D. Quantifiers for filters

This section is devoted to the introduction and one application of a very convenient terminological device for dealing with filters and ultrafilters.

Definition 1. Let $\mathcal{F}$ be a filter on I, and let $\varphi(i)$ be a statement about a variable element $i \in I$. Then "for $\mathcal{F}$-almost all i, $\varphi(i)$", sometimes abbreviated $(\mathcal{F}\text{-}\forall i)\,\varphi(i)$, means $\{i \in I \mid \varphi(i)\} \in \mathcal{F}$. "For $\mathcal{F}$-frequent i, $\varphi(i)$" sometimes abbreviated $(\mathcal{F}\text{-}\exists i)\,\varphi(i)$, means $\{i \in I \mid \neg\varphi(i)\} \notin \mathcal{F}$ (where $\neg$ means negation).

Insert p. $1\frac{1}{2}$ here

Examples 2. (a) If $\mathcal{F}$ consists only of I, then "$\mathcal{F}$-almost all" means "all", and "$\mathcal{F}$-frequent" means at least one. Hence the notations $\mathcal{F}\text{-}\forall$, $\mathcal{F}\text{-}\exists$.

(b) If $\mathcal{F}$ is the cofinite filter, then "$\mathcal{F}$-almost all" means all but finitely many, and "$\mathcal{F}$-frequent" means infinitely many.

(c) If $\mathcal{F}$ is the filter of neighborhoods of a point p in a topological space X, then $(\mathcal{F}\text{-}\forall x)\,\varphi(x)$ means that $\varphi(x)$ holds for all x sufficiently near p, and $(\mathcal{F}\text{-}\exists x)\,\varphi(x)$ means that $\varphi(x)$ holds for some x's arbitrarily near p.

Insert on p. 1.

Technically, "let $\varphi(i)$ be a statement about a variable element $i \in I$" should be interpreted as "let $\varphi(i)$ be a formula and i a free variable ranging over I". It is not required that i actually occur in φ.

(d) If $\mathcal{F}$ is the principal ultrafilter concentrated at i_0, then $(\mathcal{F}\text{-}\forall i)\,\varphi(i)$ and $(\mathcal{F}\text{-}\exists i)\,\varphi(i)$ both mean $\varphi(i_0)$.

The following proposition summarizes the formal properties of these filter quantifier

<u>Proposition 3.</u> Let $\mathcal{F}$ be a filter on I, and let $\varphi(i)$, $\psi(i)$ be statements about $i \in I$. Insert p. $2\frac{1}{2}$

(c) $(\mathcal{F}\text{-}\exists i)\,\varphi(i) \iff \neg(\mathcal{F}\text{-}\forall i)\neg\varphi(i)$.

(d) $(\mathcal{F}\text{-}\forall i)\,\varphi(i) \iff \neg(\mathcal{F}\text{-}\exists i)\neg\varphi(i)$.

(e) $(\mathcal{F}\text{-}\forall i)\,\varphi(i) \iff (\exists A \in \mathcal{F})(\forall i \in A)\,\varphi(i)$.

(f) $(\mathcal{F}\text{-}\exists i)\,\varphi(i) \iff (\forall A \in \mathcal{F})(\exists i \in A)\,\varphi(i)$.

(g) $(\forall i)\varphi(i) \Rightarrow (\mathcal{F}\text{-}\forall i)\varphi(i) \Rightarrow (\mathcal{F}\text{-}\exists i)\,\varphi(i) \Rightarrow (\exists i)\,\varphi(i)$.

(h) If i does not occur in φ, then $(\mathcal{F}\text{-}\forall i)\varphi \iff (\mathcal{F}\text{-}\exists i)\varphi \iff \varphi$.

~~If $(\mathcal{F}\text{-}\forall i)(\varphi(i) \Rightarrow \psi(i))$.~~

(i) If $(\mathcal{F}\text{-}\forall i)(\varphi(i) \Rightarrow \psi(i))$, hence in particular if $(\forall i)(\varphi(i) \Rightarrow \psi(i))$, then $(\mathcal{F}\text{-}\forall i)\,\varphi(i) \Rightarrow (\mathcal{F}\text{-}\forall i)\,\psi(i)$ and $(\mathcal{F}\text{-}\exists i)\,\varphi(i) \Rightarrow (\mathcal{F}\text{-}\exists i)\,\psi(i)$.

Insert on p. 2 (not a new ¶)

Let $\varphi(j)$ be the result of substituting for i in $\varphi(i)$ a variable j that does not occur in $\varphi(i)$.

