardinals. Then every projective set is determined.

**Theorem**

The following are equivalent.

1. Every projective set is determined.
2. For each $n < \omega$ there is a countable (iterable) transitive set $M$ such that
   
   \[ M \models \text{ZFC} + \text{“There exist } n \text{ Woodin cardinals”}, \]

**PD** is the missing (and true) axiom for $\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, \cdot, +, \in \rangle$

Is there such an axiom for $V$ itself?
Mathematical truth and two (very) controversial claims

- Large cardinal axioms predict facts about our world.

**Prediction**

There will be no contradiction discovered from PD (by any means) before the year 3010.

There will be no contradiction discovered from PD (by any means) before all the Clay Millennium problems have been solved.

- A far more controversial claim.

**Claim**

*Consistency claims for large cardinal axioms require a conception of the Universe of Sets in which large cardinals axioms are true.*

- This ultimately requires that questions such as that of the Continuum Hypothesis also be resolved
  - or an explanation of the exact nature of the ambiguity.
Basic notions: The cumulative hierarchy

- If $X$ is a set then $\mathcal{P}(X)$ denotes the set of all subsets of $X$:

$$\mathcal{P}(X) = \{ Y \mid Y \subseteq X \}.$$ 

The von Neumann cumulative hierarchy of sets

1. $V_0 = \emptyset$.
2. (Successor case) $V_{\alpha+1} = \mathcal{P}(V_\alpha)$.
3. (Limit case) $V_\alpha = \bigcup \{ V_\beta \mid \beta < \alpha \}$.

- $V_\omega$ is bi-interpretable with $\langle \mathbb{N}, \cdot, + \rangle$.
- $V_{\omega+1}$ is bi-interpretable with $\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, \cdot, +, \in \rangle$.
- The Continuum Hypothesis is a question of $V_{\omega+2}$. 
The effective cumulative hierarchy: $L$

The definable power set

For each set $X$, $\mathcal{P}_{\text{Def}}(X)$ denotes the set of all $Y \subseteq X$ such that $X$ is logically definable in the structure $(X, \in)$ from parameters in $X$.

- (Axiom of Choice) $\mathcal{P}_{\text{Def}}(X) = \mathcal{P}(X)$ if and only if $X$ is finite.
- $\mathcal{P}_{\text{Def}}(V_{\omega+1}) \cap \mathcal{P}(\mathbb{R})$ is exactly the projective sets.

Gödel’s constructible universe, $L$

Define $L_\alpha$ by induction on $\alpha$ as follows.

1. $L_0 = \emptyset$,
2. (Successor case) $L_{\alpha+1} = \mathcal{P}_{\text{Def}}(L_\alpha)$,
3. (Limit case) $L_\alpha = \bigcup \{ L_\beta \mid \beta < \alpha \}$.

$L$ is the class of all sets $X$ such that $X \in L_\alpha$ for some ordinal $\alpha$. 
The axiom $V = L$, the projective sets, and large cardinals

**Theorem**

Assume $V = L$.

1. (Gödel) *Every projective set* $A \subseteq \mathbb{R} \times \mathbb{R}$ *can be uniformized by a projective function.*
2. (Gödel) *There is a projective set which is not Lebesgue measurable:*
   - there is a projective wellordering of the reals.
3. (Scott) *There are no measurable cardinals:*
   - there are no (interesting) large cardinals.

**Corollary**

$V \neq L.$
The inner model problem

For a specified large cardinal axiom, produce a generalization of Gödel’s construction of $L$ which is compatible with the given large cardinal axiom.

- Fundamental issue: how to make this problem precise.

Goals

- Understand the hierarchy of large cardinals.
- Use this to understand the Universe of Sets.
  - perhaps even to find an ultimate version of $L$. 
Basic notions: Enlargements of \( L \)

**Definition**

Suppose \( Y \) is a set (or class). Then

1. \( L_0[Y] = \emptyset \),
2. (Successor case) \( L_{\alpha+1}[Y] = \mathcal{P}_{\text{Def}}(Z) \) where
   \[
   Z = L_\alpha[Y] \cup \{Y \cap L_\alpha[Y]\},
   \]
3. (Limit case) \( L_\alpha[Y] = \bigcup \{L_\beta[Y] \mid \beta < \alpha\} \).

\( L[Y] \) is the class of all sets \( a \) such that \( a \in L_\alpha[Y] \) for some ordinal \( \alpha \).

- If \( Y \cap L = \emptyset \) then \( L[Y] = L \).
- For every set \( X \) there is a set \( Y \) such that \( X \in L[Y] \).
  - This is equivalent to the Axiom of Choice.
Basic notions: More enlargements of $L$

**Definition**

Suppose $X$ is a transitive set. Then

1. $L_0(X) = X$, 
2. (Successor case) $L_{\alpha+1}(X) = \mathcal{P}_{\text{Def}}(L_\alpha(X))$, 
3. (Limit case) $L_\alpha(X) = \bigcup \{L_\beta(X) \mid \beta < \alpha\}$.

$L(X)$ is the class of all sets $a$ such that $a \in L_\alpha(X)$ for some $\alpha$. □

- In general $X \notin L[X]$, but necessarily $X \in L(X)$.

Interesting cases for analyzing $L(X)$:
- $X = \mathbb{R}$
  - $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ is a transfinite extension of the hierarchy of the projective sets.
- $X = \mathcal{P}(\mathbb{R})$. Understanding $L(\mathcal{P}(\mathbb{R}))$ would resolve CH.

**Suppose** $X$ is a transitive set and $Y$ is a set (or class). Then combining the definitions of $L(X)$ and $L[Y]$ one obtains $L(X)[Y]$. 
Universally Baire sets

Speculation

In general the enlargements of $L$ produced by the Inner Model Program have companion enlargements of $L(\mathbb{R})$.

