

# SEVEN TREES IN ONE

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ABSTRACT. Following a remark of Lawvere, we explicitly exhibit a particularly elementary bijection between the set  $T$  of finite binary trees and the set  $T^7$  of seven-tuples of such trees. “Particularly elementary” means that the application of the bijection to a seven-tuple of trees involves case distinctions only down to a fixed depth (namely four) in the given seven-tuple. We clarify how this and similar bijections are related to the free commutative semiring on one generator  $X$  subject to  $X = 1 + X^2$ . Finally, our main theorem is that the existence of particularly elementary bijections can be deduced from the provable existence, in intuitionistic type theory, of any bijections at all.

## INTRODUCTION

This paper was motivated by a remark of Lawvere [8], which implies that there is a particularly elementary coding of seven-tuples of binary trees as single binary trees. In Section 1, we explicitly exhibit such a coding and discuss the sense in which it is particularly elementary. In Section 2, we discuss the algebra behind this situation, which explains why “seven” appears here. Section 3 connects, in a somewhat more general context, the algebraic manipulations of Section 2 with the elementary codings of Section 1. Finally, in Section 4, we prove a meta-theorem saying that such particularly elementary constructions can be extracted from existence proofs carried out in the much more liberal context of constructive type theory.

Throughout this paper, we use *tree* to mean specifically a finite binary tree in which the immediate successors (= children) of any node are labeled left and right. Even if a node has only one child, the child must have a label. We admit the empty tree, denoted by 0, but every non-empty tree has a unique root, from which every node can be reached by repeatedly passing to children. The *depth* of a node is defined as the number of nodes on the path joining it to the root (so the root has depth 1), and the largest depth of any node is the *depth* of the tree. (If we need to assign a depth to the empty tree, we assign 0.) As this terminology suggests, we visualize trees as growing downward, the root being at the top. We use the notation  $[t_1, t_2]$  for the tree consisting of a root, a left subtree (consisting of the left child, if any, and all its descendants) isomorphic to  $t_1$ , and a right subtree isomorphic to

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$t_2$ . For example,  $[0, 0]$  consists of just the root. By the *leftward path* of a tree, we mean the set of those nodes such that neither the node nor any ancestor of it is a right child, i.e., the set consisting of the root (if any), its left child (if any), its left child (if any), etc.

## 1. A VERY EXPLICIT BIJECTION

Our first theorem is due to Lawvere, who mentions it (though without proof and not quite in the same form) in [8].

**Theorem 1.** *There is a very explicit bijection between the set of all seven-tuples of trees and the set of all trees.*

Before we prove the theorem, we must explain the meaning of “very explicit,” for without this phrase the theorem is trivial. The set  $T$  of all trees (as defined in the introduction) is clearly countably infinite and is therefore in one-to-one correspondence with the set  $T^7$  of seven-tuples of trees (and also with  $T^k$  for any finite  $k \geq 1$ ). Since a bijection from  $T$  to the set of natural numbers can be given explicitly, so can the bijection  $T^7 \rightarrow T$ . This is *not* the content of the theorem. “Very explicit” means, roughly, that the value of the bijection  $f$  at a seven-tuple  $\vec{t} = (t_1, \dots, t_7)$  of trees can be determined by (1) inspecting these seven trees down to a depth  $n$  that depends only on  $f$ , not on the particular trees, (2) depending on these seven partial trees, constructing a part  $p$  of  $f(\vec{t})$ , and (3) taking the subtrees that were below depth  $n$  in the  $t_i$ ’s (and were thus ignored at step (1)) and attaching them to certain leaves of  $p$ . In other words, any structure occurring below depth  $n$  in the  $t_i$ ’s is simply copied into  $f(\vec{t})$ ; the “real work” done by  $f$  involves the  $t_i$ ’s only to depth  $n$ .

For a more precise definition of “very explicit” (which the impatient reader can skip for now, as it will not be needed until Section 3), first define a *pattern* to be a tree in which some of the leaves are labeled with distinct symbols. (This labeling has nothing to do with the left-right labeling that is part of the structure of every tree.) An *instance* of a pattern is obtained by replacing each labeled leaf by some tree; the function assigning to each label the corresponding tree is called the *substitution* leading to the instance. (Note that the substitution can have the empty tree  $0$  as a value; the corresponding labeled node is then removed from the pattern when the instance is formed.) Similarly, we define *7-patterns* to be seven-tuples of patterns with all labels distinct, and we define their instances under substitutions, which are seven-tuples of trees, by replacing every labeled node by the image of its label under the substitution. A *very explicit function*  $f : T^7 \rightarrow T$  is given by (1) a finite indexed family  $(\vec{p}_i)_{i \in I}$  of 7-patterns such that every seven-tuple  $\vec{t}$  of trees is an instance of exactly one  $\vec{p}_i$  (under exactly one substitution) and (2) a family  $(q_i)_{i \in I}$  of patterns, indexed by the same  $I$ , such that each  $q_i$  contains the same labels as  $\vec{p}_i$ . To apply  $f$  to a seven-tuple  $\vec{t}$  of trees, find the unique  $\vec{p}_i$  and the unique substitution yielding  $\vec{t}$  as an instance, and then apply the same substitution to  $q_i$ .

A non-trivial example will occur in the proof of Theorem 1. For a trivial but instructive example, note that there is a very explicit function  $T^2 \rightarrow T$  sending

every pair  $(t_1, t_2)$  of trees to the tree  $[t_1, t_2]$  consisting of a root with left subtree  $t_1$  and right subtree  $t_2$ . (In the notation of the preceding definition, it is given by a singleton  $I$ , so we can omit the subscripts  $i$ ; by  $\vec{p}$  consisting of two one-point trees with the points labeled, say, 1 and 2; and by  $q$  a tree consisting of the root, a left child labeled 1, and a right child labeled 2.) This very explicit map is one-to-one but not quite surjective, as the empty tree is not in its range. There is also a very explicit injection in the opposite direction,  $T \rightarrow T^2$ , sending each tree  $t$  to the pair  $(t, 0)$ . One could apply the Cantor-Schröder-Bernstein argument to this pair of injections to obtain a bijection  $T^2 \rightarrow T$ , but the result is not very explicit. In fact, the bijection one obtains is just like the injection  $T^2 \rightarrow T$  described above except that, if the output is just a leftward path (i.e., a tree where no node has a right child) then the path is shortened by one node. Unlike the injections, this bijection may need to look down to arbitrary depth in its input trees to determine whether the output is a leftward path. We shall see in Section 3 that there is no very explicit bijection  $T^2 \rightarrow T$ .

*Proof of Theorem 1.* We shall define the desired bijection  $f : T^7 \rightarrow T$  by cases depending on the structure of its input  $\vec{t} = (t_1, \dots, t_7)$  (down to depth 4). To improve readability, we shall write integers  $k$  in place of the trees  $t_k$ .

*Case 1.* At least one of the first four trees is non-empty. Output the tree  $[[[[[7, 6], 5], 4], 3], 2], 1]$ .

*Case 2.* Trees 1 through 4 are empty, but tree 5 is not, say  $5 = [5a, 5b]$ . Output the tree  $[[[[0, 7], 6], 5a], 5b]$

*Case 3.* Trees 1 through 5 are empty, but tree 6 is not. Output the tree  $[[[[[6, 7], 0], 0], 0], 0]$

*Case 4.* Trees 1 through 6 are empty, but the leftward path in tree 7 has at least 4 nodes. So  $7 = [[[[7a, 7b], 7c], 7d], 7e]$ . Output the tree  $[[[[[0, 7a], 7b], 7c], 7d], 7e]$ .

*Case 5.* Otherwise, output tree 7.

This  $f$  clearly fits the rough description above of very explicit functions. To see that it fits the precise description, we should list the appropriate patterns  $\vec{p}_i$  and  $q_i$ . There are eleven  $\vec{p}$ 's and eleven corresponding  $q$ 's — four for Case 1 (according to which of trees 1 through 4 is the first nonempty one), one each for Cases 2, 3, and 4, and four for Case 5 (according to whether the leftward path of tree 7 has 0, 1, 2, or 3 nodes). We refrain from explicitly exhibiting the results of this splitting of cases.

We must still verify that we have defined a bijection, i.e., that every tree arises exactly once as the output of this construction. For this purpose, note that the leftward path of the output has length

- $\geq 6$  in Case 1,
- 4 in Case 2,
- $\geq 6$  in Case 3 (remember that tree 6 is non-empty here),
- 5 in Case 4, and
- $\leq 3$  in Case 5.

