A. What are ultrafilters?

Ultrafilters can be defined in numerous equivalent ways. The definition given here was chosen for its simplicity. Equivalent and perhaps more informative characterizations will be discussed afterward.

Definition 1. Let I be any set. An ultrafilter on (or over) I is a family $\mathcal{U}$ of subsets of I such that, for all subsets X and Y of I,

(i) $X \cap Y \in \mathcal{U} \iff X \in \mathcal{U}$ and $Y \in \mathcal{U}$,

and (ii) $I - X \in \mathcal{U} \iff X \notin \mathcal{U}$.

Example 2. For any element $i \in I$, the principal or trivial ultrafilter concentrated at i is defined to be $\{X \subseteq I \mid i \in X\}$. Conditions (i) and (ii) reduce, in this case, to the definitions of intersection and complement.

This example shows that there are ultrafilters on all nonempty sets; there are none on the empty set because of (ii). The more interesting question whether non-principal ultrafilters exist will be taken up in the next section.

One sometimes thinks of an ultrafilter on I as describing an "ideal element" of I, which belongs to exactly those subsets of I that are in the ultrafilter. ~~From this point of view, the definition~~ Definition 1 says that such ideal elements behave correctly with respect to intersection and complement, i.e., an ideal element belongs to $X \cap Y$ if and only if it belongs to both X and Y, and ~~similarly~~ it belongs to $I - X$ if and only if it doesn't belong to X. The principal ultrafilters are, in this picture, identified with the actual elements of I. This rather vague notion of ideal element can be made precise in at least two ways, one model-theoretic and the other topological; see Chapters 2 and 3.

The following proposition summarizes some immediate consequences of the definition of ultrafilters and elementary Boolean algebra.

<u>Proposition 3</u>. Let $\mathcal{U}$ be an ultrafilter on I.

(a) If $X \in \mathcal{U}$ and $X \subseteq Y \subseteq I$ then $Y \in \mathcal{U}$.

(b) $I \in \mathcal{U}$.

(c) $\emptyset \notin \mathcal{U}$.

(d) For all subsets X and Y of I,

$$X \cup Y \in \mathcal{U} \iff X \in \mathcal{U} \text{ or } Y \in \mathcal{U}.$$

Insert Ⓐ from p. $2\frac{1}{2}$

<u>Proof</u>. (a): $X \cap Y = X \in \mathcal{U}$, so by part (i) of the definition $Y \in \mathcal{U}$.

(b) and (c): By part (ii) of the definition, either $I \in \mathcal{U}$ and $\emptyset \notin \mathcal{U}$ as desired, or $I \notin \mathcal{U}$ and $\emptyset \in \mathcal{U}$. The second alternative contradicts (a).

~~(d) The statement $X \cup Y \notin \mathcal{U}$ is equivalent by (ii) to~~

$$\cancel{(I-X) \cap (I-Y) = I-(X \cup Y) \in \mathcal{U},}$$

~~hence by (i) to~~

$$\cancel{I-X \in \mathcal{U} \text{ and } I-Y \in \mathcal{U},}$$

(A) Insert on p. 2

(e) If $X \subseteq I$ and $X \cap Y \neq \emptyset$ for all $Y \in \mathcal{U}$, then $X \in \mathcal{U}$.

Return to p. 2

(B) Insert on p. 3

(e) The assumption, applied with $Y = I - X$, shows that $I - X \notin \mathcal{U}$. So $X \in \mathcal{U}$ by part (ii) of the definition. $\square$

(d): Apply (ii), then (i), then (ii) again to get

$$X \cup Y \in \mathcal{U} \iff (I-X) \cap (I-Y) = I-(X \cup Y) \notin \mathcal{U}$$

$$\iff I-X \notin \mathcal{U} \text{ or } I-Y \notin \mathcal{U}$$

$$\iff X \in \mathcal{U} \text{ or } Y \in \mathcal{U}.$$

Insert (B) from p. $2\frac{1}{2}$

The essential point in the proof of 3(d) is that union is definable in terms of intersection and complement, so that, knowing from Definition 1 how ultrafilters behave with respect to the latter operations, one can infer how they behave with respect to unions. All other ~~operation of~~ Boolean ~~algebra~~ operations are also definable in terms of intersection and complement and therefore satisfy ~~propositions as~~ analogs of 3(d). For example, an ultrafilter $\mathcal{U}$ on I contains the symmetric difference $X \triangle Y$ of two subsets of I if and only if it contains exactly one of X and Y. All these analogs reduce, in the case of principal ultrafilters, to the definitions of the Boolean operations.

