Forcing axioms and unsolvable problems

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Ambiguous notions of set

The two sources for ambiguity in Set Theory

1. *Large cardinal axioms*
2. *Cohen extensions.*

- The large cardinal hierarchy seems to be a rich coherent structure.
- Are there axioms which can mitigate or eliminate the ambiguities which arise because of Cohen's method?
Forcing axioms

Definition

Suppose \( \mathcal{B} \) is a complete Boolean algebra and \( \kappa \) is a cardinal. The \( \kappa \)-Baire Category Theorem holds for \( \mathcal{B} \) if the following holds in \( \Omega \) where \( \Omega \) is the Stone space of \( \mathcal{B} \).

- Suppose \( \mathcal{A} \) is a family of open dense subsets of \( \Omega \) and \( |\mathcal{A}| \leq \kappa \). Then \( \cap \mathcal{A} \) is dense in \( \Omega \).

- The \( \omega \)-Baire Category Theorem holds for all complete Boolean algebras.

Question

For which complete Boolean algebras \( \mathcal{B} \) can the \( \omega_1 \)-Baire Category Theorem hold for \( \mathcal{B} \)?

- The \( \omega_1 \)-Baire Category Theorem cannot hold for all complete Boolean algebras.
ccc complete Boolean algebras

**Definition**

A complete Boolean algebra $\mathcal{B}$ is ccc if whenever

$$A \subset \mathcal{B}$$

is an antichain of nonzero elements, $A$ is countable.

**Examples.**

1. $\mathcal{B}$ is the measure algebra of borel subsets of $[0, 1]$ modulo the ideal of borel sets of measure 0.
2. $\mathcal{B}$ is the Boolean algebra of borel subsets of $[0, 1]$ modulo the ideal of meager borel sets.
Martin’s Axiom

Definition

**Martin’s Axiom:** The $\omega_1$-Baire Category Theorem holds for all ccc complete Boolean algebras.

(This is really $\text{MA}_{\omega_1}$)

Lemma (Martin’s Axiom)

*The ideal of meager subsets of $[0,1]$ is closed under unions of cardinality $\omega_1$.*

Lemma (Martin’s Axiom)

*The ideal of Lebesgue null subsets of $[0,1]$ is closed under unions of cardinality $\omega_1$.*

Corollary (Martin’s Axiom)

$2^{\aleph_0} > \aleph_1$. 
Suslin’s Hypothesis

Suppose that \((L, <)\) is a dense, order-complete, linear order without endpoints. Then

\[(L, <) \cong (\mathbb{R}, <)\]

if and only if there is no uncountable family of pairwise disjoint open intervals of \((L, <)\).

Suslin’s Hypothesis is independent of the ZFC axioms.

Theorem (Martin’s Axiom: Martin, Solovay)

Suslin’s Hypothesis holds.
More consequences of Martin’s Axiom

**Theorem (Martin’s Axiom: Solovay)**

*Suppose that $X \subset \mathbb{R}$ and $|X| = \omega_1$. Then $\mathcal{P}(X) = \{A \cap X \mid A \text{ is borel}\}$. □*

**Corollary (Martin’s Axiom)**

$2^{\aleph_0} = 2^{\aleph_1}$. □
Limitations of Martin’s Axiom

**Definition**

\((\mathbb{N}^\mathbb{N}, <_F)\) is the partial order of all functions,

\[ f : \mathbb{N} \to \mathbb{N} \]

where \( f <_F g \) if \( f(k) < g(k) \) for all but finitely many \( k \in \mathbb{N} \).

**Definition**

A gap in \((\mathbb{N}^\mathbb{N}, <_F)\) is a pair \((A, B)\) such that

1. \( A \cup B \) is a chain of \((\mathbb{N}^\mathbb{N}, <_F)\),
2. \( f <_F g \) for all \( f \in A \) and for all \( g \in B \),
3. There is no function \( h \in \mathbb{N}^\mathbb{N} \) such that
   
   \[ f <_F h <_F g \]

   for all \( f \in A \) and \( g \in B \).
Definition

A gap \((A, B)\) in \((\mathbb{N}^\mathbb{N}, <_F)\) is a \((\kappa, \lambda)\)-gap if

1. \(\text{cof}(A) = \kappa\).
2. \(\text{coi}(B) = \lambda\).

- If either \(\kappa\) or \(\lambda\) are 1 then both are 1.
  - The \((1, 1)\)-gaps are the trivial gaps.