This shows that no two cases, with the possible exception of Cases 1 and 3, could produce the same tree as output. The possible exception is not a real exception,

because in Case 1 at least one of the first four nodes on the leftward path has a right successor (because those successor subtrees are 1 through 4, of which at least one is non-empty by case hypothesis) whereas in Case 3 this is not the case. Thus, given an arbitrary tree  $t$ , we can determine the unique case that might produce it as output. Once this is done, it is straightforward to check that each case actually does produce all the trees thereby assigned to it, and that it produces them exactly once each.  $\square$

Theorem 1 would remain true if we extended the concept of tree to allow infinite trees, and the proof would be unchanged. The reason is that a very explicit bijection can be applied to infinite trees just as to finite trees, since below a certain depth it merely copies the input into appropriate places in the output. More unusual notions of “tree” can be handled similarly. For example, we could allow trees to have finitely many infinite paths; we could allow infinite paths only if, beyond some node on a path, all further nodes are left children; we could fix a set  $C$  of “colors” and allow infinite trees in which every infinite path is assigned a color from  $C$  (and we could impose continuity requirements on the coloring); etc. In each case, Theorem 1 remains true as long both occurrences of “trees” are interpreted the same way.

## 2. ALGEBRA

The proof of Theorem 1 raises at least two questions. Where did it come from? And why seven? The reader is invited to try some “easier” numbers in place of seven, say five or two; no analogous proof will be forthcoming, so there is indeed something special about seven. (Of course there *is* an analogue for thirteen. Given thirteen trees, apply Theorem 1 to code the first seven as a single tree, and then code this tree with the other six of the original inputs. The same trivial observation handles any number congruent to 1 modulo 6.)

To see why seven is special, we first give an argument to establish Theorem 1 in the style of eighteenth-century analysis, where meaningless computations (e.g., manipulating divergent series as though they converged absolutely and uniformly) somehow gave correct results. This argument begins with the observation that a tree either is 0 or splits naturally into two subtrees (by removing the root). Thus the set  $T$  of trees satisfies  $T = 1 + T^2$ . (Of course equality here actually means an obvious isomorphism. Note that the same equation holds also for the variant notions of tree mentioned at the end of Section 1.) Solving this quadratic equation for  $T$ , we find  $T = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ . (The reader who objects that this is nonsense has not truly entered into the eighteenth-century spirit.) These complex numbers are primitive sixth roots of unity, so we have  $T^6 = 1$  and  $T^7 = T$ . And this is why seven-tuples of trees can be coded as single trees.

Although this computation is nonsense, it has at least the psychological effect of suggesting that something like Theorem 1 has a better chance of being true for seven than for five or two. To improve the effect from psychological to mathematical, we attempt to remove the nonsense while keeping the essence of the computation. Incidentally, Lawvere’s remark that led to this paper was phrased in terms of such

a meaningless computation giving a correct result: “I was surprised to note that an isomorphism  $x = 1 + x^2$  (leading to complex numbers as Euler characteristics if they don’t collapse) always induces an isomorphism  $x^7 = x$ ” [8] p. 11. This remark was an application of the method of Schanuel [10] for developing objective meaning for such computations.

One might hope for a meta-theorem to the effect that, when such a meaningless computation leads from a meaningful equation (like  $T = 1 + T^2$ ) to another meaningful equation (like  $T^7 = T$ ) then the latter honestly follows from the former. This would be somewhat analogous to Hilbert’s program for making constructive sense of infinitary mathematics by showing that, when a detour through the infinite leads from one finitistically meaningful statement to another, then the latter honestly (finitistically) follows from the former. Unfortunately, our situation (like Hilbert’s) is more subtle. After all,  $T^6 = 1$  is a meaningful equation, saying that there is only one six-tuple of trees, which is false. So any rehabilitation of the computation above must, in particular, explain why the ultimate conclusion  $T^7 = T$  is right while the penultimate  $T^6 = 1$  is not.

As a first step toward rehabilitation, we observe that we can avoid complex numbers by working entirely within the ring of polynomials with integer coefficients. (The possibility of such a step is guaranteed by general facts about polynomial rings and ideals, but here is the step explicitly.) We simplify  $T^7$  by repeatedly reducing its degree, using the equation  $T = 1 + T^2$  in the form  $T^2 = T - 1$ . The result is

$$T^7 = T^6 - T^5 = -T^4 = -T^3 + T^2 = T.$$

The complex numbers are gone, but the computation is still meaningless (when we remember that  $T$  is a set) because of negative coefficients, and we could still get  $T^6 = 1$  just by reducing all exponents by one.

The next step is to eliminate the negative terms by adding new terms ( $T^5$ ,  $T^4$ , and  $T^3$ ) to cancel them. The resulting calculation, which takes place in the semiring  $\mathbb{N}[T]/(T = 1 + T^2)$ , reads

$$\begin{aligned} T^7 + T^5 + T^4 + T^3 &= T^6 + T^4 + T^3 = T^5 + T^3 \\ &= T^5 + T^4 + T^2 = T^5 + T^4 + T^3 + T. \end{aligned}$$

Everything here is meaningful and correct when  $T$  is interpreted as the set of trees (rather than a formal indeterminate subject to  $T = 1 + T^2$ ) and equality is interpreted as obvious isomorphism (or, better, as very explicit bijection). Unfortunately, instead of getting a very explicit bijection from  $T^7$  to  $T$ , we have the extra terms  $T^5 + T^4 + T^3$  on both sides. Can we get rid of them?

There is a general way to convert a bijection  $f : A + X \rightarrow B + X$  into a bijection  $f' : A \rightarrow B$  when  $X$  is finite, namely the Garsia-Milne involution principle [6]. The idea is to define  $f'(a)$  by first applying  $f$  to  $a$ ; if the result is in  $B$ , accept it as the output for  $f'$ ; if not, then it is in  $X$ , so we can apply  $f$  again; if the result is in  $B$ , accept it as the output for  $f'$ ; if not, then it is in  $X$ , so we can apply  $f$  again; and so forth until we finally get an output in  $B$ . The finiteness of  $X$  is used

to ensure that the process terminates. In our situation,  $X = T^5 + T^4 + T^3$  is not finite. One can try to apply the Garsia-Milne construction anyway, hoping that the process terminates even though no finiteness forces it to. Unfortunately, it does not terminate.

We can do better by considering the last computation displayed above. It is in two parts. In the first part,  $T^7 + T^5 + T^4 + T^3$  is simplified to  $T^5 + T^3$ ; that is,  $T^7 + T^4$  is eliminated. In this elimination, the presence of  $T^5$  was essential as a sort of catalyst, since the  $T^5$  is removed at the first step and restored at the second.  $T^3$ , on the other hand is irrelevant to this half of the calculation. Clearly, the same argument shows that, in any polynomial in  $\mathbb{N}[T]/(T = 1 + T^2)$ , we can delete or introduce  $T^{k+3} + T^k$  provided  $T^{k+1}$  is also present as a catalyst. The second half of the displayed computation introduces  $T^4 + T$ , using  $T^3$  as a catalyst; the same argument would allow us to introduce or delete  $T^{k+3} + T^k$  provided  $T^{k+2}$  is also present as a catalyst. Thus, we can delete or introduce two powers of  $T$  whose exponents differ by exactly three, provided one of the two intervening powers is also present as a catalyst.

This result can be significantly improved by noticing that any positive power of  $T$  can serve as a catalyst; it need not be between the two that are being deleted or introduced. To see this, suppose we want to delete or introduce  $T^{k+3} + T^k$ , and we have another positive power  $T^r$  present in the polynomial. If  $r$  is  $k+1$  or  $k+2$ , then we already know  $T^r$  can serve as the desired catalyst. If not, then use  $T = 1 + T^2$  to replace  $T^r$  with  $T^{r-1} + T^{r+1}$ . So now we have two powers of  $T$ , in one of which the exponent is closer to the desired  $k+1$  or  $k+2$ . Apply the same procedure to that power, leaving the other one alone, and repeat the process until finally you get the desired catalyst. Use it to delete or introduce  $T^{k+3} + T^k$ , and then reverse the previous steps to recover the original  $T^r$  by reassembling the terms into which the first part of the procedure decomposed it. (Note that we needed  $r$  to be strictly positive in order to replace  $T^r$  with  $T^{r-1} + T^{r+1}$ .)

Now we can, in  $\mathbb{N}[T]/(T = 1 + T^2)$ , convert  $T^7$  into  $T^7 + T^4 + T$  (introducing  $T^4 + T$  with  $T^7$  as catalyst) and then into  $T$  (deleting  $T^7 + T^4$  with  $T$  as catalyst). We refrain from writing out explicitly the computation in  $\mathbb{N}[T]/(T = 1 + T^2)$  obtained by the method of the preceding paragraph. It consists of twenty steps: four to convert  $T^7$  into a usable catalyst  $T^3$  (plus extra terms), two to introduce  $T^4 + T$  using this catalyst, four to collect the catalyst and extra terms back into a  $T^7$ , four to convert  $T$  into a usable catalyst  $T^5$  (plus extra terms), two to delete  $T^7 + T^4$  using this catalyst, and four to collect the catalyst and extra terms back into a  $T$ . Notice that the proof of  $T^7 = T$  does not become a proof of  $T^6 = 1$  if we reduce the exponents; unlike  $T$ , 1 cannot serve as a catalyst.

