C. Generators

This section concerns the minimum number of sets needed to generate an ultrafilter ~~[illegible]~~ and, to a lesser extent, the number of ultrafilter on a set. Some of the results here will be superseded in later sections, but the ideas involved in their proofs will be useful in other contexts. The first proposition disposes of a trivial case.

Proposition 1. ~~Let U be an ultrafilter on I. Then~~ An ultrafilter is principal if and only if it has a base (or subbase) consisting of finitely many sets (or a single set).

The parenthetical alternatives are independent, so the proposition makes four assertions. ~~depending on which of the alter~~

Proof. It suffices to show that principality implies the strongest of the four alternatives, a base consisting of one set, and is implied by the weakest, a finite subbase. The former is obvious, since the principal ultrafilter concentrated at i has a base consisting of $\{i\}$. For the latter, consider the intersection A of the sets in a finite subbase of $\mathcal{U}$. Then $\mathcal{U}$ consists of the supersets of A, and of course A, being in an ultrafilter, is nonempty. If A had two or more elements, then it could be partitioned into two proper subsets; as neither of these subsets is in $\mathcal{U}$, Proposition A3 would be contradicted. So A is a singleton $\{i\}$, and $\mathcal{U}$ is therefore the principal ultrafilter concentrated at i. $\square$

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Theorem 2. If a non-principal ultrafilter $\mathcal{U}$ is generated by κ sets, then it contains a set of cardinality $< \kappa$.

Proof. Suppose $\mathcal{U}$ and κ were a counterexample. By the preceding proposition, κ is infinite, so the base $\mathcal{B}$ of $\mathcal{U}$ consisting of intersections of finitely many generators has cardinality κ (because a set of infinite cardinality κ has only κ finite subsets). Let $\mathcal{B}$ be well-ordered with order-type κ, i.e., so that each element has $< \kappa$ predecessors. By induction on this well-ordering, assign to each $B \in \mathcal{B}$ two elements, x_B and y_B, of B ~~in such a way that~~ that are distinct from each other and from all the elements chosen a previous stages of the induction. As the number of predecessors of B in the well-ordering is some cardinal $\lambda < \kappa$, the number of previously chosen elements 2λ

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Proposition $1\frac{1}{2}$. If a filter $\mathcal{F}$ is generated by κ sets, then it has a base of cardinality κ.

Proof. If κ is infinite, the desired base can be taken to consist of all intersections of finitely many generators; this works because a set of infinite cardinality κ has only κ finite subsets. If κ is finite, let A be the intersection of all the given generators. Then $\{A\}$ is a base for $\mathcal{F}$. To get a base of size κ, use A together with $\kappa - 1$ of the given generators. $\square$

Theorem 2. If a non-principal ultrafilter is generated by κ sets, then it contains a set of cardinality $< \kappa$.

Proof. Suppose $\mathcal{U}$ and κ were a counterexample. By Proposition 1, κ is infinite. By Proposition $1\frac{1}{2}$, ~~By the preceding proposition~~ $\mathcal{U}$ has a base $\mathcal{B}$ of cardinality κ.

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is also $< \kappa$ because κ is infinite. By assumption, every set in $\mathcal{U}$, in particular B, has cardinality $\geq \kappa$, so B has two (in fact κ) elements distinct from all previous choices. Thus, the desired x_B and y_B exist. ~~Then the sets $X = \{x_B \mid B \in \mathcal{B}\}$ and $Y = \{y_B \mid B \in \mathcal{B}\}$ each intersect every set in $\mathcal{B}$, hence also every set in $\mathcal{U}$. So they are both in $\mathcal{U}$, by Proposition A3(e). But this is absurd, as X and Y are disjoint. $\square$~~ ~~Then no $B \in \mathcal{B}$ is included in the set $X = \{x_B \mid B \in \mathcal{B}$~~ Let $X = \{x_B \mid B \in \mathcal{B}\}$. No $B \in \mathcal{B}$ is included in X, for $y_B \in B - X$. Nor is any $B \in \mathcal{B}$ included in the complement of X, for $x_B \in B \cap X$. Thus, neither X nor its complement is in $\mathcal{U}$, contrary to the definition (A1) of ultrafilter. $\square$

In the terminology of A18, this theorem says that a uniform ultrafilter on a set of cardinality κ requires at least κ^+ sets to generate it. ~~The following corollary is the case $\kappa = \omega$, where "uniform" reduces to "non-principal".~~

Corollary 3. ~~[illegible]~~ A non-principal ultrafilter can not be generated by countably many sets.