The (nontrivial) gaps must be specified by functions

\[
f : \mathbb{N} \to \mathbb{N}
\]

such that \(\lim_{n \to \infty} f(n) = \infty\).

- One could also just consider \((\mathbb{Q}^\mathbb{N}, <_F)\).
The gap structure of \((\mathbb{N}^\mathbb{N}, <_F)\)

**Lemma**

There are no \((\omega, \omega)\) gaps in \((\mathbb{N}^\mathbb{N}, <_F)\).

**Corollary (CH)**

There is an \((\omega, \omega_1)\)-gap in \((\mathbb{N}^\mathbb{N}, <_F)\).

**Lemma (Martin’s Axiom)**

There is no \((\omega, \omega_1)\)-gap in \((\mathbb{N}^\mathbb{N}, <_F)\).

**Corollary (Martin’s Axiom + 2^{\aleph_0} = \aleph_2)**

There is an \((\omega, \omega_2)\)-gap in \((\mathbb{N}^\mathbb{N}, <_F)\).
Question

Assume Martin’s Axiom and $2^\aleph_0 = \aleph_2$.

- Are there $(\omega_1, \omega_2)$-gaps in $(\mathbb{N}^\mathbb{N}, <_F)$?

Theorem (Hausdorff)

There is an $(\omega_1, \omega_1)$-gap in $(\mathbb{N}^\mathbb{N}, <_F)$.

Theorem (Kunen)

The following are each independent of Martin’s Axiom assuming $2^\aleph_0 = \aleph_2$.

1. There is an $(\omega_1, \omega_2)$-gap in $(\mathbb{N}^\mathbb{N}, <_F)$.
2. There is an $(\omega_2, \omega_2)$-gap in $(\mathbb{N}^\mathbb{N}, <_F)$.

- The resolution of the gap structure requires a stronger version of Martin’s Axiom.
Stationary sets in $\omega_1$

**Definition**

1. A cofinal set $C \subseteq \omega_1$ is **closed and unbounded** if if for all limit ordinals $\alpha < \omega_1$, if $C \cap \alpha$ is cofinal in $\alpha$ then $\alpha \in C$.
2. A set $S \subset \omega_1$ is **stationary** if

   $$S \cap C \neq \emptyset$$

   for all closed unbounded sets $C \subset \omega_1$.

**Assuming the Axiom of Choice, there exist sets $S \subset \omega_1$ such that both $S$ and $\omega_1 \setminus S$ are stationary.**
Stationary set preserving

**Definition**

A complete Boolean algebra $\mathbb{B}$ is *stationary set preserving* if the following holds for all $c \in \mathbb{B}$ with $c > 0$, for all sequences

$$\langle b_\alpha : \alpha < \omega_1 \rangle$$

of elements of $\mathbb{B}$, and for all stationary sets $S \subseteq \omega_1$.

- If $c \leq \vee \{ b_\alpha | \beta < \alpha < \omega_1 \}$ for all $\beta < \omega_1$,

then there exists $\eta \in S$ and $0 < d \leq c$ such that

- $d \leq \vee \{ b_\alpha | \beta < \alpha < \eta \}$ for all $\beta < \eta$.

- Every ccc Boolean algebra is stationary set preserving.
Theorem (Foreman, Magidor, Shelah)

Suppose that $\mathcal{B}$ is a complete Boolean algebra and that the $\omega_1$-Baire Category Theorem holds for $\mathcal{B}$. Then $\mathcal{B}$ is stationary set preserving.

Definition (Foreman, Magidor, Shelah)

**Martin’s Maximum**: The $\omega_1$-Baire Category Theorem holds for all stationary set preserving complete Boolean algebras.
Theorem (Foreman, Magidor, Shelah)

Suppose there is a supercompact cardinal. Then there is a stationary set preserving complete Boolean algebra $\mathbb{B}$ such that

$$V^\mathbb{B} \models \text{Martin’s Maximum}. \qed$$

Theorem (Foreman, Magidor, Shelah)

Assume Martin’s Maximum. Then $2^{\aleph_0} = \aleph_2$. 
Martin’s Maximum and gaps in $(\mathbb{N}^\mathbb{N}, <_\mathcal{F})$

**Theorem (Martin’s Maximum)**

1. There is no $(\omega_2, \omega_2)$ gap in $(\mathbb{N}^\mathbb{N}, <_\mathcal{F})$.
2. There is no $(\omega_1, \omega_2)$ gap in $(\mathbb{N}^\mathbb{N}, <_\mathcal{F})$.