We can now answer the first of the questions at the beginning of this section — where did the bijection in the proof of Theorem 1 come from? It came from this twenty-step computation. Wherever the computation used  $T = 1 + T^2$ , apply the obvious bijection between the sets  $T$  and  $\{0\} \cup T^2$ , and the whole computation will give a composite bijection which is the one used to prove Theorem 1 (except that I interchanged left and right once, to make the description of the proof easier).

As for the second question — why seven? — we have a partial answer. The

technique that led to the proof of Theorem 1 works only for numbers  $k$  that are congruent to 1 modulo 6, because it is only for these numbers that  $T^k = T$  is true in the semiring  $\mathbb{N}[T]/(T = 1 + T^2)$ , or even in the rings  $\mathbb{Z}[T]/(T = 1 + T^2)$  or  $\mathbb{C}[T]/(T = 1 + T^2)$ , i.e., because it is only for these values of  $k$  that  $T^k = T$  is satisfied by the roots  $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$  of  $T = 1 + T^2$ . A more complete answer to the second question would involve showing that very explicit bijections between  $T^k$  and  $T$  can exist only when the corresponding polynomials are equal in  $\mathbb{N}[T]/(T = 1 + T^2)$ . This will be done, in greater generality, in the next section.

Because of the importance of the semiring  $\mathbb{N}[T]/(T = 1 + T^2)$  in computations like the preceding ones, we present a normal form for its elements.

**Theorem 2.** *Every element of  $\mathbb{N}[T]/(T = 1 + T^2)$  is uniquely expressible in the form  $a + bT^2 + cT^4$  with non-negative integer coefficients  $a$ ,  $b$ , and  $c$  such that either at least one coefficient is 0 or else  $a \geq b = c = 1$ .*

*Proof.* Every element of the semiring  $\mathbb{N}[T]/(T = 1 + T^2)$  is a polynomial in  $T$  with non-negative integer coefficients. We show how to simplify such a polynomial to the form claimed in the theorem. Since  $T^7 = T$ , all terms of degree 7 or more can be reduced to lower degree, so we may assume the polynomial has degree at most 6. The degree can be reduced to at most 4 by means of the equations

$$T^6 = T^5 + T^7 = T^5 + T \quad \text{and} \quad T^5 = 1 + T^3 + T^5 = 1 + T^4,$$

where  $1 + T^3$  was introduced in the second equation with catalyst  $T^5$ . Furthermore, we can eliminate all terms of odd degree, since  $T = 1 + T^2$  and  $T^3 = T^2 + T^4$ . So our polynomial has the form  $a + bT^2 + cT^4$ . We check next that the coefficient restrictions in the theorem can be enforced.

For this purpose, consider the polynomial  $Q = 1 + T^2 + T^4$ . In the presence of a catalyst (any positive power of  $T$ ),  $Q$  vanishes, for it equals  $T + T^4$  which a catalyst can delete. Now consider our general polynomial  $a + bT^2 + cT^4$ . If at least one of the coefficients is 0, then it has the desired form, so suppose all three coefficients are at least 1. Then the polynomial is  $Q + (a - 1) + (b - 1)T^2 + (c - 1)T^4$ . If either  $b - 1$  or  $c - 1$  is positive, we have a catalyst and can remove  $Q$ . Repeating the process of splitting off a  $Q$  and deleting it as long as a catalyst is available, we ultimately get either a polynomial of the form  $a + bT^2 + cT^4$  with a zero coefficient (if, at some stage, we cannot split off a  $Q$ ) or one of the form  $d + Q$  with  $d \in \mathbb{N}$  (if, at some stage, we have split off a  $Q$  but have no catalyst to remove it). This completes the proof that every polynomial can be put in the required form; it remains to show uniqueness.

If two expressions of the form described in the theorem, say  $a + bT^2 + cT^4$  and  $a' + b'T^2 + c'T^4$ , are equal in  $\mathbb{N}[T]/(T = 1 + T^2)$ , then they are also equal as complex numbers when  $T$  is interpreted as the primitive sixth root of unity  $t = \frac{1}{2} + i\frac{\sqrt{3}}{2}$  (since  $\mathbb{C}$  is a commutative semiring, and  $t = 1 + t^2$ , and  $\mathbb{N}[T]/(T = 1 + T^2)$  with  $T$  is the initial commutative semiring with a solution of  $T = 1 + T^2$ ).  $t^2$  is a primitive cube root of unity; its minimal polynomial is  $1 + z + z^2$ . So this minimal polynomial would have to divide  $(a - a') + (b - b')z + (c - c')z^2$ , which means that  $a - a'$ ,  $b - b'$ ,

and  $c - c'$  are all equal. If they are all zero, then  $a + bT^2 + cT^4$  and  $a' + b'T^2 + c'T^4$  are identical as expressions, which is what we needed to prove. So suppose that the differences  $a - a'$ ,  $b - b'$ , and  $c - c'$  are all positive. (If they are all negative, interchange primed and unprimed in what follows.) A fortiori,  $a$ ,  $b$ , and  $c$  are all positive, so  $a + bT^2 + cT^4$ , being of the form in the theorem, has  $b = c = 1$  and  $a \geq 1$ , i.e., it is  $(a - 1) + Q$ . As  $a - a'$ ,  $b - b'$  and  $c - c'$  are equal and positive, we must have  $b' = c' = 0$  and  $a' = a - 1$ .

Our task is thus reduced to showing that  $a'$  and  $a' + Q$  are not equal in  $\mathbb{N}[T]/(T = 1 + T^2)$ . For this purpose, notice that the set consisting of the natural numbers and the cardinal  $\aleph_0$  is a commutative semiring (with the usual addition and multiplication of cardinal numbers) in which  $\aleph_0 = 1 + \aleph_0^2$ . So an equation  $a' = a' + Q$  in  $\mathbb{N}[T]/(T = 1 + T^2)$  would imply the same equation in this cardinal semiring with  $T$  interpreted as  $\aleph_0$ . But in this interpretation such an equation is false, since  $a'$  is interpreted as the integer  $a'$  while  $a' + Q$  is interpreted as  $\aleph_0$ . So the equation cannot hold in  $\mathbb{N}[T]/(T = 1 + T^2)$ .  $\square$

### 3. ALGEBRAIC EQUIVALENCE AND VERY EXPLICIT BIJECTIONS

Let us call two polynomials,  $P(X)$  and  $Q(X)$ , with non-negative integer coefficients *algebraically equivalent* if the equation  $P(X) = Q(X)$  holds in the semiring  $\mathbb{N}[X]/(X = 1 + X^2)$ . We call the indeterminate  $X$  rather than  $T$  to avoid confusion, since we shall need to discuss it and the set  $T$  of trees in the same context.

A polynomial  $P(X) \in \mathbb{N}[X]$  has a natural interpretation as an operation  $S \mapsto P(S)$  on sets (well-defined up to canonical bijections); sums and products of polynomials correspond to disjoint unions and Cartesian products of sets. (We can interpret the numerical coefficients in  $P(X)$  as canonically chosen sets of the corresponding cardinalities, or we can eliminate these coefficients in favor of sums of repeated terms.) Thanks to the canonical bijection from  $T$  to  $1 + T^2$ , algebraically equivalent polynomials, when applied to the set  $T$  of trees, give canonically isomorphic sets. In fact, we shall show, as half of the next theorem, that the canonical isomorphism is very explicit in the sense of Section 1.

Let us call two polynomials  $P(X)$  and  $Q(X)$  *combinatorially equivalent* if there is a very explicit bijection from  $P(T)$  to  $Q(T)$ . Of course, we must extend the definition of “very explicit,” given in Section 1 for the special case of  $T^7$  and  $T$ , to the general case of  $P(T)$  and  $Q(T)$ , but this is straightforward. If  $P(X) = \sum_k c_k X^k$ , then we regard  $P(T)$  as the set of tagged  $k$ -tuples of trees,  $\vec{t} = (t_1, \dots, t_k, \tau)$  where  $k$  ranges over the same finite set as in the sum defining  $P(X)$  and where the tag  $\tau$  is an integer in the range  $1 \leq \tau \leq c_k$ . For brevity, we call such a tagged  $k$ -tuple a  $P$ -tuple. A  $P$ -pattern is a similarly tagged  $k$ -tuple of patterns (in the sense defined in Section 1) in which all the labels are distinct, and the notion of an *instance* of a pattern with respect to a substitution is defined just as in Section 1. A *very explicit function*  $f$  from  $P(T)$  to  $Q(T)$  is given by an indexed family  $(\vec{p}_i)_{i \in I}$  of  $P$ -patterns and a similarly indexed family  $(\vec{q}_i)_{i \in I}$  of  $Q$ -patterns such that, for each  $i \in I$ , the same labels occur in  $\vec{p}_i$  as in  $\vec{q}_i$  and such that every element  $\vec{t}$  of  $P(T)$  is an instance of a unique  $\vec{p}_i$  (under a unique substitution). Then  $f(\vec{t})$  is defined as the result of

applying the same substitution to  $\vec{q}_i$  (for the same  $i$ ).