The choice of intersection and complement as the basic operations is an arbitrary one. Any other complete set of Boolean operations would serve as well, and for one such set, union and complement, this fact is worth writing out explicitly.

<u>Corollary 4</u>. A family $\mathcal{U}$ of subsets of I is an ultrafilter on I if and only if, for all subsets X and Y of I,

$$X \cup Y \in \mathcal{U} \iff X \in \mathcal{U} \text{ or } Y \in \mathcal{U}, \text{ and}$$

$$I - X \in \mathcal{U} \iff X \notin \mathcal{U}. \ \square$$

Since there are complete sets consisting of a single Boolean operation, one could define ultrafilters by means of a single equivalence, for example that $\mathcal{U}$ contains ~~the complement~~ $I-(X \cup Y)$ if and only if it contains neither X nor Y.

<u>Corollary 5</u>. Let $\mathcal{U}$ be an ultrafilter on I and let $X_1, \ldots, X_n$ be finitely many subsets of I.

(a) $X_1 \cap \ldots \cap X_n \in \mathcal{U} \iff$ for each $i = 1, 2, \ldots, n$, $X_i \in \mathcal{U}$.

(b) $X_1 \cup \ldots \cup X_n \in \mathcal{U} \iff$ for some $i = 1, 2, \ldots, n$, $X_i \in \mathcal{U}$.

(c) Any function defined on a set in $\mathcal{U}$ and taking only finitely many values is constant on some set in $\mathcal{U}$.

<u>Proof</u>. For (a) and (b), use induction on n. Part (ii) of the definition and Proposition 3(d) give the induction step. Parts (b) and (c) of Proposition 3 give the base case, $n=0$. For (c), suppose f is defined on a set $A \in \mathcal{U}$ and takes only the values $\cancel{a_1, \ldots, a_n}$ $v_1, \ldots, v_n$. The $\Rightarrow$ half of (b), with $X_i = \{a \in A \mid f(a) = v_i\}$ gives the desired conclusion. $\square$

~~Proof. Induction on n~~

~~*Proof* by induction on n. Part (ii) of the definition and Proposition 3(d) give the induction step. Proposition 3(b,c) gives the base case, $n=0$. $\square$~~

The following definitions concern some useful weakenings of the concept of ultrafilter. They will be used to give equivalent characterizations of ultrafilters, but they are also of considerable importance in their own right.

Definition 6. A family $\mathcal{S}$ of subsets of a set I has the finite intersection property (abbreviated FIP) if every finite subfamily has a nonempty intersection.

Definition 7. A nonempty family $\mathcal{B}$ of subsets of a set I is a filterbase if $\emptyset \notin \mathcal{B}$ and, for every X and Y in $\mathcal{B}$, there exists $Z \in \mathcal{B}$ such that $Z \subseteq X \cap Y$.

Definition 8. A nonempty family $\mathcal{F}$ of subsets of a set I is a filter on I if

(i) $\emptyset \notin \mathcal{F}$

(ii) $X \in \mathcal{F}$ and $Y \in \mathcal{F} \Rightarrow X \cap Y \in \mathcal{F}$

(iii) $X \in \mathcal{F}$ and $X \subseteq Y \subseteq I \Rightarrow Y \in \mathcal{F}$.

Clearly, every ultrafilter is a filter, every filter is a filterbase, and every filterbase has the finite intersection property. Typical examples of filters are the family of all (not necessarily open) neighborhoods of a particular point in a topological space and the family of subsets of the real line whose complements have Lebesgue measure zero. The following considerably simpler example will be important in connection with ultrafilters.

Definition 9. Let I be an infinite set. The Fréchet filter or cofinite filter on I is the family of subsets of I whose complements (in I) are finite. Such subsets are called cofinite.