**Theorem (Martin’s Maximum)**

Suppose that

$$\pi : C([0,1]) \rightarrow A$$

is an algebra homomorphism of $C([0,1])$ into a Banach algebra $A$.

Then $\pi$ is continuous.
Definition

A set \( X \subset \mathbb{R} \) is uniformly of cardinality \( \omega_1 \) if for each (nonempty) open interval \( I \), \( X \cap I \) has cardinality \( \omega_1 \).

Lemma

Any two countable dense subsets of \( \mathbb{R} \) are order-isomorphic.

Lemma (CH)

There are two subsets of \( \mathbb{R} \) of uniform cardinality \( \omega_1 \) which are not order-isomorphic.

Theorem (Martin’s Maximum: Baumgartner)

Any two subsets of \( \mathbb{R} \) of uniform cardinality \( \omega_1 \) are order isomorphic.
Second generation results

Lemma
There are 2 infinite total orders such that every infinite total order contains an isomorphic copy of them.

> \((\mathbb{Z}^+, <) \) and \((\mathbb{Z}^-, <)\)

Theorem (Martin’s Maximum: J. Moore)
There are 5 uncountable total orders such that every uncountable total order contains an isomorphic copy of one of them.

Conjecture (Moore)
Suppose there is a basis for the uncountable orders of cardinality strictly less that \(2^{\aleph_1}\). Then \(2^{\aleph_0} = \aleph_2\).
The Brown-Douglas-Filmore Problem

Suppose $H$ is a separable (infinite dimensional) Hilbert space. The Calkin Algebra is the $C^*$-algebra,

$$\mathcal{B}(H)/\mathcal{K}(H)$$

where $\mathcal{B}(H)$ is the algebra of bounded operators and $\mathcal{K}(H)$ is the algebra of compact operators. If $U$ is a unitary operator, then the map

$$\pi(A) = U^* AU$$

induces an automorphism of $\mathcal{B}(H)/\mathcal{K}(H)$, These are the *inner automorphisms*.

Question (Brown-Douglas-Filmore)

*Suppose that*

$$\pi : \mathcal{B}(H)/\mathcal{K}(H) \to \mathcal{B}(H)/\mathcal{K}(H)$$

*is an automorphism. Must $\pi$ be an inner automorphism?*
Set theoretic entanglements

Theorem (CH: Phillips and Weaver)

There is an automorphism of $\mathcal{B}(H)/\mathcal{K}(H)$ which is not an inner automorphism.

Theorem (Martin’s Maximum: I. Farah)

Every automorphism of $\mathcal{B}(H)/\mathcal{K}(H)$ is an inner automorphism.
Summary: Consequences of Martin’s Maximum

Assume Martin’s Maximum.

1. Suslin’s Hypothesis.
2. Suppose $X, Y \subset \mathbb{R}$ are each dense and locally of cardinality $\omega_1$. Then $X$ and $Y$ are order isomorphic.
3. There is a 5 element basis for the uncountable linear orders.
4. Every homomorphism $\pi : C([0,1]) \to \mathcal{A}$ of $C[0,1]$ into a Banach algebra is automatically continuous.
5. Every automorphism of the Calkin Algebra is an inner automorphism.
Is Martin’s Maximum true?

**Definition**

Suppose that $\kappa$ is an infinite regular cardinal. $H(\kappa)$ is the set of all sets $X$ such that there is a transitive set $Y$ such that

1. $X \in Y$,
2. $|Y| < \kappa$.

Two interesting cases.

1. $H(\omega_1)$. This is logically equivalent to $V_{\omega + 1}$.
2. $H(\omega_2)$. Assuming CH this is logically equivalent to $V_{\omega + 2}$. 
The structure \((H(\omega_2), I_{\text{NS}})\)

**Definition**

\(I_{\text{NS}}\) is the ideal of all non-stationary subsets of \(\omega_1\).

**Lemma (Martin’s Maximum)**

Suppose that \(\varphi\) is a \(\Pi_2\) sentence and that there is a stationary set preserving Boolean algebra \(B\) such that

\[ V^B \models "H(\omega_2) \models \varphi" \]

Then \(H(\omega_2) \models \varphi\).