*Remark.* If the very explicit function  $f$  is a bijection from  $P(T)$  to  $Q(T)$ , then every element of  $Q(T)$  occurs exactly once as an instance of a  $\vec{q}_i$  (in the notation of the preceding definition). It follows that the inverse bijection is also very explicit, as we can interchange the roles of the  $\vec{p}_i$  and the  $\vec{q}_i$ . Thus, it makes no difference whether “very explicit bijection” is interpreted in the obvious way as “very explicit function that happens to be a bijection” or as “very explicit function with very explicit two-sided inverse.”

Our interest will be focused on very explicit bijections, and for the study of these our definition of “very explicit” seems adequate. If we were to study very explicit functions in general, it would be advisable to liberalize the definition of “very explicit” by allowing labels to be repeated in a  $\vec{q}_i$  and allowing a label to occur in  $\vec{p}_i$  without occurring in  $\vec{q}_i$ . In either of these cases, the very explicit function cannot be a bijection.

**Theorem 3.** *Two polynomials  $P(X), Q(X) \in \mathbb{N}[X]$  are combinatorially equivalent if and only if they are algebraically equivalent.*

*Proof.* Suppose first that  $P(X)$  and  $Q(X)$  are algebraically equivalent. Since  $\mathbb{N}[X]$  is the free commutative semiring on one generator, its quotient  $\mathbb{N}[X]/(X = 1 + X^2)$  is the initial algebra in the variety  $\mathcal{V}$  of commutative semirings with a distinguished element  $X$  subject to  $X = 1 + X^2$ . Algebraic equivalence therefore means that the equation  $P(X) = Q(X)$  is satisfied by the whole variety  $\mathcal{V}$ . (Note that here  $X$  is a constant for the distinguished element of a  $\mathcal{V}$ -algebra.) By Birkhoff’s theorem [3, Theorem 14.19], there is an equational deduction of  $P(X) = Q(X)$  from  $X = 1 + X^2$  and the axioms for commutative semirings, using as rules of inference only substitution of equals for equals and substitution of terms for variables. All the substitutions for variables can be done at the beginning, as substitutions into axioms. So we get a deduction of  $P(X) = Q(X)$  from  $X = 1 + X^2$  and variable-free instances of the commutative semiring axioms, using only substitution of equals for equals.

Consider the following property of equations: When the two sides are applied as operations to  $T$ , the two resulting sets have a bijection between them such that both the bijection and its inverse are very explicit. It is trivial to check that the equation  $X = 1 + X^2$  and all variable-free instances of the axioms of commutative semirings enjoy this property and that the property is preserved by substitution of equals for equals. It therefore follows that  $P(X) = Q(X)$  has this property, which implies combinatorial equivalence.

For the converse, suppose that  $P(X)$  and  $Q(X)$  are combinatorially equivalent. Specifically, let  $f$  be a very explicit bijection from  $P(T)$  to  $Q(T)$ , and let  $(\vec{p}_i)_{i \in I}$  and  $(\vec{q}_i)_{i \in I}$  be as in the definition of “very explicit” preceding the theorem.

It will be convenient, both for this proof and for Section 4, to introduce certain standard collections of patterns. (We temporarily deal with patterns in the sense of Section 1, single trees; we will return to tuples and  $P$ -patterns later.) For each positive integer  $n$ , let  $S_n$  be the set of patterns  $p$  of depth  $\leq n + 1$  such that all nodes at level  $n$  have exactly two children, all nodes at level  $n + 1$  are labeled

(with distinct labels, as required by the definition of pattern), and no nodes at levels  $\leq n$  are labeled. If  $p \in S_n$  then its instances are exactly those trees that are identical with  $p$  down to depth  $n$ , with no constraints on what (if anything) happens at greater depth. In keeping with this, we can define  $S_0$  to consist of just one pattern, which consists of a single, labeled node. Clearly, for every  $n$ , every tree is an instance of a unique member of  $S_n$ .

If  $r$  is any pattern of depth  $< n$ , then we can associate to it a set  $\hat{r} \subseteq S_n$  having collectively the same instances as  $r$ . One can produce  $\hat{r}$  from  $r$  by the following step-by-step procedure, which we call *developing*  $r$  (to depth  $n$ ). Choose any labeled leaf in  $r$  and replace  $r$  with two new patterns,  $r'$  in which this leaf has been deleted, and  $r''$  in which this leaf has been given two labeled children (and has lost its own label, as it is no longer a leaf). Apply the same construction to  $r'$  and  $r''$ , using labeled leaves at depths  $\leq n$ . Iterate the process until all the patterns have their labels at level  $n + 1$ . At this stage, they are easily seen to be in  $S_n$ . Furthermore, the instances of  $r$  are precisely the instances of  $r'$  and those of  $r''$  (according to whether the substitution gives the label of the chosen leaf the value 0 or not). And no tree is an instance of both  $r'$  and  $r''$ . It follows inductively that, at each stage of the development, the patterns have disjoint sets of instances and the union of these sets contains precisely the instances of  $r$ . Thus, the final set of patterns obtained by this development is  $\hat{r}$ .

We need to extend the preceding concepts from patterns to  $P$ -patterns (and  $Q$ -patterns). By  $S_n(P)$  we mean the set of all  $P$ -patterns whose component patterns lie in  $S_n$ . (Recall that a  $P$ -pattern is a tagged  $k$ -tuple of patterns, so this makes sense.) Every  $P$ -tuple of trees is an instance of a unique pattern in  $S_n(P)$  (for each fixed  $n$ ).  $P$ -patterns can be developed componentwise; that is, the development of  $(r_1, \dots, r_k, \tau)$  to depth  $n$  ( $>$  the depths of all the  $r_j$ 's) consists of the patterns  $(z_1, \dots, z_k, \tau) \in S_n(P)$  with each  $z_j$  in  $\hat{r}_j \subseteq S_n$ . Again, the members of the development of  $\vec{r}$  have disjoint sets of instances and the union of these sets contains precisely the instances of  $\vec{r}$ .

Let us return to the families  $(\vec{p}_i)_{i \in I}$  and  $(\vec{q}_i)_{i \in I}$  determining our very explicit bijection  $f$ . Fix  $n$  greater than the depths of all the patterns in all these  $P$ - and  $Q$ -patterns. Since every  $P$ -tuple is an instance of exactly one  $\vec{p}_i$ , it follows from the preceding discussion that the sets  $\hat{p}_i \subseteq S_n(P)$  obtained by developing  $\vec{p}_i$  to depth  $n$  are disjoint and their union is  $S_n(P)$ .

By the *weight* of a pattern  $r$ , we mean the monomial  $X^l \in \mathbb{N}[X]/(X = 1 + X^2)$ , where  $l$  is the number of labels in  $r$ . By the *weight* of a (tagged) tuple of patterns (e.g., a  $P$ - or  $Q$ -pattern), we mean the product of the weights of its component patterns, i.e.,  $X$  with exponent the total number of labels in the tuple. By the *weight* of a set of patterns or of (tagged) tuples of patterns, we mean the sum of the weights of its members.

One step in the development of a pattern  $r$  replaces it by a set of two patterns  $r'$  and  $r''$  where, if  $r$  has weight  $X^l$ , then  $r'$  has weight  $X^{l-1}$  (as one labeled leaf was deleted) and  $r''$  has weight  $X^{l+1}$  (as one labeled leaf lost its label but was given two labeled children). So the weight of the set obtained is  $X^{l-1} + X^{l+1} = X^l$ , since the weights are in a semiring where  $1 + X^2 = X$ . So one step of development leaves the

weight unchanged. It follows by induction that development of a pattern to depth  $n$  leaves the weight unchanged;  $r$  and  $\hat{r}$  have the same weight. It further follows easily that the same applies to (tagged) tuples of patterns.