~~Proposition 10. An ultrafilter $\mathcal{U}$ on an infinite set I is non-principal if and only if it includes the Fréchet~~

Proposition 10. Let $\mathcal{U}$ be an ultrafilter on an infinite set I. The following are equivalent.

(a) $\mathcal{U}$ is non-principal.

(b) $\mathcal{U}$ contains no singleton $\{i\}$.

(c) $\mathcal{U}$ contains no finite set.

(d) $\mathcal{U}$ includes the cofinite filter on I.

<u>Proof</u>. (c)$\Rightarrow$(b)$\Rightarrow$(a) trivially. Definition 1(ii) gives (c)$\Leftrightarrow$(d), and Corollary 5(b) gives (b)$\Rightarrow$(c). Finally, to prove (a)$\Rightarrow$(b), suppose $\mathcal{U}$ contains a singleton $\{i\}$ and therefore (by 3(a)) contains all subsets of I that contain i. The subsets of I that don't contain i are the complements of those that do, hence cannot be in $\mathcal{U}$ by 1(ii). So $\mathcal{U}$ is the principal ultrafilter concentrated at i. $\square$

~~The following proposition gives a connection, beyond the implications mentioned just after Definition 8, between filters, filterbases, and the finite intersection property.~~

~~<u>Proposition 11</u> (a) If $\mathcal{B}$ is a filterbase on I, then~~

~~$$\{X\subseteq I \mid B\subseteq X \text{ for some } B\in\mathcal{B}\}$$~~

~~is a filter on I~~

The following definitions establish a connection, beyond the implications mentioned just after Definition 8, between filters, filterbases, and families with the finite intersection property.

<u>Definition 11</u>. Let $\mathcal{B}$ be a filterbase on I. The <u>filter generated by</u> $\mathcal{B}$ is

$$\{X \subseteq I \mid \text{For some } B \in \mathcal{B},\ B \subseteq X\}.$$

$\mathcal{B}$ is said to be a <u>base</u> for this filter.

<u>Definition 12</u>. Let $\mathcal{S}$ be a family of subsets of I with the finite intersection property. The <u>filterbase generated by</u> $\mathcal{S}$ consists of all the intersections of finite subfamilies of $\mathcal{S}$. The <u>filter generated by</u> $\mathcal{S}$ is the filter generated (in the sense of Definition 11) by this filterbase. $\mathcal{S}$ is said to be a <u>subbase</u> for this filter.

It is easy to check that the filterbase generated by a family with the FIP is a filterbase (it is nonempty, even if $\mathcal{S}$ is empty, because I is the intersection of the empty subfamily), that the filter generated by such a family is a filter, and that the filter generated by a filterbase $\mathcal{B}$ (as in Definition 11) is the same as the filter generated by $\mathcal{B}$ considered (as in Definition 12) as merely a family with FIP. ~~these definitions are not misleading.~~ The filter generated by $\mathcal{B}$ or $\mathcal{S}$ is the smallest filter that includes $\mathcal{B}$ or $\mathcal{S}$, but the corresponding statement about the filterbase generated by $\mathcal{S}$ is in general false.

Proposition 12.5. Let $\mathcal{F}$ be a filter on I and $\mathcal{B}$ a family of subsets of I. $\mathcal{B}$ is a base for $\mathcal{F}$ if and only if $\mathcal{B} \subseteq \mathcal{F}$ and every set in $\mathcal{F}$ has a subset in $\mathcal{B}$.

Proof. The "only if" part is immediate from Definition 11. ~~and for the "if" part the only nontrivial thing to check is~~ For the "if" part, ~~notice that~~ suppose $\mathcal{B}$ satisfies the given condition. Then $\mathcal{B}$ is nonempty because $\mathcal{F}$ is, and, if X and Y are in $\mathcal{B}$, then ~~they are~~ they are in $\mathcal{F}$, so $X \cap Y \in \mathcal{F}$, and therefore $X \cap Y$ has a subset in $\mathcal{B}$. This shows that $\mathcal{B}$ is a filterbase. Now the first part of the given condition on $\mathcal{B}$ implies that the filter it generates is included in $\mathcal{F}$, and the second part implies the reverse inclusion.

