

E. Images of filters

Definition 1. If \mathcal{F} is a filter on I and if $f: I \rightarrow J$, then

$$f(\mathcal{F}) = \{X \subseteq J \mid f^{-1}(X) \in \mathcal{F}\},$$

is called the image of \mathcal{F} under f , and f is said to map \mathcal{F} to $f(\mathcal{F})$

The following propositions give the basic properties of this construction.

Proposition 2. Let \mathcal{F} be a filter on I and let $f: I \rightarrow J$.

- (a) $f(\mathcal{F})$ is a filter on J .
- (b) If \mathcal{F} is an ultrafilter, then so is $f(\mathcal{F})$.
- (c) ~~\mathcal{F}~~ $\{f(A) \mid A \in \mathcal{F}\}$ is a basis for $f(\mathcal{F})$.
- (d) If f is surjective, then $\{f(A) \mid A \in \mathcal{F}\} = f(\mathcal{F})$.

Proof. (a) ~~$f(\mathcal{F})$ is nonempty because it contains J , as $f^{-1}(J) = I \in \mathcal{F}$.~~
 ~~$\emptyset \notin f(\mathcal{F})$ because $f^{-1}(\emptyset) = \emptyset$.~~

Proof. (a) follows from the fact that the operation $f^{-1}: \mathcal{P}(J) \rightarrow \mathcal{P}(I)$ preserves the ~~construction~~ ~~the~~ notions involved in the definition of "filter", namely \emptyset , \subseteq , and \cap . (b) follows similarly because f^{-1} also preserves complements. For (c), it

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(e) $f(\mathcal{F})$ is a principal ultrafilter if and only if f is constant on a set in \mathcal{F} .

Return to p. 1

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(e) is immediate from the definition and A10.

Return to p. 2

suffices to check that the family $\mathcal{B} = \{f(A) \mid A \in \mathcal{F}\}$ satisfies the condition in Proposition A12.5. Since $f^{-1}(f(A)) \supseteq A$ and \mathcal{F} is closed under supersets, $\mathcal{B} \subseteq f(\mathcal{F})$. If $X \in f(\mathcal{F})$ then $f^{-1}(X) \in \mathcal{F}$, so \mathcal{B} contains $f(f^{-1}(X))$, which is a subset of X . ~~For (d), it suffices, in view of (c), to~~ ^{Insert (B) from p. 12} If f is surjective, then $f(f^{-1}(X)) = X$, so (d) follows. \square

Proposition 3. Let \mathcal{F} be a filter on I , let $f: I \rightarrow J$, and let $\varphi(j)$ be a statement about elements j of J . Then

$$\begin{aligned}
 & \del{(f(\mathcal{F}) - \forall j) \varphi(j) \iff (J - \forall j)} \\
 & (f(\mathcal{F}) - \forall j) \varphi(j) \iff (J - \forall i) \varphi(f(i)) \\
 & (f(\mathcal{F}) - \exists j) \varphi(j) \iff (J - \exists i) \varphi(f(i)).
 \end{aligned}$$

Proof. $(f(\mathcal{F}) - \forall j) \varphi(j) \iff \{j \in J \mid \varphi(j)\} \in f(\mathcal{F})$
 $\iff \{i \in I \mid \varphi(f(i))\} = f^{-1}\{j \in J \mid \varphi(j)\} \in \mathcal{F}$

The second equivalence follows from the first if one substitutes for φ its negation and then applies Proposition D3(c). \square

Proposition 4. If $\mathcal{F} \subseteq \mathcal{G}$ are filters on I and $f: I \rightarrow J$, then $f(\mathcal{F}) \subseteq f(\mathcal{G})$. \square

Proposition 5(a) The identity function $\text{id}: I \rightarrow I$ ~~maps~~ maps each filter on I to its

(b) If $f: I \rightarrow J$ and $g: J \rightarrow K$ then, for all filters \mathcal{F} on I , $g(f(\mathcal{F})) = g \circ f(\mathcal{F})$

Proof. ~~These~~ These statements are trivial consequences of $\text{id}^{-1}(X) = X$

and $(g \circ f)^{-1}(X) = f^{-1}(g^{-1}(X))$. \square

Digression 6.