Proof. Combine the theorem with Proposition A10 (a)$\Leftrightarrow$(c). $\square$

Theorem 2 makes possible a straightforward construction of "many" non-principal ultrafilters. Since a stronger result, with a less straightforward proof, ~~although~~ is in   , the present result is given in less than full generality.

<u>Proposition 4</u>. On a countably infinite set I, there are at least $2^{\aleph_1}$ non-principal ultrafilters.

<u>Proof</u>. Let $\mathcal{S}$ be the cofinite filter on I. For each of the $2^{\aleph_1}$ functions $f: \aleph_1 \to 2$, build an ultrafilter by means of the following slight modification of the construction in B7. Consider the α^{th} <u>essential</u> stage (B8) of the construction. That is, suppose $\mathcal{F}_\gamma$ has been defined, (Z is the γ^{th} requirement,) both $\mathcal{F}_\gamma \cup \{Z\}$ and $\mathcal{F}_\gamma \cup \{I-Z\}$ have the FIP, and the previous essential stages form a sequence of order type α. If $\alpha \geq \aleph_1$, let $\mathcal{F}_{\gamma+1}$ ~~where Y is the immediate successor of~~ be generated by $\mathcal{F}_\gamma \cup \{Z\}$, just as in B7. If, on the other hand, $\alpha < \aleph_1$, then let $\mathcal{F}_{\gamma+1}$ be generated by $\mathcal{F}_\gamma \cup \{Z\}$ or by $\mathcal{F}_\gamma \cup \{I-Z\}$ according to whether $f(\alpha)$ is 0 or 1. Exactly as in B7, this construction produces an ultrafilter $\mathcal{U}^f$.

To prove that all these ultrafilters are distinct, consider $f \neq g : \aleph_1 \to 2$, let α be the first ordinal with $f(\alpha) \neq g(\alpha)$, and let $\mathcal{F}^f_\gamma$ and $\mathcal{F}^g_\gamma$ be the filters obtained, by stage γ, in the constructions using f and g. In the construction using f, there must be an α^{th} essential stage, for otherwise there would be only countably many essential stages, Remark 88 would imply that $\mathcal{U}^f$ is countably generated (as $\mathcal{S}$ is countable), and Corollary 3 would imply that $\mathcal{U}^f$ is principal, which is absurd as $\mathcal{U}^f \supseteq \mathcal{S}$. Let γ (the γ^{th} requirement) be the α^{th} essential stage, i.e., let this stage be the adjunction of Z or $I-Z$ to $\mathcal{F}^f_\gamma$ to form $\mathcal{F}^f_{\gamma+1}$. Up to and including $\mathcal{F}^f_\gamma$, the construction using f is also the construction using g, for only the values $f(\beta)$ for $\beta < \alpha$ were used and these coincide with the values $g(\beta)$. So $\mathcal{F}^f_\gamma = \mathcal{F}^g_\gamma$ and stage γ is the α^{th} essential one for g as well as for f. Since $f(\alpha) \neq g(\alpha)$, one of the two constructions will adjoin Z at this stage, the other $I-Z$. Thus, of $\mathcal{U}^f$ and $\mathcal{U}^g$, one contains Z and the other contains $I-Z$. In particular, $\mathcal{U}^f \neq \mathcal{U}^g$. $\square$

Remark $4\frac{1}{2}$. Let $\underset{\sim}{c}$ denote the cardinality of the continuum, i.e. the cardinality $2^{\aleph_0}$ of $P(\omega)$. Since an ultrafilter on ω is a subset of $P(\omega)$, there are at most $2^{\underset{\sim}{c}}$ ultrafilters. If the continuum hypothesis, $\underset{\sim}{c} = \aleph_1$, holds, then by Proposition 4, there are exactly $2^{\underset{\sim}{c}}$ non-principal ultrafilters on ω. In fact, this is so even if $\underset{\sim}{c} \neq \aleph_1$; see below