These observations imply that the set  $S_n$  has weight  $X$ , because  $S_0$  trivially has weight  $X$  and  $S_{n+1}$  is obtainable by developing  $S_n$ . It follows that  $S_n(P)$  has weight  $P(X)$ , because the contribution to the weight from the  $k$ -tuples with a particular tag  $\tau$  is

$$\begin{aligned} \sum_{\vec{r} \in (S_n)^k} \text{weight}(\vec{r}) &= \sum_{\vec{r} \in (S_n)^k} \prod_{j=1}^k \text{weight}(r_j) = \\ \prod_{j=1}^k \sum_{r \in S_n} \text{weight}(r) &= \prod_{j=1}^k \text{weight}(S_n) = X^k. \end{aligned}$$

Since, as we saw above,  $S_n(P)$  is obtainable by developing the set of  $\vec{p}_i$ 's, this set also has weight  $P(X)$ . Similarly, the set of  $\vec{q}_i$ 's has weight  $Q(X)$ . But each  $\vec{p}_i$  contains exactly the same labels as the corresponding  $\vec{q}_i$ , so these two sets have the same weight. Therefore,  $P(X) = Q(X)$  in  $\mathbb{N}[X]/(X = 1 + X^2)$ .  $\square$

It follows immediately from Theorem 3 that, if  $P(X)$  and  $Q(X)$  are combinatorially equivalent, then  $P(t) = Q(t)$  where  $t = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ . Thus, the complex number  $P(t)$  serves as an invariant of the combinatorial equivalence class of  $P(X)$ . This is what Lawvere referred to as “complex Euler characteristics” in the passage, quoted in Section 2, that motivated this paper. In fact, Theorem 3 and the proof of Theorem 2 show that the semiring of combinatorial equivalence classes of polynomials is embedded in the product of the complex field and the semiring of cardinals  $\leq \aleph_0$  (by sending the indeterminate  $X$  to  $\langle t, \aleph_0 \rangle$ ). Lawvere has pointed out that for this purpose one could replace the cardinal semiring with a three-element semiring, for all the finite, non-zero cardinals can be identified without damaging the embedding. The resulting three elements form the system of “dimensions” associated to our problem by Schanuel’s general construction [10], so that the present solution of the word problem shows in particular that again in our case “Euler characteristic and dimension” are jointly injective.

#### 4. CONSTRUCTIVE SET THEORY

We show in this section that the algebraic equivalence of  $P(X)$  and  $Q(X)$  can be deduced from the mere provability of “there is a bijection from  $P(T)$  to  $Q(T)$ ” provided this provability is from sufficiently restricted assumptions. The restrictions we need are two: The underlying logic is constructive and the only assumption about  $T$  is the existence of a bijection  $T \rightarrow 1 + T^2$ . (Actually, the second restriction can be relaxed a bit by allowing a stronger assumption about  $T$ .) On the other hand, we allow the use of higher-order logic, so many set-theoretic methods are available.

More precisely, let  $\mathcal{L}$  be a higher-order theory in the sense of [4] or a local set theory in the sense of [1], generated by a natural number object and an additional

ground type  $T$  subject to the axiom “there is a bijection  $1 + T^2 \rightarrow T$ .” Alternatively, we could let  $\mathcal{L}$  be intuitionistic Zermelo-Fraenkel set theory augmented with a constant  $T$  and the same axiom.

**Theorem 4.** *Let  $P(X)$  and  $Q(X)$  be polynomials with non-negative integer coefficients. Suppose it is provable in  $\mathcal{L}$  that there is a bijection  $P(T) \rightarrow Q(T)$ . Then  $P(X)$  and  $Q(X)$  are algebraically equivalent.*

Before proving the theorem, we make several remarks. First, the converse of the theorem is easy to prove. By Theorem 3, algebraic equivalence implies the existence of a very explicit bijection  $P(T) \rightarrow Q(T)$  when  $T$  is the set of trees. But the very-explicitness makes it possible to apply the bijection to arbitrary sets for which a bijection  $1 + T^2 \rightarrow T$  is given, and this application can be carried out in constructive set or type theory. Alternatively, we can proceed as in the proof of Theorem 3, considering for all equations  $P(X) = Q(X)$  the property “it is provable in  $\mathcal{L}$  that there is a bijection  $P(T) \rightarrow Q(T)$ ,” noticing that this property is enjoyed by the equation  $1 + X^2 = X$  and by all variable-free instances of the axioms for commutative semirings, noticing further that the property is preserved by substitution of equals for equals, and concluding that the property holds of all equations true in  $\mathbb{N}[X]/(X = 1 + X^2)$ .

The remaining remarks are intended to justify the restrictions we place on the logic and the assumptions on  $T$  in the theory  $\mathcal{L}$ .

If we allowed full classical set theory, with the axiom of choice, then the assumption  $T \cong 1 + T^2$  implies that  $T$  is infinite and therefore  $T$ ,  $1 + T$ ,  $T + T$ , and  $T^2$  all have the same cardinality. It follows that every non-constant polynomial is equivalent, in the sense of provable bijection, to  $T$ . In other words, with this stronger set theory, the corresponding notion of algebraic equivalence would be equality not in  $\mathbb{N}[X]/(X = 1 + X^2)$  but in  $\mathbb{N}[X]/(X = 1 + X = X + X = X^2)$ , a semiring isomorphic to the  $\mathbb{N} \cup \{\aleph_0\}$  example used at the end of the proof of Theorem 2.

If we work in classical set theory without the axiom of choice, so that addition and multiplication of infinite cardinals are no longer trivial, we still get the same conclusion with a bit more work. From  $T \cong 1 + T^2$  and its immediate consequence  $T^2 \cong T + T^3$ , we infer that each of  $T$  and  $T^2$  can be embedded in the other. By the Cantor-Schröder-Bernstein Theorem, whose proof does not require the axiom of choice, we have  $T \cong T^2$ . Then from  $1 \leq 2 \leq T$  (where  $\leq$  means embeddability, the usual inequality relation on cardinals) we get  $T \leq T + T \leq T^2 \leq T$ , so another application of the Cantor-Schröder-Bernstein Theorem gives  $T \cong T + T$ .

If we use intuitionistic rather than classical logic, then the Cantor-Schröder-Bernstein Theorem is no longer available and, as Theorem 4 shows, the argument in the preceding paragraph breaks down. Even in intuitionistic logic, however, if we assume that  $T$  is the set of trees (rather than some arbitrary set with a bijection  $1 + T^2 \rightarrow T$ ) then the argument in the preceding paragraph works, since the bijections produced by the Cantor-Schröder-Bernstein Theorem can, in this case, be constructively defined. For example, we described, just before the proof of Theorem 1, a bijection  $T^2 \rightarrow T$ , and that description is intuitionistically legitimate. The main point here is that the case distinction, whether a tree is just a leftward path,

is decidable because the tree is finite. There is a similarly constructive bijection  $T + T \rightarrow T$  when  $T$  is the set of finite trees. It sends any tree  $t$  from the first copy of  $T$  to  $[0, t]$ , and it sends any  $t$  from the second copy of  $T$  to  $[t, 0]$  unless  $t$  is of the form  $0$  or  $[p, q]$  or  $[[p, q], 0]$  or  $[[[p, q], 0], 0]$  or  $\dots$  with  $p$  and  $q$  both  $\neq 0$ , in which case it sends  $t$  to  $t$ . Again, it is the finiteness of the trees that makes the case distinction decidable and the definition constructively correct.

*Proof of Theorem 4.* We begin by giving a more useful description of sets  $T$  with bijections  $f : 1 + T^2 \rightarrow T$ . The bijection is determined by specifying a distinguished element of  $T$  and a binary operation on  $T$ ; the distinguished element is the value of  $f$  at the unique element of  $1$ , and the operation is the restriction of  $f$  to  $T^2$ . To match the notation used earlier for trees, we write the distinguished element as  $0$  and the operation as  $[-, -]$ . Thus,  $T$  (with the structure  $f$ ) is an algebra with one constant and one binary operation. That  $f$  is a bijection means that this algebra must satisfy the following system  $\mathcal{T}$  of axioms, which, for later convenience, we write as geometric sequents (as defined in [1, page 250] or [7, Section 6.5]), indeed, whenever possible, as universal Horn formulas.

- (1)  $0 = [x, y] \implies \mathbf{false}$
- (2)  $[x, y] = [x', y'] \implies x = x'$
- (3)  $[x, y] = [x', y'] \implies y = y'$
- (4)  $\mathbf{true} \implies x = 0 \vee \exists y \exists z x = [y, z]$

The algebra of finite trees is initial in the variety of algebras of signature  $\{0, [-, -]\}$ , and, since it satisfies the axioms of  $\mathcal{T}$ , it is also initial in the category of models of  $\mathcal{T}$ . Any model  $M$  of  $\mathcal{T}$  can be regarded as an algebra of generalized trees, in that it has an element  $0$  corresponding to the empty tree, and all its other elements are uniquely of the form  $[y, z]$  and can therefore be pictured as consisting of a root with two subtrees,  $y$  and  $z$ , attached to it. Among such algebras are, for example, the collections of trees in any of the generalized senses mentioned at the end of Section 1.