In the language of category theory, this proposition says that there is a functor $\text{Filters}: \text{Sets} \rightarrow \text{Sets}$, where $\text{Filters}(I)$ is the set of filters on I and $\text{Filters}(f)$ sends a filter \mathcal{F} on the domain of f to $f(\mathcal{F})$. Proposition 4 ~~lets~~ lets one view this functor as taking values in the category of partially ordered sets, and in fact one can lift it further, to the category of complete lattices.

Proposition 5 also suggests that filters can themselves be viewed as objects of a category, the morphisms from \mathcal{F} to \mathcal{G} being the functions ~~of~~ f such that $f(\mathcal{F}) = \mathcal{G}$. A ~~more~~ better category can be obtained by making two ~~operator~~ changes. First, let morphisms $\mathcal{F} \rightarrow \mathcal{G}$ be functions with $f(\mathcal{F}) \supseteq \mathcal{G}$. Second, identify two morphisms $\mathcal{F} \rightarrow \mathcal{G}$ if they agree on a set \mathcal{F} .

Although category-theoretic language will be ~~avoided in the main develop~~ confined to digressions such as this, it may help to motivate certain points in the main development. For example, the ~~same~~ identification at the end of the last paragraph leads to the definition of isomorphism ~~that follows~~ below. That identifica

Definition 7. Let \mathcal{F} and \mathcal{G} be filters on I and J respectively. A function $f: I \rightarrow J$ is an isomorphism from \mathcal{F} to \mathcal{G} if ~~if~~ $f(\mathcal{F}) = \mathcal{G}$ and for some $g: J \rightarrow I$

is made possible by the following proposition (which is not part of the digression).

Proposition 7. Let \mathcal{F} be a filter on I , and let $f, g: I \rightarrow J$. If $(\mathcal{F} - \forall i) f(i) = g(i)$, then $f(\mathcal{F}) = g(\mathcal{F})$.

Proof. If $X \in f(\mathcal{F})$, then $(\mathcal{F} - \forall i) f(i) \in X$. Combining this with the hypothesis $(\mathcal{F} - \forall i) f(i) = g(i)$ yields, by $\exists j, i$, $(\mathcal{F} - \forall i) g(i) \in X$, which means $X \in g(\mathcal{F})$. \square

The converse of this proposition is false. For example, every finite-to-one function f from an infinite set to itself maps the cofinite filter to itself.

Definition 8. An isomorphism from a filter \mathcal{F} on I to a filter \mathcal{G} on J is a function $f: I \rightarrow J$ such that $f(\mathcal{F}) = \mathcal{G}$ and, for some $g: J \rightarrow I$, called an inverse of f , $(\mathcal{F} - \forall i) g(f(i)) = i$ and $(\mathcal{G} - \forall j) f(g(j)) = j$. If there is an isomorphism from \mathcal{F} to \mathcal{G} , then \mathcal{F} and \mathcal{G} are said to be isomorphic, $\mathcal{F} \cong \mathcal{G}$.

The following proposition serves to justify the terminology.

Proposition 9. ~~Lemma~~ (a) If f is an isomorphism from \mathcal{F} to \mathcal{G} with inverse g , then g is an isomorphism from \mathcal{G} to \mathcal{F} .

(b) Isomorphism is an equivalence relation on filters. \mathcal{G}

(c) All inverses of the same isomorphism to \mathcal{G} agree at almost all arguments.

Proof. (a) The only thing to check is that $g(\mathcal{G}) = \mathcal{F}$, but, by Propositions 5 and 7, $g(\mathcal{G}) = g(f(\mathcal{F})) = g \circ f(\mathcal{F}) = \text{id}(\mathcal{F}) = \mathcal{F}$. For (b), use (a) to get symmetry and use Proposition 5 to get reflexivity (with id as the isomorphism) and transitivity (by composing isomorphisms). ~~Insert lemma~~ p. 5¹

That Definition 8 requires the equations $g \circ f = \text{id}$ and $f \circ g = \text{id}$ to hold almost everywhere (with respect to \mathcal{F} and \mathcal{G}), rather than everywhere, is a reflection of the idea, mentioned in connection with A15, that ~~also~~ whatever happens outside a set is

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For (c), suppose g and g' are inverses of f . So $(\mathcal{F}-\forall i) g(f(i)) = i = g'(f(i))$. By

Proposition 3, since $f(\mathcal{F}) = \mathcal{G}$, $(\mathcal{G}-\forall j) g(j) = g'(j)$. \square

Return to p. 5.

the filter under consideration is negligible. ~~But~~ The last part of the following theorem allows us, in many cases, to replace "almost everywhere" by "everywhere" in the definition of isomorphism.