~~Digression~~ Digression 5. A similar construction gives 2^{κ^+} uniform ultrafilters on a set of size κ, but a bit more work is involved. It is necessary to construct uniform ultrafilters, not merely non-principal ones, for only then will Theorem 2 and Remark 88 ensure that there are at least κ^+ essential stages. ~~It is thought~~ So it is natural to start with $\mathcal{S}$ being the family of subsets of I whose complements have cardinality $< \kappa$. Unfortunately, $\mathcal{S}$ may well have more than κ members, so the application of Theorem 2 and Remark 88 still doesn't work. One needs an improved version of Theorem 2 saying that no ultrafilter on a set of size κ can be generated by the family $\mathcal{S}$ ~~[illegible]~~ (as above) plus a family $\mathcal{S}'$ of κ or fewer other sets. This can be proved similarly to Theorem 2, choosing two elements x_B and y_B from each set B in the filterbase generated by $\mathcal{S}'$ (not $\mathcal{S} \cup \mathcal{S}'$), but coming back to each such B κ times in the course of the induction, so as to produce κ distinct x's and y's in each B. Then, if S is a finite intersection of sets from $\mathcal{S}$ (so in fact $S \in \mathcal{S}$) then $B \cap S$ still contains plenty of x's and y's, and one can proceed as in Theorem 2.

For regular κ, a simpler expedient is available. Well-order I with order-type κ

and let S consist of the final segments of this ordering. Then every ultrafilter that includes S is uniform (this would fail if κ were singular), yet S has cardinality only κ, so the proof of Proposition 4 still works.

The following definition introduces two of the so-called cardinal invariants of the continuum; see [ ] for more information about these. One of these two invariants will be used immediately, to strengthen Theorem 2. The other will not be needed until later but is so closely related to the first that it is appropriate to define them together.

<u>Definition 6</u>. A family $\mathfrak{X}$ of functions from ω to ω is <u>unbounded</u> (resp. <u>dominating</u>) if, for every $f:\omega\to\omega$ there exists $g\in\mathfrak{X}$ such that $f(n)\leq g(n)$ for infinitely many n (resp. for all but finitely many n). The <u>bounding number</u> $\mathfrak{b}$ (resp. <u>dominating number</u> $\mathfrak{d}$) is defined to be the smallest cardinality of any unbounded (resp. dominating) family.

<u>Remark 7</u>. The set of functions from ω to ω is pre-ordered by the relation of eventual dominance,

$$f \leq^* g \iff \text{For all but finitely many } n,\ f(n) \leq g(n).$$

A family of functions is dominating if and only if it is cofinal in this pre-ordering; it is unbounded if and only if it has no upper bound. The dominating number would be unchanged if, in the definition of dominating family, one deleted "but finitely many", since any dominating family in the old sense yields one in the new sense of the same cardinality simply by adjoining all functions that differ only finitely from those already present. In other words, the dominating number is also the smallest cardinality of a cofinal set with respect to the ordering

$$f \leq g \iff \text{For all } n,\ f(n) \leq g(n).$$

In contrast, the smallest cardinality of a set with no upper bound in $\leq$ is not the bounding number but $\aleph_0$. (Take the set of constant functions.)

The bounding and dominating numbers satisfy

$$\aleph_1 \leq \underset{\sim}{b} = cf(\underset{\sim}{b}) \leq cf(\underset{\sim}{d}) \leq \underset{\sim}{d} \leq 2^{\aleph_0}.$$

The last two inequalities here are trivial. The first inequality, which says that any countable family $\{f_n \mid n \in \omega\}$ of functions $\omega \to \omega$ is $\leq^*$-bounded, is proved by diagonalization. ~~Any function g such that $g(k)$~~ The function g defined by

$$g(k) = 1 + \max\{f_n(k) \mid n \leq k\}$$

eventually dominates each f_n. The remaining inequality (resp. the equation) are true because a dominating (resp. unbounded) family $\mathcal{X}$ cannot be the union of fewer than $\underset{\sim}{b}$ non-dominating (resp. bounded) families $\mathcal{X}_i$. Indeed, if f_i is, for each i, a witness to the failure of $\mathcal{X}_i$ to be dominating (resp. unbounded), then the fewer than $\underset{\sim}{b}$ functions f_i have an upper bound f with respect to $\leq^*$, and this f witnesses the same failure for $\mathcal{X}$.