We shall prove the theorem by constructing a specific topos model of  $\mathcal{L}$ . The hypothesis of the theorem says that in this model there must be a bijection  $P(T) \rightarrow Q(T)$ , and an analysis of what this means will lead to the desired conclusion. Perhaps the most natural topos model of  $\mathcal{L}$  is the classifying topos ([7, Section 6.5]) of  $\mathcal{T}$ , with  $T$  interpreted as the generic model in this topos. For technical reasons, however, it is easier to work with the classifying topos of a slightly stronger theory,  $\mathcal{T}'$ , obtained from  $\mathcal{T}$  by adding the following axiom for every term  $t$  that contains the variable  $x$  but is not just  $x$ .

- (5)  $t = x \implies \mathbf{false}$

This additional axiom schema says that no (generalized) tree in a model of  $\mathcal{T}'$  can be a proper subtree of itself. It is satisfied by the initial algebra (of finite trees), but not by, for example, the algebra of all finite and infinite binary trees, where the full binary tree (every node of which has two children) satisfies  $x = [x, x]$ .

Notice that, apart from making the proof easier, the addition of axiom schema (5) to  $\mathcal{T}$  slightly improves the theorem. The theorem remains true (with the same proof) if we replace  $\mathcal{L}$  by the stronger theory  $\mathcal{L}'$  where  $T$  is assumed to satisfy (5).

This strengthening of  $\mathcal{L}$  weakens the hypothesis of Theorem 4 and thus strengthens the theorem.

As indicated above, we shall work with the classifying topos  $\mathcal{E}$  of  $\mathcal{T}'$ . In it, there is a generic (or universal) model  $G$  of  $\mathcal{T}'$ ; this implies that, for any model  $M$  of  $\mathcal{T}'$  in any Grothendieck topos  $\mathcal{F}$ , there is a geometric morphism  $\mu : \mathcal{F} \rightarrow \mathcal{E}$  whose inverse image functor  $\mu^*$  sends  $G$  to  $M$ . (It implies more than this, but this, along with the explicit construction described below, will suffice for our purposes.) Being a Grothendieck topos,  $\mathcal{E}$  gives an interpretation of higher order logic (as in [4]) and local set theories ([1]), with natural number object, and intuitionistic Zermelo-Fraenkel set theory ([5]), so by interpreting  $T$  as  $G$  we obtain a model of  $\mathcal{L}$  in the internal logic of  $\mathcal{E}$ . By the hypothesis of Theorem 4, it must be internally true in  $\mathcal{E}$  that there is a bijection from  $P(G)$  to  $Q(G)$ .

The next part of the proof consists of studying  $\mathcal{E}$  and  $G$  in sufficient detail to draw useful conclusions from this internal information. We begin by describing  $\mathcal{E}$  explicitly as the topos of sheaves over a specific site. As explained in [9], it is convenient to first build the classifying topos for the universal Horn axioms and then obtain  $\mathcal{E}$  as a sheaf subtopos. For  $\mathcal{T}'$ , axioms (1), (2), (3), and (5) are universal Horn sentences. The classifying topos for this subtheory is, according to [2], the topos of presheaves on the dual of the category  $\mathcal{A}$  of finitely presented models of (1), (2), (3), and (5). So we need to analyze such models  $\langle A \mid E \rangle$ , where  $A$  is a finite set of generators and  $E$  is a finite, consistent set of equations between terms built from the generators, 0, and  $[-, -]$ .

We show first that every such model is free, i.e., is isomorphic to  $\langle B \rangle = \langle B \mid \emptyset \rangle$  for some finite set  $B$ . To see this, we systematically simplify the given set  $E$  of equations as follows. (Technically, the simplification is an inductive process; at each step, we either decrease the cardinality of  $A$  or we leave this cardinality unchanged but decrease the total length of all the equations in  $E$ .) If 0 occurs as one side of an equation in  $E$ , then the other side must be 0 or a member of  $A$ ; it cannot be of the form  $[t_1, t_2]$  because then the equation would be inconsistent by (1). If it is 0, then the equation  $0 = 0$  can be deleted from  $E$  as it is always true. If it is a member  $a$  of  $A$ , then we can delete  $a$  from  $A$ , delete the equation  $0 = a$  (or  $a = 0$ ) from  $E$ , and replace all occurrences of  $a$  in the rest of  $E$  by 0. The result is a simpler presentation of the same algebra. So we may assume from now on that 0 does not occur as a side of an equation in  $E$ . Suppose next that an element  $a$  of  $A$  occurs as a side of an equation in  $E$ . If the other side is also  $a$ , then the equation  $a = a$  can simply be omitted. Otherwise, the other side must be a term  $t$  not involving  $a$ , for if it involved  $a$  then the equation would contradict (5). So we can delete  $a$  from  $A$ , delete  $a = t$  (or  $t = a$ ) from  $E$ , and replace all other occurrences of  $a$  in  $E$  by  $t$ . Again, we have a simpler presentation of the same algebra. So we may assume that each equation in  $E$  has the form  $[t_1, t_2] = [t_3, t_4]$ ; but such an equation can, thanks to (2) and (3), be replaced with the two equations  $t_1 = t_3$  and  $t_2 = t_4$ , of lesser total length. So again we get a simpler presentation of the same algebra. Repeating these steps, we find that the process must terminate, for the size of  $A$  cannot decrease infinitely often, and, after it stops decreasing, the total length of  $E$  cannot decrease infinitely often. But the only way the process can stop is if  $E$

has become empty. This proves that  $\langle A \mid E \rangle$  is isomorphic to  $\langle B \mid \emptyset \rangle$  for some  $B$  (a subset of  $A$ ).

By virtue of this simplification, we may regard  $\mathcal{A}$  as consisting of only the free algebras  $\langle A \rangle$  on finite sets  $A$  of generators. We may also suppose that the only sets  $A$  occurring are of the form  $\{1, 2, \dots, k\}$  for natural numbers  $k$ , since every  $A$  is isomorphic to one of these. We write  $\langle k \rangle$  for  $\langle \{1, 2, \dots, k\} \rangle$ .

The elements of  $\langle k \rangle$  are the variable-free terms of the language having the constant symbol  $0$ , the binary operation  $[-, -]$ , and constant symbols for the generators  $1, 2, \dots, k$ . They can be identified with trees in which leaves may (but need not) be labeled with integers in the range from  $1$  to  $k$ . So they are like patterns (defined in Section 1) except for the restriction on the possible labels and the fact that several leaves are allowed to have the same label. We call them *k-labeled trees*. Note in particular that the members of  $\langle 0 \rangle$  are simply the trees.

A morphism in  $\mathcal{A}$  from  $\langle k \rangle$  to  $\langle l \rangle$  is, since  $\langle k \rangle$  is free, simply a map from  $\{1, 2, \dots, k\}$  into  $\langle l \rangle$ , i.e., a  $k$ -tuple of  $l$ -labeled trees. To compose this with some  $\langle l \rangle \rightarrow \langle m \rangle$ , i.e., with an  $l$ -tuple of  $m$ -labeled trees, take the  $k$ -tuple of  $l$ -labeled trees and replace, in each of its component trees, each leaf labeled  $j$  with the  $j$ th  $m$ -labeled tree in the given  $l$ -tuple. The identity morphism of  $\langle k \rangle$  is the  $k$ -tuple whose  $i$ th member is a single node labeled  $i$ .

The topos of set-valued functors on  $\mathcal{A}$  is the classifying topos for the universal Horn theory axiomatized by (1), (2), (3), and (5). The universal model  $U$  is the underlying set functor. For details about this, see for example [2].

To obtain the classifying topos for the full theory  $\mathcal{T}'$ , we must pass to the subtopos of sheaves for the Grothendieck topology “forcing” the remaining axiom, (4). See [9, 11] for more information about forcing topologies. In the case at hand, the topology in question is that described in Part (1) of the following lemma, whose other parts give useful alternative ways of viewing this topology. In connection with Parts (2) and (3) of the lemma, recall that in the proof of Theorem 3 we introduced, for any polynomial  $P(X)$ , a set  $S_n(P)$  of tagged tuples of patterns; we shall need this for the polynomials  $X^k$ . In this special case of monomials, all tags are  $1$ , so they can be omitted, and all the tuples are  $k$ -tuples. So the elements of  $S_n(X^k)$  can be taken to be simply  $k$ -tuples of patterns and thus, by suitable choice of labels, morphisms  $\langle k \rangle \rightarrow \langle l \rangle$  for certain  $l$ 's.

**Lemma.** *The following four Grothendieck topologies on the dual of  $\mathcal{A}$  coincide.*

- (1) *The smallest topology for which  $\langle 1 \rangle$  is covered by the set of two morphisms  $\langle 1 \rangle \rightarrow \langle 0 \rangle : 1 \mapsto 0$  and  $\langle 1 \rangle \rightarrow \langle 2 \rangle : 1 \mapsto [1, 2]$*
- (2) *The smallest topology in which, for each  $n$ , each  $\langle k \rangle$  is covered by the set  $S_n(X^k)$ .*
- (3) *The topology where the covering sieves of any  $\langle k \rangle$  are those sieves that include  $S_n(X^k)$  for some  $n$ .*
- (4) *The topology where the covering sieves of any  $\langle k \rangle$  are those sieves that include a finite family of maps  $\langle k \rangle \rightarrow \langle l_i \rangle$  such that every map  $\langle k \rangle \rightarrow \langle 0 \rangle$  factors through a map from the finite family.*

*(It is part of the assertion of the lemma that the collections of sieves described in*

(3) and (4) are topologies.)