Theorem 13. Let $\mathcal{U}$ be a family of subsets of I. The following are equivalent.

(a) $\mathcal{U}$ is an ultrafilter.

(b) $\mathcal{U}$ has the finite intersection property (is a filterbase, is a filter) and, for all $X \subseteq I$, either $X \in \mathcal{U}$ or $I - X \in \mathcal{U}$.

(c) $\mathcal{U}$ is a maximal family of subsets of I with the finite intersection property (a maximal filterbase on I, a maximal filter on I).

~~In parts (b)~~
~~Each of parts (b) and (c)~~
The parenthesized parts in (b) and (c) mean that each of these two statements can be read in three ways, so the theorem asserts the equivalence of seven statements. The proof will provide a few more.

~~Proof. (a) $\Rightarrow$ (b) follows, even for the strongest reading of (b) with "is~~ a ~~filter",~~

<u>Proof</u>. (a) implies the strongest form, with "is a filter", of (b) by Proposition 3 and Definition 1. Conversely, assume the weakest reading, with "has the FIP" of (b). Half of 1(ii) is satisfied by assumption, and the other half holds because X and $I-X$, being disjoint, cannot both be in a family with FIP. As for 1(i), suppose first that $X \cap Y \in \mathcal{U}$. Then the FIP prevents $\mathcal{U}$ from containing $I-X$ or $I-Y$, so it must contain X and Y. On the other hand, if $\mathcal{U}$ contains X and Y then the FIP prevents it from containing $I-(X \cap Y)$, so it must contain $X \cap Y$. This proves that (a)$\Leftrightarrow$(b).

In particular, the three readings of (b) are equivalent and can be used interchangeably. For (b) $\Leftrightarrow$ (c), it is convenient to work with the following weakened and strengthened versions of (c).

(c^-) $\mathcal{U}$ has the FIP and is not properly included in any filter on I.

(c^+) $\mathcal{U}$ is a filter on I and is not properly included in any family of subsets of I with the FIP.

Clearly, (c^+) implies all three readings of (c), and each of these implies (c^-). It therefore suffices to prove (c^-) $\Rightarrow$ (b) $\Rightarrow$ (c^+).

Assume (c^-). To prove the "FIP" reading of (b), it suffices to consider an arbitrary $X \subseteq I$ and prove that X or $I - X$ is in $\mathcal{U}$. Suppose X were a ~~contrary~~ counterexample. If $\mathcal{U} \cup \{X\}$ ~~were~~ had the FIP, the filter it generates would properly ~~contain~~ include $\mathcal{U}$ (since it contains X), contrary to assumption. So there must exist $A_1, \ldots, A_m \in \mathcal{U}$ such that $A_1 \cap \ldots \cap A_m \cap X = \emptyset$, i.e., such that

~~$A_1 \cap \ldots \cap A_m \cap X = \emptyset$.~~

$A_1 \cap \ldots \cap A_m \subseteq I - X$. The same argument ~~starting with~~ with $I - X$ in place of X produces $B_1, \ldots, B_n \in \mathcal{U}$ such that $B_1 \cap \ldots \cap B_n \subseteq X$. But then

$$A_1 \cap \ldots \cap A_m \cap B_1 \cap \ldots \cap B_n \subseteq X \cap (I - X) = \emptyset,$$

~~contradicts~~ which contradicts the FIP of $\mathcal{U}$.

Finally, assume the "filter" reading of (b). This assumption includes the first part of (c^+), so it remains to prove the second part, namely that if $\mathcal{U} \subseteq \mathcal{S}$ and $\mathcal{S}$ has the FIP, then $\mathcal{S} \subseteq \mathcal{U}$. Consider any such $\mathcal{S}$ and any $X \in \mathcal{S}$. The FIP prevents $\mathcal{S}$ from containing $I - X$, so $I - X \notin \mathcal{U}$. Then, by (b), $X \in \mathcal{U}$. This proves $\mathcal{S} \subseteq \mathcal{U}$. $\square$

<u>Corollary 14.</u> No ultrafilter on I is properly included in another.