(a) $(\mathcal{F}\text{-}\forall i)\varphi(i) \Longleftrightarrow (\mathcal{F}\text{-}\forall j)\varphi(j)$

(b) $(\mathcal{F}\text{-}\exists i)\varphi(i) \Longleftrightarrow (\mathcal{F}\text{-}\exists j)\varphi(j)$

Return to p. 2

(j) $(\mathcal{F}\text{-}\forall i)(\varphi(i) \wedge \psi(i)) \iff (\mathcal{F}\text{-}\forall i)\varphi(i) \wedge (\mathcal{F}\text{-}\forall i)\psi(i)$.

(k) $(\mathcal{F}\text{-}\exists i)(\varphi(i) \vee \psi(i)) \iff (\mathcal{F}\text{-}\exists i)\varphi(i) \vee (\mathcal{F}\text{-}\exists i)\psi(i)$.

(l) If $(\mathcal{F}\text{-}\forall i)(\varphi(i) \iff \psi(i))$, hence in particular if $\forall i\,(\varphi(i) \iff \psi(i))$, then $(\mathcal{F}\text{-}\forall i)\varphi(i) \iff (\mathcal{F}\text{-}\forall i)\psi(i)$, and $(\mathcal{F}\text{-}\exists i)\varphi(i) \iff (\mathcal{F}\text{-}\exists i)\psi(i)$.

Proof. (a) and (b) are trivial because $\{i \in I \mid \varphi(i)\} = \{j \in I \mid \varphi(j)\}$ and similarly for $\neg\varphi$. (c) is immediate from the definitions, and (d) follows by substituting $\neg\varphi$ for φ and negating both sides. The $\Rightarrow$ half of (e) is trivial, while the $\Leftarrow$ half follows from clause (iii) in the definition (A8) of filters: If $\{i \mid \varphi(i)\}$ includes a set $A \in \mathcal{F}$, then it is in $\mathcal{F}$ also. (f) follows from (e) and (c), and can also be proved directly by observing that $\{i \mid \varphi(i)\}$ intersects every set in $\mathcal{F}$ if and only if its complement $\{i \mid \neg\varphi(i)\}$ is not in $\mathcal{F}$ (by clause (iii) of the definition again). The first and last implications in (g) amount to the statement that $I \in \mathcal{F}$, while the middle implication says that $\{i \mid \varphi(i)\}$ and its complement are not both in $\mathcal{F}$. For (h), notice that, since i does not occur in φ, $\{i \mid \varphi(i)\}$ is I or $\emptyset$ according to whether φ is true or false. Then (h) follows from $I \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$. ~~Both (j) and (k)~~

The first part of the $\Leftarrow$ half of (i) as well as (j) follow from $\mathcal{F}$ being closed under finite intersections. The $\Rightarrow$ half of (j) follows from $\mathcal{F}$ being closed under supersets. The other half of (i) follows from the half already proved, and (k) follows from (j), by way of (c), expressing $\mathcal{F}$-$\exists$ in terms of $\mathcal{F}$-$\forall$. Finally (l) follows from (i); since $\Leftrightarrow$ can be expressed in terms of $\Rightarrow$ and $\wedge$, and (j) allows the $(\forall\text{-}\mathcal{F}_i)$ in the hypothesis to be distributed over $\wedge$. The "hence in particular" in (i) and (l) follows from the first implication in (g). $\square$

Insert p. $4\frac{1}{2}$ here.

<u>Proposition 4.</u> With $\mathcal{F}$ and $\varphi(i)$ as before, suppose that $\mathcal{G}$ is another filter on I and $\mathcal{F} \subseteq \mathcal{G}$. Then

~~$(\forall\text{-}\mathcal{F}_i)\varphi(i) \Rightarrow (\forall\text{-}\mathcal{G}_i)\varphi(i)$, and $(\exists\text{-}\mathcal{G}_i)$~~

$$(\mathcal{F}\text{-}\forall_i)\varphi(i) \Rightarrow (\mathcal{G}\text{-}\forall_i)\varphi(i), \text{ and } (\mathcal{G}\text{-}\exists_i)\varphi(i) \Rightarrow (\mathcal{F}\text{-}\exists_i)\varphi(i).$$

Furthermore, either of these implications, for all φ, implies that $\mathcal{F} \subseteq \mathcal{G}$.

<u>Proof.</u> Immediate from the definitions. $\square$

Since an ultrafilter $\mathcal{U}$ contains a subset of I if and only if it does not contain its complement, the quantifiers $\mathcal{U}$-$\forall$ and $\mathcal{U}$-$\exists$ are equivalent, so the $\forall$ and $\exists$ can safely be omitted in this context. (This equivalence between the two sorts of quantifiers clearly characterizes ultrafilters among filters.)