*Proof.* We first show that the family of sieves in described in (4) is a Grothendieck topology. It clearly contains the maximal sieve on any object (use the family consisting of just the identity map). If it contains a sieve  $R$  on  $\langle k \rangle$ , witnessed by a finite family of  $\langle k \rangle \rightarrow \langle l_i \rangle$  as in (4), and if  $f : \langle k \rangle \rightarrow \langle m \rangle$  is any morphism, then those pushouts of the  $\langle k \rangle \rightarrow \langle l_i \rangle$  along  $f$  that exist in  $\mathcal{A}$  witness that (4) also contains the sieve of morphisms out of  $\langle m \rangle$  whose composites with  $f$  are in  $R$ , i.e., the pullback of  $R$  along  $f$  in the sense of the dual category. (We use here that, for any pair of maps in  $\mathcal{A}$  with the same domain, if they can be completed to a commutative square then they have a pushout. This is a general property of categories of models of universal Horn theories.) Finally, the alleged topology is closed under composition, because if a family of maps  $\langle k \rangle \rightarrow \langle l_i \rangle$  and, for each  $i$ , another family of maps  $\langle l_i \rangle \rightarrow \langle m_{ij} \rangle$  satisfy the requirements in (4), then so does the family of all  $\langle k \rangle \rightarrow \langle m_{ij} \rangle$ . Thus, (4) is a topology.

This topology contains the sieve generated by the two maps in (1). Indeed, these two maps themselves serve as the finite family required in (4), for any morphism  $\langle 1 \rangle \rightarrow \langle 0 \rangle$  must send 1 to either the empty tree 0 or a non-empty tree  $[t_1, t_2]$ , and in the former case it factors through the specified map  $\langle 1 \rangle \rightarrow \langle 0 \rangle$  while in the latter case it factors through the specified map  $\langle 1 \rangle \rightarrow \langle 2 \rangle$  (via the map sending 1 and 2 to  $t_1$  and  $t_2$ ). Therefore, the topology (4) includes the topology (1).

We show next that (1) includes (2); of course it suffices to show that the generating covers  $S_n(X^k)$  in (2) are also covers with respect to (1). For this purpose, note that  $S_n(X^k)$  can be obtained from the identity map of  $\langle k \rangle$  (a  $k$ -tuple of distinctly labeled, one-element trees) by repeated use of the development process described in the proof of Theorem 3. That is, a tree  $r$  with a particular labeled node is replaced by two trees,  $r'$  where that node has been removed and  $r''$  where that node has been given two distinctly labeled children and has lost its own label. Notice that, in each of the  $k$ -tuples arising in this development process, no labels are repeated. We show that applying one development step to a cover in the topology (1), with no repeated labels in its  $k$ -tuples, yields again a cover. This will clearly imply that  $S_n(X^k)$ , obtained by repeated development of a trivial cover, is itself a cover, as desired. So consider one development step applied to such a cover. Suppose it replaces a node labeled  $i$  in some (unique) component of a  $k$ -tuple,  $\langle k \rangle \rightarrow \langle l \rangle$ , in the given covering. Then the pushouts in  $\mathcal{A}$  of the two maps in (1) along  $\langle 1 \rangle \rightarrow \langle l \rangle : 1 \mapsto i$  cover  $\langle l \rangle$  (since a topology on the dual of  $\mathcal{A}$  is closed under pullbacks in this dual). So in the given cover of  $\langle k \rangle$  we may replace the map  $\langle k \rangle \rightarrow \langle l \rangle$  by its composites with these two pushouts, and we still have a cover of  $\langle k \rangle$ . But this replacement is precisely the development step at label  $i$ . This completes the proof that topology (1) includes (2).

It is trivial that all the sieves described in (3) are in topology (2). That (3) is a topology will follow once we show that it includes (4), for then all four items listed in the lemma are equal.

So, to complete the proof, we consider an arbitrary sieve  $R$  containing a finite family of maps  $f_i : \langle k \rangle \rightarrow \langle l_i \rangle$  as in (4), and we show that this sieve is in (3). Fix an integer  $n$  greater than the depths of all the labeled trees occurring in the  $k$ -tuples  $f_i$ .

We would like to develop all these  $k$ -tuples to depth  $n$  and show that the resulting  $k$ -tuples are all in  $R$  and include all of  $S_n(X^k)$ . Some caution is needed, however, since the same label may occur several times in an  $f_i$ , and development has not even been defined for such an  $f_i$ . We begin by extending the notion of development to the case of repeated labels. If label  $z$  occurs several times in a  $k$ -tuple  $\vec{r}$  (either in the same component tree or in different components) and if all its occurrences are at depths  $\leq n$ , then we develop  $\vec{r}$  at  $z$  by replacing it by two  $k$ -tuples,  $\vec{r}'$  where all nodes labeled  $z$  have been deleted, and  $\vec{r}''$  where each node labeled  $z$  has been (unlabeled and) given two children, the left one being labeled  $z_l$  and the right one  $z_r$ , where these are two labels not yet occurring in  $\vec{r}$ . In other words, we carry out development in the previous sense at all  $z$ -labeled nodes in parallel, treating them all identically. As in the proof that (1) includes (2), it is easy to see that this definition makes  $\vec{r}'$  and  $\vec{r}''$ , regarded as morphisms out of  $\langle k \rangle$ , the composites of  $\vec{r}$  with two other morphisms; in particular,  $\vec{r}'$  and  $\vec{r}''$  belong to the sieve  $R$  if  $\vec{r}$  does. By repeated development, the given family of maps  $f_i$  becomes a new family of maps  $f'_j$  with the property that each tree in it has depth at most  $n+1$ , and every label that occurs in any  $f'_j$  has at least one occurrence at depth  $n+1$  in  $f'_j$  (for if all occurrences were at depth  $\leq n$  then we could develop further at that label). After development is finished, any  $f'_j$  without repeated labels is a member of  $S_n(X^k)$ .

We shall show that all members of  $S_n(X^k)$  must be among the  $f'_j$ 's; as pointed out above, this will suffice to complete the proof, since each  $f'_j$  is in  $R$ . Suppose, toward a contradiction, that  $\vec{p} \in S_n(X^k)$  is not among the  $f'_j$ 's. Form an instance  $\vec{t}$  of  $\vec{p}$  by a substitution that replaces the labels in  $\vec{p}$  with distinct trees of all of the same depth  $d > n$ . This  $\vec{t}$  is an instance of at least one of the original  $f_i$ 's, since all  $k$ -tuples of trees are such instances. By considering the development process one step at a time, one easily sees that  $\vec{t}$  is an instance of some  $k$ -tuple at each stage of the development; in particular it is an instance of some  $f'_j$  at the final stage. That  $f'_j$  cannot have distinct labels, for then it would be in  $S_n(X^k)$ , whereas the unique element of  $S_n(X^k)$  having  $\vec{t}$  as an instance is  $\vec{p}$  which is not an  $f'_j$ . So  $\vec{t}$  is an instance of an  $f'_j$  in which some label, say  $z$ , occurs at least twice. Let  $s$  be the tree substituted for  $z$  in instantiating  $f'_j$  to  $\vec{t}$ . Then  $s$  occurs at least twice as a subtree in components of  $\vec{t}$ , namely wherever  $z$  occurred as a label in  $f'_j$ , and at least one of these subtrees has its root at depth  $n+1$  in  $\vec{t}$ . So when  $\vec{t}$  was obtained by instantiating  $\vec{p}$ , some label must have been replaced by  $s$ ; thus, by our choice of that substitution,  $s$  has depth  $d > n$ . Our choice of substitution also ensures that  $s$  cannot have a second occurrence with its root at depth  $n+1$  in  $\vec{t}$ , for at depth  $n+1$  we replaced all nodes in  $\vec{p}$  with different trees. Nor can a second occurrence of  $s$  in  $\vec{t}$  have its root at depth greater than  $n+1$ , for then that occurrence would extend to depth greater than  $n+d$ , which is the maximum depth occurring in  $\vec{t}$ . So  $s$  must have a second occurrence in  $\vec{t}$  with its root  $x$  at depth  $e \leq n$ . In  $\vec{p}$ , there is a node at the same position that  $x$  has in  $\vec{t}$  (as the substitution leading from  $\vec{p}$  to  $\vec{t}$  affects only depths  $> n$ ), and we call it  $x$  also. If  $x$  has any labeled descendants in  $\vec{p}$ , then in  $\vec{t}$  those descendants, at level  $n+1$ , were replaced by trees of depth