<u>Proof.</u> Such an inclusion would contradict the maximality in 13(c). $\square$

The following easy proposition is surprisingly useful in that it often allows one to ignore phenomena outside a set of the filter under consideration.

~~Proposition 15. Let $\mathcal{F}$ be a filter on I, let $A \in \mathcal{F}$, and suppose that X and Y are subsets of I such that $X \cap A = Y \cap A$. Then $X \in \mathcal{F} \iff Y \in \mathcal{F}$.~~

~~Proof. As $A \in \mathcal{F}$, we have $X \in \mathcal{F} \iff$~~

<u>Proposition 15.</u> Let $\mathcal{F}$ be a filter on I.

(a) For any subsets X and Y of I,

$$X \cap Y \in \mathcal{F} \iff X \in \mathcal{F} \text{ and } Y \in \mathcal{F}.$$

(b) Suppose that $A \in \mathcal{F}$ and ~~supp~~ that X and Y are subsets of I with $X \cap A = Y \cap A$. Then $X \in \mathcal{F} \iff Y \in \mathcal{F}$.

<u>Proof.</u> The $\Leftarrow$ half of (a) is ~~part of~~ clause (ii) of the definition of filter, and the $\Rightarrow$ half follows immediately from clause (iii). For (b), the hypotheses and (a) yield $X \in \mathcal{F} \iff X \cap A \in \mathcal{F} \iff Y \cap A \in \mathcal{F} \iff Y \in \mathcal{F}$. $\square$

Part (b) of this proposition says that, if $A \in \mathcal{F}$, then in determining whether $X \in \mathcal{F}$ the part of X outside A is irrelevant. ~~Because of this, one sometimes~~ This would not be so if $A \notin \mathcal{F}$ (take $X = A$, $Y = I$ for a counterexample). Because of this, one sometimes ~~says that $\mathcal{F}$~~ expresses that $A \in \mathcal{F}$ by saying that $\mathcal{F}$ <u>concentrates</u> (or <u>is concentrated</u>) on A. This terminology coheres with that in Example 2. $\mathcal{U}$ is concentrated at i if and only if it concentrates on $\{i\}$.

An alternative way to view ultrafilters ~~in terms~~, which has considerable intuitive value, is as finitely additive measures. If $\mathcal{U}$ is an ultrafilter on I, then one can view $\mathcal{U}$ as assigning to each $X \subseteq I$ the measure $\mu(X) = 1$ if $X \in \mathcal{U}$ and the measure $\mu(X) = 0$ if $X \notin \mathcal{U}$. In this terminology, the preceding paragraph says that one can, for certain purposes, ignore sets of measure zero (a familiar idea in measure theory).

<u>Proposition 16.</u> For any ultrafilter $\mathcal{U}$ on I, the function μ defined above is a finitely additive two-valued measure on the power set $\mathcal{P}(I)$ of I. That is,

$$\mu : \mathcal{P}(I) \to \{0, 1\},$$

$$\mu(I) = 1, \text{ and}$$

whenever $X_1, \ldots, X_n$ are disjoint subsets of I,

$$\mu(X_1 \cup \ldots \cup X_n) = \mu(X_1) + \ldots + \mu(X_n).$$

Conversely, every such μ arises in this way from a unique ultrafilter $\mathcal{U}$ on I.

<u>Proof</u>. The first two stated properties of μ are obvious. For the third, finite additivity, notice first that, because the X_i are disjoint, at most one of them can belong to $\mathcal{U}$. If one of them does, then so does their union, by 5(b), and both sides of the additivity equation are 1. Otherwise, the union isn't in $\mathcal{U}$, by 5(b) again, so both sides are 0.