Insert on p. 4

A filter $\mathcal{F}$ on I is completely determined if the quantifier $\mathcal{F}$-$\forall$ is known, for $\mathcal{F} = \{X \subseteq I \mid (\mathcal{F}\text{-}\forall_i)\ i \in X\}$. The quantifiers that ~~determine filters~~ arise as $\mathcal{F}$-$\forall$ for some filter on I are precisely those that satisfy the following parts of Proposition 3: (a), the first implication in (g), (j), and the implication in (l) from $\forall_i\,(\varphi(i) \Leftrightarrow \psi(i))$ to $(\mathcal{F}\text{-}\forall_i)\,\varphi(i) \Leftrightarrow (\mathcal{F}\text{-}\forall_i)\,\psi(i)$. (The first and last of these should probably be demanded of anything claiming the name "quantifier".)

It should be emphasized, to avert possible confusion, that, unlike the ordinary universal and existential quantifiers, filter quantifiers do not in general commute even if they are of the same type. For example, if $\mathcal{F}$ is the cofinite filter on ω (or any extension of that filter), then $(\mathcal{F}\text{-}\forall_i)(\mathcal{F}\text{-}\forall_j)\, i<j$ is true, but $(\mathcal{F}\text{-}\forall_j)(\mathcal{F}\text{-}\forall_i)\, i<j$ is false.

Definition 5. For $\mathcal{U}$ an ultrafilter on I and $\varphi(i)$ as above, "for $\mathcal{U}$-most i, $\varphi(i)$", sometimes abbreviated as $(\mathcal{U}i)\,\varphi(i)$, means $\{i \in I \mid \varphi(i)\} \in \mathcal{U}$, i.e., $(\mathcal{U}\text{-}\forall i)\,\varphi(i)$, or equivalently $(\mathcal{U}\text{-}\exists i)\,\varphi(i)$.

The following corollary is a transcription of Proposition 3 into this notation.

Corollary 6. Let $\mathcal{U}$ be an ultrafilter on I, and let $\varphi(i)$, $\psi(i)$, and $\varphi(j)$ be as in Proposition 3. Then

(a) $(\mathcal{U}i)\,\varphi(i) \iff (\mathcal{U}j)\,\varphi(j)$

(b) $(\mathcal{U}i)\,\varphi(i) \iff \neg(\mathcal{U}i)\neg\varphi(j)$; hence $\neg(\mathcal{U}j)\,\varphi(i) \iff (\mathcal{U}i)\neg\varphi(i)$

(c) $(\mathcal{U}i)\,\varphi(i) \iff (\exists A \in \mathcal{U})(\forall i \in A)\,\varphi(i) \iff (\forall A \in \mathcal{U})(\exists i \in A)\,\varphi(i)$.

(d) $(\forall i)\,\varphi(i) \Rightarrow (\mathcal{U}i)\,\varphi(i) \Rightarrow (\exists i)\,\varphi(i)$.

(e) If i does not occur in φ, then $(\mathcal{U}i)\,\varphi \iff \varphi$.

(f) If $(\mathcal{U}i)(\varphi(i) \Rightarrow \psi(i))$, hence in particular if $(\forall i)(\varphi(i) \Rightarrow \psi(i))$, then $(\mathcal{U}i)\,\varphi(i) \Rightarrow (\mathcal{U}i)\,\psi(i)$. Similarly with $\iff$ in place of $\Rightarrow$.

(g) $(\mathcal{U}i)(\varphi(i) \wedge \psi(i)) \iff (\mathcal{U}i)\varphi(i) \wedge (\mathcal{U}i)\psi(i)$.

(h) $(\mathcal{U}i)(\varphi(i) \vee \psi(i)) \iff (\mathcal{U}i)\varphi(i) \vee (\mathcal{U}i)\psi(i)$. $\square$

Corollary 7. For any ultrafilter $\mathcal{U}$, the quantifier $(\mathcal{U}i)$ commutes with all finitary propositional connectives.

Proof. For the connectives $\neg, \wedge,$ and $\vee$ this is the content of parts (b), (g), and (h) of the preceding corollary. It follows for all other finitary connectives, since these can be defined using $\neg, \wedge,$ and $\vee$. $\square$

Thus, for example, "$(\mathcal{U}i)$[Exactly ~~at least~~ two of $\varphi_1(i), \varphi_2(i), \varphi_3(i)$ hold]" is equivalent to "Exactly ~~At least~~ two of $(\mathcal{U}i)\varphi_1(i)$, $(\mathcal{U}i)\varphi_2(i)$, and $(\mathcal{U}i)\varphi_3(i)$ hold."