- Assuming Martin’s Maximum in an enhanced form, this extends to the structure \((H(\omega_2), I_{\text{NS}})\).
Observation

*Martin’s Maximum is attempting to maximize the $\Pi_2$-theory of $(H(\omega_2), I_{NS})$.*

Definition

A $\Pi_2$-sentence $\varphi$ is $\Omega$-satisfiable for $(H(\omega_2), I_{NS})$ if there is a complete Boolean algebra $\mathbb{B}$ such that

$$V^\mathbb{B} \models "(H(\omega_2), I_{NS}) \models \varphi."$$

- No requirement that $\mathbb{B}$ be stationary set preserving.

Question

*Suppose $\varphi_1$ and $\varphi_2$ are each $\Pi_2$-sentences which are $\Omega$-satisfiable for $(H(\omega_2), I_{NS})$. Must $\varphi_1 \land \varphi_2$ be $\Omega$-satisfiable for $(H(\omega_2), I_{NS})$?*
Theorem (Π₂-Maximality)

Assume there is a proper class of Woodin cardinals. There is a partial order $\mathbb{P}_{\text{max}} \in L(\mathbb{R})$ such that the following hold.

1. $\mathbb{P}_{\text{max}}$ is homogeneous and $\omega$-closed.
2. $L(\mathbb{R})^{\mathbb{P}_{\text{max}}} \models \text{ZFC}.$
3. Suppose that $\varphi$ is a $\Pi_2$-sentence which is $\Omega$-satisfiable for $(H(\omega_2), I_{\text{NS}})$. Then
   
   $L(\mathbb{R})^{\mathbb{P}_{\text{max}}} \models “(H(\omega_2), I_{\text{NS}}) \models \varphi”.$

$\square$
Observation

*Most of the applications of Martin’s Maximum which I have discussed involve only the $\Pi_2$-consequences of Martin’s Maximum for the structure*

$$(H(\omega_2), \mathcal{I}_{NS}).$$

The Axiom ($\ast$)

1. $L(\mathbb{R}) \models \text{AD}$.
2. *There is an $L(\mathbb{R})$-generic filter $G \subset \mathbb{P}_{\text{max}}$ such that*

$$\mathcal{P}(\omega_1) \subset L(\mathbb{R})[G].$$

One needs to extend the influence of the Axiom ($\ast$) to $H(c^+)$ if one wants to recover all the consequences of Martin’s Maximum which I have discussed.
The Sealing Theorem

Suppose that there is a proper class of Woodin cardinals. A natural question is whether

\[ \Gamma = L(\Gamma) \cap \mathcal{P}(\mathbb{R}) \]

where \( \Gamma \) is the collection of all universally Baire sets \( A \subseteq \mathbb{R} \).

Theorem (Generic Sealing)

Suppose \( \delta \) is a supercompact cardinal and that \( V[G] \subseteq V[H] \) are set generic extensions of \( V \) such that \( V_{\delta+2} \) is countable in \( V[G] \).

Let \( \Gamma_G \) be the collection of universally Baire sets as computed in \( V[G] \) and let \( \Gamma_H \) be the collection of universally Baire sets as computed in \( V[H] \). Then there is an elementary embedding

\[ \pi : L(\Gamma_G) \rightarrow L(\Gamma_H) \]

such that \( \pi|\Gamma_G \) is the canonical map. \( \square \)
The ultimate forcing axiom?

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<thead>
<tr>
<th>The Axiom $(\ast)^+$</th>
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<tbody>
<tr>
<td>There is a proper class of measurable Woodin cardinals. Let $\Gamma$ be the collection of all universally Baire sets.</td>
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<tr>
<td>1. $\Gamma = L(\Gamma) \cap \mathcal{P}(\mathbb{R})$.</td>
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<th>Theorem</th>
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<td>The Axiom $(\ast)^+$ implies the Axiom $(\ast)$.</td>
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| All the consequences of Martin’s Maximum which I have discussed follow from Axiom $(\ast)^+$. |
Claim

The Axiom (∗) is the ultimate forcing axiom as far as the structure of $H(\omega_2)$ is concerned. This is certified by:
- The $\Pi_2$-Maximality Theorem.
- The extension of (∗) to (∗)$^+$. 

Theorem

Assume there is a proper class of Woodin cardinals and that the Axiom (∗) holds.

Then $H(\omega_2)$ is logically bi-interpretable with $H(\omega_1)$. 
Summary

Conclusion

The ultimate forcing axiom logically reduces $H(\omega_2)$ to $H(\omega_1)$. 