For the converse, the only ~~possible choice of $\mathcal{U}$~~ possibility for $\mathcal{U}$ is $\{X \subseteq I \mid \mu(X) = 1\}$. ~~Finite additivity of μ implies that Since $\mu(X) + \mu(I-X) = \mu(I) = 1$, exactly one of X and $I-X$ belongs to $\mathcal{U}$.~~ For any $X \subseteq I$, the equation

$$\mu(X) + \mu(I-X) = \mu(I) = 1$$

implies that exactly one of X and $I-X$ is in $\mathcal{U}$. If $X \in \mathcal{U}$ and $X \subseteq Y \subseteq I$, then

$$\mu(Y) = \mu(X) + \mu(Y-X) \geq \mu(X) = 1,$$

so $Y \in \mathcal{U}$. That is, $\mathcal{U}$ is closed under supersets. In particular, if $\mathcal{U}$ contains one of X and Y, then it contains $X \cup Y$. On the other hand, if $X \cup Y \in \mathcal{U}$ then

$$1 = \mu(X \cup Y) = \mu(X) + \mu(Y - X),$$

so either $\mathcal{U}$ contains X or it contains $Y - X$ and therefore Y. By Corollary 4, it follows that $\mathcal{U}$ is an ultrafilter. $\square$

Digression 17

This connection between ultrafilters and finitely additive two-valued measures suggests asking about ultrafilters that correspond to countably additive measures; after all, countable additivity is the usual assumption in measure theory. Essentially the same proof as above shows that μ is countably additive if and only if, whenever the corresponding $\mathcal{U}$ contains the union of countably many sets, then it contains at least one of those sets. ~~As in Corollary [illegible] closed under countable intersections~~ Using complementation to pass back and forth between unions and intersections (as in 3(d) and 4), one sees that this property of $\mathcal{U}$ is equivalent to being closed under countable intersections.

More generally, for an arbitrary infinite cardinal number κ, an ultrafilter $\mathcal{U}$ is called ~~Definition 7. Let κ be an infinite cardinal number. An ultrafilter $\mathcal{U}$ is~~ $<\kappa$-*complete* if it ~~is closed under intersections of fewer than κ sets or~~ contains the intersection of every subfamily of $\mathcal{U}$ of cardinality smaller than κ. (Some authors omit the $<$ and call such ultrafilters κ-complete.)

Thus, all ultrafilters are $<\aleph_0$-complete, ~~countable completeness is~~ countably additive two-valued measures correspond to $<\aleph_1$-complete ultrafilters (also called countably complete ultrafilters), and principal ultrafilters are $<\kappa$-complete for all κ. A non-principal ultrafilter on a set I cannot be $<|I|^+$-complete, for it contains the $|I|$ sets that are complements of singletons but does not contain their (empty) intersection. ~~which is empty~~ In particular, non-principal ultrafilters on countable sets, ~~countable~~ the primary object of study in this book, cannot have any greater degree of completeness than the $<\aleph_0$-completeness enjoyed by all ultrafilters. Any treatment here of such completeness is therefore a digression and will be kept much shorter than what the intrinsic importance of the subject deserves.

The basic facts about countably (or more) complete ultrafilters are as follows. If there exists a countably complete non-principal ultrafilter on a set I, then the smallest possible cardinality κ for such an I is very large (inaccessible and larger than many inaccessible cardinals), and κ is a <u>measurable cardinal</u>, which means that there exists a $<\kappa$-complete non-principal ultrafilter on κ. ~~The existence of measurable cardinals cannot~~ The usual axioms of set theory do not imply that measurable (or even inaccessible) cardinals exist, or even that it is consistent (with the usual axioms) to assume that they do. It is therefore perfectly safe to assume that there are no measurable cardinals and therefore no countably complete non-principal ultrafilters. The riskier hypothesis that measurable cardinals do exist is, however, a very interesting and fruitful one, implying much information about, for example, the structure of the real line and its definable subsets. This hypothesis is consistent if and only if the weaker hypothesis that there is a real-valued (rather than two-valued) non-~~trivial~~ atomless measure defined on all subsets of some set is consistent. Measurable cardinals play a central role in the theory of large cardinals and, thanks to recent equiconsistency results, even in the combinatorics of small cardinals like $\aleph_1$. ~~More information about the [illegible]~~ Good introductions to the theory of measurable cardinals are [

Digression 1?

~~[illegible]~~ Another concept that deserves to be defined here because of its importance in general ultrafilter theory, and despite its lack of significance for ultrafilters on countable sets, is uniformity.)