The next proposition is a useful strengthening of Corollary 6(d) in the case of non-principal ultrafilters on ω. An analog holds, with the same proof, for uniform ultrafilters on any cardinal

Proposition 8. Let $\mathcal{U}$ be a non-principal ultrafilter on ω and $\varphi(i)$ any statement about $i \in \omega$. Then for every $n \in \omega$,

$$(\forall i > n)\,\varphi(i) \Rightarrow (\mathcal{U}i)\,\varphi(i) \Rightarrow (\exists i > n)\,\varphi(i)$$

Proof. This just says that $\mathcal{U}$ contains all supersets of $\{i \in \omega \mid i > n\}$ and ~~does not contain~~ no subsets of $\{i \in \omega \mid i \leq n\}$, which is clear by A10. $\square$

Application 9

Somewhat surprisingly, these rather formal considerations can be used to obtain a non-trivial result (not involving ultrafilters), ~~a weak form of~~ Ramsey's theorem. The mathematical substance of the following proof is in the existence theorem 82 for non-principal ultrafilters, but the quantifiers introduced in this section make a valuable psychological contribution by ~~making~~ making otherwise unwieldy expressions amenable to easy manipulation. (The reader is invited to transcribe the proof without using the ultrafilter quantifiers; the value of the quantifiers should become evident.)

In Ramsey theory, one uses the notation $[X]^r$ for the collection of ~~subsets~~ subsets of X of cardinality r.

Ramsey's Theorem (weak form). Let $[\omega]^r$ be partitioned into finitely many sets $C_1, \ldots, C_k$, and let $n \in \omega$ with $n \geq r$ be given. Then there exists an n-element subset $X \in [\omega]^n$ such that $[X]^r$ lies in a single C_i.

Proof. Let $x_1, \ldots, x_n, y_1, \ldots, y_r$ be variables ranging over ω and let $\mathcal{U}$ be a non-principal ultrafilter on ω. The hypothesis of the theorem gives

$$\forall y_1 \, \forall y_2 > y_1 \ldots \forall y_r > y_{r-1} \; \bigvee_{t=1}^{k} \{y_1, \ldots, y_r\} \in C_t,$$

where $\bigvee_{i=1}^{k}$ means the disjunction ("or") of the statements following it. The restrictions in the quantifiers, that $y_1 < y_2 < \ldots < y_r$, are there to ensure that the y's are distinct so the $\{y_1, \ldots, y_r\} \in [\omega]^r$. By Proposition 8, it follows that

$$(\mathcal{U}y_1)(\mathcal{U}y_2) \cdots (\mathcal{U}y_r) \; \bigvee_{t=1}^{k} \{y_1, \ldots, y_r\} \in C_t,$$

which is equivalent, by Corollary 6(k), to

$$\bigvee_{t=1}^{k} (\mathcal{U}y_1)(\mathcal{U}y_2) \cdots (\mathcal{U}y_r) \; \{y_1, \ldots, y_r\} \in C_t.$$

For any choice of r elements $i_1 < \ldots i_r$ from $\{1, \ldots, n\}$, the formula

$$(\mathcal{U}y_1)(\mathcal{U}y_2) \cdots (\mathcal{U}y_r) \; \{y_1, \ldots, y_r\} \in C_i$$

can be transformed, by renaming each y_j as x_{i_j} and adding vacuous quantifiers on the remaining x variables, into

$$(\mathcal{U}x_1) \ldots (\mathcal{U}x_n) \; \{x_{i_1}, \ldots, x_{i_r}\} \in C_i,$$

and the result of this transformation is, by Corollary 6(a,c) equivalent to the original formula. But then the original formula is also equivalent to the conjunction of these (equivalent) transforms, over all choices of $i_1, \ldots, i_r$. Thus, the previously established fact that

$$\bigvee_{t=1}^{k} (\mathcal{U}y_1)(\mathcal{U}y_2)\cdots(\mathcal{U}y_r)\ \{y_1, \ldots, y_r\} \in C_t$$

is equivalent to

$$\bigvee_{t=1}^{k} \ \bigwedge_{1 \le i_1 < i_2 < \cdots < i_r \le n} (\mathcal{U}x_1)\cdots(\mathcal{U}x_n)\ \{x_{i_1}, \ldots, x_{i_r}\} \in C_t$$

and therefore, by Corollary 6(g), to

$$\bigvee_{t=1}^{k} (\mathcal{U}x_1)\cdots(\mathcal{U}x_n) \bigwedge_{1 \le i_1 < i_2 < \cdots < i_r \le n} \{x_{i_1}, \ldots, x_{i_r}\} \in C_t .$$

By Proposition 8, it follows that

$$\bigvee_{t=1}^{k} (\exists x_1)(\exists x_2 > x_1)\ldots(\exists x_n > x_{n-1}) \bigwedge_{1 \le i_1 < i_2 < \cdots < i_r \le n} \{x_{i_1}, \ldots, x_{i_r}\} \in C_t .$$

For the $t, x_1, x_2, \ldots, x_n$ whose existence is asserted here, all r-element subsets $\{x_{i_1}, \ldots, x_{i_r}\}$ of the n-element set $\{x_1, \ldots, x_n\}$ are in C_t. $\square$

A standard compactness argument allows one to weaken the hypothesis of this theorem as follows.