Apart from the inequalities and equation in the preceding paragraph, there are no constraints on the behavior of $\underset{\sim}{b}$ and $\underset{\sim}{d}$ in models of set theory. In particular, it is consistent that $\underset{\sim}{b}$ be much larger than $\aleph_1$. Thus, the following theorem of Solomon [ ] genuinely improves upon Theorem 2, for ultrafilters on ω. Some of the concepts introduced in this proof will be needed again later and are therefore explicitly labeled ~~as definitions~~ as a definitions and a lemma for future reference.

<u>Theorem 8</u>. A family of fewer than $\underset{\sim}{b}$ sets cannot generate a non-principal ultrafilter on ω.

<u>Proof</u>. Suppose S were a counterexample. By Proposition 1, the cardinality of S is infinite, so the filterbase $\mathcal{B}$ that it generates has the same cardinality $< \underset{\sim}{b}$.

<u>Definition 9</u>. For any infinite $X \subseteq \omega$ and any $n \in \omega$, $next_X(n)$ is the smallest element of X that is $\geq n$.

By definition of $\underset{\sim}{b}$, the functions $next_B$ for $B \in \mathcal{B}$ do not form an unbounded family, so there is an f such that

for each $B \in \mathcal{B}$, for all but finitely many n, $f(n) > next_B(n)$.

By increasing f if necessary, arrange that f is strictly increasing and $f(0) > 0$.

<u>Definition 10</u>. For any strictly increasing $f : \omega \to \omega$ with $f(0) > 0$, the <u>blocks</u> associated to f are the intervals

$$[a_n, a_{n+1}) = \{x \in \omega \mid a_n \leq x < a_{n+1}\}$$

where $a_0 = 0$ and $a_{n+1} = f(a_n)$, i.e., $a_n = f^n(0)$.

<u>Lemma 11</u>. With the notation of the preceding definition, $a_n < a_{n+1}$ for all n, the ~~[illegible]~~ blocks are nonempty and constitute a partition of ω, and ~~[illegible] $[a_n, a_{n+1}]$~~ f maps each block $[a_n, a_{n+1})$ into the next one $[a_{n+1}, a_{n+2})$.

<u>Proof</u>. The inequality $a_n < a_{n+1}$ follows by induction on n from the assumed properties of f, and the second assertion is an immediate consequence of this inequality. Finally, since f is strictly increasing, if $a_n \leq x < a_{n+1}$ then $a_{n+1} = f(a_n) \leq f(x) < f(a_{n+1}) = a_{n+2}$. $\square$

For the f that arose in the proof of Theorem 8, the associated blocks have the following crucial property. If $B \in \mathcal{B}$, then B intersects all but finitely many of the blocks. Indeed, for each $B \in \mathcal{B}$, the choice of f ensures that, for all sufficiently large n,

$$a_n \leq \mathrm{next}_B(a_n) < f(a_n) = a_{n+1}.$$

~~One of the two complementary sets~~

This contradicts the fact that one of the two complementary sets $\bigcup_{n \text{ even}} [a_n, a_{n+1})$ and $\bigcup_{n \text{ odd}} [a_n, a_{n+1})$ is in the ultrafilter generated by $\mathcal{B}$ and therefore includes a set from $\mathcal{B}$. $\square$

In view of Theorem 8, the proof of Proposition 4 can be carried out with $\underset{\sim}{b}$ in place of $\aleph_1$. Thus, the number of ultrafilters on ω is at least $2^{\underset{\sim}{b}}$. ~~A stronger result is given~~ This is improved to $2^{\underset{\sim}{c}}$ in   below.