- Even though we cannot yet define the ultimate enlargement of $L$ perhaps we can define the ultimate enlargement of $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$.

Definition (Feng-Magidor-Woodin)

A set $A \subseteq \mathbb{R}^n$ is universally Baire if for all topological spaces $\Omega$ and for all continuous functions $\pi : \Omega \to \mathbb{R}^n$, the preimage of $A$ by $\pi$ has the property of Baire in the space $\Omega$.

- Universally Baire sets are necessarily Lebesgue measurable and have the property of Baire.
An abstract generalization of the projective sets

**Theorem**

*Suppose that there is a proper class of Woodin cardinals. Then every projective set is universally Baire.*

**Theorem**

*Suppose that there is a proper class of Woodin cardinals.*

1. *(Martin-Steel)* *Suppose \( A \subseteq \mathbb{R} \) is universally Baire. Then \( A \) is determined.*
2. *Suppose \( A \subseteq \mathbb{R} \) is universally Baire. Then every set \( B \in L(A, \mathbb{R}) \cap \mathcal{P}(\mathbb{R}) \) is universally Baire.*
3. *(Steel)* *Suppose \( A \subseteq \mathbb{R} \times \mathbb{R} \) is universally Baire. Then \( A \) can be uniformized by a universally Baire function.*

- The answers to Luzin’s questions are both yes for the universally Baire sets.
Measuring the complexity of universally Baire sets

Definition

Suppose $A$ and $B$ are subsets of $\mathbb{R}$.

1. $A$ is *borel reducible* to $B$, $A \leq_{\text{borel}} B$, if there is a borel function $\pi : \mathbb{R} \to \mathbb{R}$ such that
   - either $A = \pi^{-1}[B]$ or $A = \mathbb{R} \setminus \pi^{-1}[B]$.

2. $A$ and $B$ are *borel bi-reducible* if
   - $A \leq_{\text{borel}} B$ and $B \leq_{\text{borel}} A$.

3. The *borel degree* of $A$ is the equivalence class of all sets which are borel bi-reducible with $A$. 
Theorem (Martin-Steel, Martin, Wadge)

Assume there is a proper class of Woodin cardinals.

Then the borel degrees of the universally Baire sets are linearly ordered by borel reducibility and moreover this is a wellorder.

Observation

This order refines the order of generation of the universally Baire sets in any possible enlargement of $L$ adapted to define an enlargement of $L(R)$:

Suppose $X$ is transitive and $R \subseteq X$,

- if $A \leq_{borel} B$ and $B \in \mathcal{P}_{\text{Def}}(X)$ then $A \in \mathcal{P}_{\text{Def}}(X)$. 
Conclusion

Solutions to the inner model problem, adapted to produce enlargements of $L(\mathbb{R})$, must define initial segments of the universally Baire sets.

Speculation

Perhaps using universally Baire sets one can directly define solutions to the Inner Model Problem for all large cardinal axioms.

- Shifting the focus of the Inner Model Program to an analysis of structures one can already define.
- Removing the incremental nature of the Inner Model Program.
- Allowing for a definition of an ultimate version of $L$. 
Gödel’s transitive class HOD

**Definition**

HOD is the class of all sets $X$ such that there exist $\alpha \in \text{Ord}$ and $Y \subseteq \alpha$ such that

1. $Y$ is definable in $V_\alpha$ without parameters,
2. $X \in L[Y]$.

▶ (ZF) The Axiom of Choice holds in HOD.

**Definition**

Suppose that $A \subseteq \mathbb{R}$. Then $\text{HOD}^{L(A, \mathbb{R})}$ is the class HOD as defined within $L(A, \mathbb{R})$. 
Definition
Suppose that \( A \subseteq \mathbb{R} \) is universally Baire.

Then \( \Theta^{L(A,\mathbb{R})} \) is the supremum of the ordinals \( \alpha \) such that there is a surjection, \( \pi : \mathbb{R} \rightarrow \alpha \), such that \( \pi \in L(A, \mathbb{R}) \).

- \( \Theta^{L(A,\mathbb{R})} \) is another measure of the complexity of \( A \).
- \( \Theta^{L(A,\mathbb{R})} \) bounds the influence of \( L(A, \mathbb{R}) \) on the structure of the ordinals.

Theorem
Suppose that there is a proper class of Woodin cardinals and that \( A \) is universally Baire.

Then \( \Theta^{L(A,\mathbb{R})} \) is a Woodin cardinal in \( \text{HOD}^{L(A,\mathbb{R})} \).
HOD$^{L(A,\mathbb{R})}$ and the Inner Model Program

**Theorem (Steel)**

Suppose that there is a proper class of Woodin cardinals and let $\delta = \Theta^{L(\mathbb{R})}$.

Then $HOD^{L(\mathbb{R})} \cap V_\delta$ is a Mitchell-Steel inner model.

**Theorem**

Suppose that there is a proper class of Woodin cardinals.

Then $HOD^{L(\mathbb{R})}$ is not a Mitchell-Steel inner model.

There is another class of solutions to the inner model problem for large cardinals.

- previously unknown.
(Conjecture) The axiom scheme for $V = \text{ultimate } L$

There is a proper class of Woodin cardinals. Further for each sentence $\varphi$, if $\varphi$ holds in $V$ then there is a universally Baire set $A \subseteq \mathbb{R}$ such that

$$\text{HOD}^{L(A,\mathbb{R})} \cap V_\Theta \models \varphi$$

where $\Theta = \Theta^{L(A,\mathbb{R})}$.

(meta) Conjecture

This axiom will be validated on the basis of compelling and accepted principles of infinity just as the axiom PD has been.

- This axiom will reduce all questions of Set Theory to axioms of infinity.