<u>Ramsey's Theorem</u> (finite form). For each r, k and $n \geq r$ (in ω), there exists $M \in \omega$ such that, if $[M]^r$ is partitioned into k sets $C_0, \ldots, C_{k-1}$, then there exists an n-element subset $X \in [M]^n$ such that $[X]^r$ lies in a single C_i.

<u>Proof</u>. Suppose not, and let r, k, n be a counterexample. For each $M \in \omega$, fix a partition of $[M]^r$ into k pieces such that no X of the desired sort exists. Let $f_M : [M]^r \to k$ be the function which is constant on each C_i with value i. The family $K = \{f_M \mid M \in \omega\}$ satisfies the hypotheses of Rado's selection principle (B10) with $A = [\omega]^r$ and $B = k$. So there exists $g : [\omega]^r \to k$ such that, for every finite $F \subseteq [\omega]^r$, g agrees on F with some f_M whose domain includes F. By the weak form of Ramsey's theorem, applied to the partition of $[\omega]^r$ into the pieces $g^{-1}\{i\}$ for $i < k$, there exists $X \in [\omega]^n$ such that $[X]^r$ is included in one of these pieces. Thus, g is constant on the finite set $[X]^r$ and agrees, on this set, with some $f_M \in K$. But this means that $[X]^r \subseteq [M]^r$ and f_M is constant on $[X]^r$. Thus, $X \subseteq M$, i.e., $X \in [M]^n$, and, by definition of f_M, $[X]^r$ lies in a

single piece of the ~~partition~~ chosen partition of $[M]^r$. The existence of X clearly contradicts the way the partition was chosen. $\square$

Instead of improving the weak form of Ramsey's theorem by reducing the hypothes one can strengthen the conclusion: The hypothesis of the weak form implies the existence of an infinite homogeneous set X. ~~[illegible]~~ The proof is another nice illustration of ultrafilter techniques; a different proof will be given in

<u>Ramsey's Theorem (infinite form)</u>. Let $[\omega]^r$ be partitioned into finitely many sets $C_1, \ldots, C_k$. There exists an infinite $X \subseteq \omega$ such that $[X]^r$ lies in a single C_i.

<u>Proof</u>. Let $\mathcal{U}$ be a non-principal ultrafilter on ω. As in the proof of the weak form of Ramsey's Theorem, there is an index t such that

$$(\mathcal{U}y_1)(\mathcal{U}y_2)\cdots(\mathcal{U}y_r)\ \{y_1, \ldots, y_r\} \in C_t.$$

Call a finite subset $\{x_1, \ldots, x_k\}$ of ω, with $k \leq r$, <u>good</u> if and only if.

$$(\mathcal{U}y_{k+1})\cdots(\mathcal{U}y_r)\ \{x_1, \ldots, x_k, y_{k+1}, \ldots, y_r\} \in C_t.$$

Thus, the empty set is good. An r-element set is good if and only if it is in C_t. Notice that if $\{x_1, \ldots, x_k\}$ is good ~~[illegible]~~ and $k < r$ then [illegible]

$(\mathcal{U}y_{k+1})(\{x_1,\dots,x_k,y_{k+1}\}$ is good). Call a finite subset of ω <u>very good</u> if all subsets are good. The preceding remarks imply:

(a) $\emptyset$ is very good.

(b) If a set Y of cardinality $\geq r$ is very good, then $[Y]^r \subseteq C_t$.

(c) If Y is very good, then $(\mathcal{U}y)(Y \cup \{y\}$ is very good).

(For (c) it is essential that the quantifier $(\mathcal{U}y)$ distributes over the finite conjunction, over the subsets of Y, in the definition of very good.) By (a) and (c), one can inductively define an infinite sequence $\{y_0, y_1, \dots\}$ in which every finite initial segment $\{y_0,\dots,y_n\}$ is very good. Then (b) implies that $[\{y_0,y_1,\dots\}]^r \subseteq C_t$, as desired. $\square$