$d$ . So the subtree with root  $x$  in  $\vec{t}$  has depth  $n - e + 1 + d > d$ . If, on the other hand,  $x$  has no labeled descendants in  $\vec{p}$ , then the subtree with root  $x$  extends to depth at most  $n - 1 < d$  in  $\vec{p}$  (as  $\vec{p} \in S_n(X^k)$ , so any node at depth  $n$  must have two labeled children), and nothing changes in this subtree when we pass from  $\vec{p}$  to  $\vec{t}$ . But the subtree of  $\vec{t}$  with root  $x$  is  $s$ , of depth  $d$ , so both cases are contradictory. This contradiction completes the proof that the sieve  $R$  includes  $S_n(X^k)$  and is therefore a covering in (3).  $\square$

Let  $\mathcal{E}$  be the topos of sheaves on the dual of  $\mathcal{A}$  with respect to the topology described in this lemma. We write  $i$  for the canonical geometric morphism from  $\mathcal{E}$  to the presheaf topos  $\mathcal{S}^{\mathcal{A}}$ , so  $i_*$  is the inclusion functor and  $i^*$  is the associated sheaf functor. It follows from [9] that  $\mathcal{E}$  is the classifying topos for models of  $\mathcal{T}'$ , the generic model being  $G = i^*(U)$ .

The hypothesis of the theorem implies that the statement “there exists a pair of inverse bijections  $f : P(G) \rightarrow Q(G)$  and  $g : Q(G) \rightarrow P(G)$ ” is internally true in  $\mathcal{E}$ . This implies, by virtue of the internal meaning of “there exists,” that there is an epimorphism  $C \rightarrow 1$  in  $\mathcal{E}$  such that in the slice topos  $\mathcal{E}/C$  there is a pair of actual inverse isomorphisms between  $C^*(P(G))$  and  $C^*(Q(G))$ , where  $C^*$  means the inverse image along the canonical geometric morphism  $\mathcal{E}/C \rightarrow \mathcal{E}$ .

We show that  $C$  has a global element  $1 \rightarrow C$ . The fact that  $C \rightarrow 1$  is an epimorphism in the sheaf topos  $\mathcal{E}$  means that, in the presheaf topos, it has dense range, i.e., that every object of the site is covered by arrows from objects  $A$  where  $C(A)$  is inhabited. (Here “from” refers to the site, the dual of  $\mathcal{A}$ ; in terms of maps in  $\mathcal{A}$  we would have “to” instead.) But  $\langle 0 \rangle$  is covered only by its maximal sieve, as is clear by description (4) in the lemma. So  $C(\langle 0 \rangle)$  is inhabited, say by  $z$ . But  $\langle 0 \rangle$  is initial in  $\mathcal{A}$ , so  $C$  has a global section (in  $\mathcal{S}^{\mathcal{A}}$  and hence also in  $\mathcal{E}$ ) whose value at any object  $A$  is the image of  $z$  under  $C$  of the unique map  $\langle 0 \rangle \rightarrow A$ .

By taking inverse images along this section, we find that already in  $\mathcal{E}$  (without having to pass to a slice topos) we have a pair of inverse isomorphisms  $f : P(G) \rightarrow Q(G)$  and  $g : Q(G) \rightarrow P(G)$ .

Now consider, in the topos  $\mathcal{S}$  of sets, the set  $T$  of (finite, binary) trees. It is a model of  $\mathcal{T}'$  in an obvious way (in fact the initial model of  $\mathcal{T}'$ ). So it is  $\mu^*(G)$  for some geometric morphism  $\mu^* : \mathcal{S} \rightarrow \mathcal{E}$  (a point of  $\mathcal{E}$ ). Since inverse images along geometric morphisms preserve finite products and coproducts,  $\mu^*(f)$  and  $\mu^*(g)$  are inverse bijections between the sets  $P(T)$  and  $Q(T)$ . To complete the proof of the theorem, it suffices to show that they are very explicit, for then the polynomials  $P(X)$  and  $Q(X)$  are combinatorially equivalent and, by Theorem 3, algebraically equivalent, as desired.

In fact, it suffices to prove somewhat less. In the definition of “very explicit function” in Section 3, we required that no label be repeated in any  $\vec{q}_i$  (as part of the notion of pattern) and that  $\vec{q}_i$  contain exactly the same labels as  $\vec{p}_i$ . But we mentioned, just before Theorem 3, that one could relax these requirements, by allowing repeated labels in  $\vec{q}_i$  and by allowing labels to occur in  $\vec{p}_i$  without occurring in  $\vec{q}_i$ , without affecting the very explicit bijections. This is because any function that is very explicit in the liberalized sense but not in the original sense cannot be

a bijection. So in our present situation, it suffices to show that  $\mu^*(f)$  and  $\mu^*(g)$  are very explicit in the liberalized sense, since we already know that they are bijections.

Of course it suffices to treat  $\mu^*(f)$ , as the situation is symmetric between the two bijections. In fact, it suffices to treat the restriction of  $\mu^*(f)$  to one of the summands  $T^k$  in  $P(T)$ , for if each of these restrictions is very explicit then so is  $\mu^*(f)$  itself. Note that such a restriction of  $\mu^*(f)$  is  $\mu^*$  of a restriction of  $f$  to one of the summands  $G^k$  of  $P(G)$  in  $\mathcal{E}$ .

So, changing notation slightly, we have a morphism  $f : G^k \rightarrow Q(G)$  in  $\mathcal{E}$ , and we wish to show that  $\mu^*(f) : T^k \rightarrow Q(T)$  is very explicit in the liberalized sense.

Recall that  $G$  is obtained by applying the associated sheaf functor  $i^*$  to the underlying set functor  $U$  in  $\mathcal{S}^{\mathcal{A}}$ . As  $i^*$  preserves sums and products, we can write  $f : i^*(U^k) \rightarrow i^*(Q(U))$ , so  $f$  corresponds, under the adjunction  $i^* \dashv i_*$ , to a map of presheaves  $U^k \rightarrow i_*i^*(Q(U))$ . Since  $U^k$  is a representable presheaf, represented by  $\langle k \rangle$ , such a map corresponds via Yoneda's Lemma to an element of  $(i_*i^*(Q(U))) (\langle k \rangle)$ . Here  $i_*i^*(Q(U))$  is the associated sheaf of  $Q(U)$ , regarded as a presheaf. Inspecting the usual construction of associated sheaves in terms of "patched together" sections, we find that any element of  $(i_*i^*(Q(U))) (\langle k \rangle)$  can be described as follows. There is a cover of  $\langle k \rangle$ , say by maps  $\langle k \rangle \rightarrow \langle l_i \rangle$ , and to each of these maps is assigned an element  $q_i \in (Q(U)) (\langle l_i \rangle) = Q(U(\langle l_i \rangle))$  in a coherent manner. (Coherence means that if two of the maps  $\langle k \rangle \rightarrow \langle l_i \rangle$  in the cover can be completed to a commutative square by maps  $\langle l_i \rangle \rightarrow A$ , then the resulting images in  $Q(U(A))$  of the  $q_i$ 's are equal.) By the lemma, we may take the covering to be  $S_n(X^k)$  for some  $n$ ; then the maps  $\langle k \rangle \rightarrow \langle l_i \rangle$  in the covering are  $k$ -tuples of patterns (since no label is re-used in a map in  $S_n(X^k)$ ) such that every  $k$ -tuple of trees is an instance of exactly one of them. If we call these patterns  $p_i$ , then they and the corresponding  $q_i$  are exactly as required in the liberalized definition of a very explicit function. It remains to check that, if we start with  $f : G^k \rightarrow Q(G)$ , transform it to  $U^k \rightarrow i_*i^*(Q(U))$  and then to an element of  $(i_*i^*(Q(U))) (\langle k \rangle)$ , and finally represent that element on a covering to obtain a very explicit function, as just described, then that function agrees with  $\mu^*(f)$ . This verification is routine, tedious, and therefore omitted.  $\square$

Lawvere has pointed out that the proof of Theorem 4 emphasizes the distinction between the 2-categorical sort of universality that defines classifying topoi and the 1-categorical sort that defines free algebras. For the proof shows that the (2-categorically universal) generic model  $G$  of  $\mathcal{T}'$  satisfies  $P(G) \cong Q(G)$  only when  $P(X)$  and  $Q(X)$  are algebraically equivalent; in particular,  $G^2$  and  $G$  are not isomorphic in the classifying topos. In contrast, as we remarked before the proof of Theorem 4, the (1-categorically universal) free model  $T$  of  $\mathcal{T}'$  in any topos, consisting of finite trees, does satisfy  $T^2 \cong T + T \cong T$ .

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