~~Definition XX.~~ An ultrafilter $\mathcal{U}$ on a set I is *uniform* if every set in $\mathcal{U}$ has the same cardinality as I.

~~An ultrafilter on a countably infinite set is one~~ This concept is nothing new when I is countably infinite, for then an ultrafilt on I is uniform if and only if it contains no finite set, i.e., if and only if it is non-principal (by Proposition 10). On larger sets, its primary use is to avoid considering extraneously large sets. If an ultrafilter $\mathcal{U}$ on I is not uniform, so it contains a set A of smaller cardinality than I, then instead of considering $\mathcal{U}$ one can consider the ultrafilter $\mathcal{U} \cap P(A)$ on A, which completely determines $\mathcal{U}$ by Proposition 15. If ~~the~~ A is chosen to have the smallest cardinality of any set in $\mathcal{U}$, then $\mathcal{U} \cap \mathcal{P}(A)$ is a uniform ultrafilter on A. Thus, roughly speaking, any ultrafilter becomes uniform when one throws away enough irrelevant members of the set on which it lives. Insert p. 19½ here

Digression 19

~~In the definition of an ultrafilter on I,~~

~~In Definition 1, the role of the set I was merely to provide a Boolean~~

One can view the role of the set I in Definition 1 as being merely to provide a Boolean algebra $\mathcal{P}(I)$; the definition is expressed entirely in terms of this algebra and makes perfect sense in any Boolean algebra. Thus, an <u>ultrafilter in</u> a Boolean ~~algebra B is a subset U of B such that, for all x and y in B~~ algebra B is a subset $\mathcal{U}$ of B such that, for all x and y in B,

$$x \wedge y \in \mathcal{U} \iff x \in \mathcal{U} \text{ and } y \in \mathcal{U}, \text{ and}$$

$$-x \in \mathcal{U} \iff x \notin \mathcal{U}.$$

(Notice that an ultrafilter <u>on</u> I is an ultrafilter <u>in</u> $\mathcal{P}(I)$.) The function μ sending members of $\mathcal{U}$ to 1 and ~~other~~ other members of B to 0 is a Boolean homomorphism from B to $\{0,1\}$ (with its usual Boolean operations), and every homomorphism

Insert on p. 19

In any infinite set I, the subsets whose complements have strictly smaller cardinality than I constitute a filter. ~~[illegible]~~ Just as in Proposition 10, an ultrafilter on I is uniform if and only if it includes this filter.

Return to p. 19

from a Boolean algebra to $\{0,1\}$ arises in this way from a unique ultrafilter. ~~ultrafilter on B~~ The Stone representation theorem asserts (in part) that every Boolean algebra is the algebra of clopen (= closed and open) subsets of a certain (totally disconnected, compact, Hausdorff) space X; the construction of this X is to take the ultrafilters in B to be its points and then to impose a suitable topology. See Chapter 3 for the special case where $B = \mathcal{P}(I)$ or see [   ] for the general case.

One can similarly define the finite intersection property, filterbases, and filters in general Boolean algebras and prove the analogs of results, like Theorem 13, that don't mention the elements of I. Filters are of particular importance in the theory of Boolean algebras, for they are precisely the sets which arise as $f^{-1}\{1\}$ where f is a homomorphism from one non-degenerate Boolean algebra to another. Up to Boolean duality (which interchanges 1 with 0), they are thus the kernels of Boolean homomorphisms and are often called <u>dual ideals</u>. The <u>ideals</u> themselves are the sets of the form $f^{-1}\{0\}$, or equivalently of the form $\{-x \mid x \in F\}$ for filters F. Notice that ~~an ideal is prime in the ring-theoretic sense (with $\wedge$ as multiplication),~~ an ultrafilter is precisely a filter whose dual is a prime ideal in the ring-theoretic sense, i.e., if the ideal contains a Boolean product $x \wedge y$, then it contains x or y. Ultrafilters can also be characterized as those filters that are the complements (in B) of their duals. ~~For the case $B = \mathcal{P}(I)$, this characterization is Theorem 13, (a) $\Rightarrow$ (b).~~