Proposition 10. Let \mathcal{F} and \mathcal{G} be filters on I and J , respectively, and let $f: I \rightarrow J$ with $f(\mathcal{F}) = \mathcal{G}$. Then f is an isomorphism from \mathcal{F} to \mathcal{G} if and only if f is one-to-one on some ~~set~~ set in \mathcal{F} . Assume further that I and J have the same cardinality κ and that \mathcal{F} and \mathcal{G} contain sets whose complements (in I and J , respectively) have cardinality κ . Then f is an isomorphism from \mathcal{F} to \mathcal{G} if and only if there is a bijection $f': I \rightarrow J$ such that $(\mathcal{F} - \forall i) f(i) = f'(i)$

Proof. If f is an isomorphism with inverse g , then on some set in \mathcal{F} the equation $g(f(i)) = i$ holds; clearly, f is one-to-one on that set. Conversely, suppose f is one-to-one on $A \in \mathcal{F}$. ~~Define g on $f(A)$ to be the inverse of this bijection, and extend g arbitrarily to a function from J to I .~~ So f restricts to a bijection from A to $f(A)$. Define g on $f(A)$ to be the inverse of this bijection, and extend g arbitrarily to a function from J to I . Then $g(f(i)) = i$ on $A \in \mathcal{F}$ and $f(g(j)) = j$ on $f(A) \in \mathcal{G}$, so f is an isomorphism from \mathcal{F} to \mathcal{G} , with inverse g . This completes the proof of the

first half of the proposition, and the "if" direction of the second half immediately follows. For the "only if" direction, fix $A_1, A_2 \in \mathcal{F}$ and $B \in \mathcal{G}$ such that f is one-to-one on A_1 , and such that $I - A_2$ and $J - B$ have cardinality κ . Then the set $A = A_1 \cap A_2 \cap f^{-1}(B)$ is in \mathcal{F} , f is one-to-one on it, and $I - A$ and $J - f(A)$ have cardinality κ . ~~Combining the bijection $f: A \rightarrow f(A)$ with~~ The restriction of f to A is a bijection $A \rightarrow f(A)$ which, when combined with an arbitrary bijection $I - A \rightarrow J - f(A)$, yields the desired f' . \square

Corollary II. ~~If f is an isomorphism between two ultrafilters on sets of the same infinite cardinality~~

Corollary II. Let \mathcal{U}, \mathcal{V} be ultrafilters on ~~infinite~~ sets I, J of the same cardinality, and let f be an isomorphism from \mathcal{U} to \mathcal{V} . Then there is a bijection $f': I \rightarrow J$ with $(\forall i) f'(i) = f(i)$.

Proof. By the ~~previous~~ proposition, it suffices to find sets in \mathcal{U} and \mathcal{V} whose complements have the same cardinality κ as I and J . ~~If~~ I is infinite, it can be split into two subsets of cardinality κ . As \mathcal{U} is an ultrafilter, one of these sets

is in U and therefore serves as the desired set. The argument for V is the same. \square

If, on the other hand, K is finite, then U and V are principal (by A10), ~~and~~

concentrated at i and j , say. Then any bijection $f: I \rightarrow J$ sending i to j will serve. \square
 insert p 8 1/2 & then see notes for how to rearrange.

The following ~~Proposition~~ ^{Theorem} is a partial converse to Proposition 7. Notice that it is

quite ~~weak~~ weak in that it applies only to ultrafilters and ~~only~~ only when, in the

notation of Proposition 7, one of f and g is the identity. The latter restriction will be

weakened in Corollary 13, but it cannot be entirely removed.

~~insert prob 12~~

Theorem 12. Let \mathcal{U} be an ultrafilter on I and let $f: I \rightarrow I$ map

\mathcal{U} to itself. Then $(\mathcal{U}_i) f(i) = i$.

Proof. The proof is based on the following combinatorial fact. I can be partitioned into four pieces ~~A_0, A_1, A_2, A_3~~ A_0, A_1, A_2 , and A_3 such that $A_0 = \{i \in I \mid f(i) = i\}$ and each of the other three A_i satisfies $f^{-1}(A_i) \cap A_i = \emptyset$ (which is equivalent to $f(A_i) \cap A_i = \emptyset$). Once this fact is established, the theorem follows, because, for $i \neq 0$, $f^{-1}(A_i)$ and A_i cannot both be in \mathcal{U} (as they are disjoint), i.e., A_i cannot be in both $f(\mathcal{U})$ and \mathcal{U} , i.e., A_i cannot be in \mathcal{U} (as $f(\mathcal{U}) = \mathcal{U}$). So $A_0 \in \mathcal{U}$, which is the

Insert on p. 8.

8 $\frac{1}{2}$

The next proposition shows that, up to isomorphism, any function can be taken to be a projection. Although it is stated for ω , there are analogs for arbitrary sets.

Proposition 11 $\frac{1}{2}$. Let \mathcal{F} and \mathcal{G} be filters and $f: \omega \rightarrow \omega$ a function with $f(\mathcal{F}) = \mathcal{G}$. Then there exists a filter \mathcal{F}' on $\omega \times \omega$ and an isomorphism h from \mathcal{F} to \mathcal{F}' such that $1^{\text{st}}(\mathcal{F}') = \mathcal{G}$ and $1^{\text{st}}(h(x)) = f(x)$ for all x .

Proof. Define h by setting $1^{\text{st}}(h(x)) = f(x)$ and $2^{\text{nd}}(h(x)) =$ the number of $y < x$ such that $f(y) = f(x)$. ~~(Thus, $h(x) = (p, q)$ if x is the q^{th} element of $f^{-1}\{p\}$.)~~ (Thus, $h(x) = (p, q)$ if x is the q^{th} element of $f^{-1}\{p\}$.) Then define \mathcal{F}' to be $h(\mathcal{F})$. As h is one-to-one, it is an isomorphism from \mathcal{F} to \mathcal{F}' by Proposition 10. Finally, $1^{\text{st}}(\mathcal{F}') = 1^{\text{st}}(h(\mathcal{F})) = f(\mathcal{F}) = \mathcal{G}$. \square

conclusion of the theorem.

It remains to prove the combinatorial fact. As the proof is ~~not~~ ^{bracketed} much easier to see than to write, it will be interspersed with comments intended to help the reader visualize the construction of the A_i .

The iterations f^n of f are defined by $f^0(i) = i$ and $f^{n+1}(i) = f(f^n(i))$. It follows by induction that $f^{m+n}(i) = f^m(f^n(i))$. Define an equivalence relation on I by

$$i \sim j \iff (\exists m, n) f^m(i) = f^n(j).$$

Notice that, once an equation $f^m(i) = f^n(j)$ is known to be true, either exponent can be increased at will, provided the other is increased by the same amount. This implies that \sim is transitive; reflexivity and symmetry are obvious. [Consider the directed graph, with I as its set of vertices and with an edge from i to $f(i)$ for each i . The equivalence classes with respect to \sim are the components of this graph. Each component is either a cycle with a (possibly degenerate) tree rooted at each of its points (edges in the tree being directed toward the cycle) or a rootless tree ~~It will be shown that~~



In the latter case, starting at any point in the component there is an infinite path, and the rest of the component can be viewed as consisting of trees rooted at the points of the path.]

Call an element $i \in I$ cyclic if $f^m(i) = i$ for some $m \geq 1$; the smallest such m is called the order of i . Thus, A_0 will consist of all the cyclic points of order 1. Notice that, if $i \sim j$ and i is cyclic, then $f^k(j) = i$ for some k ; indeed, in the equation $f^m(i) = f^m(j)$, the exponents can be increased so that m becomes a multiple of the order of i .

By the axiom of choice, select one representative element from each equivalence class in such a way that, if an equivalence class contains any cyclic points, then its representative is cyclic. (The axiom of choice is not needed if I is ω or some other well-ordered set.) For each $i \in I$, let i^* be the representative of the equivalence class of i . Notice that $f(i)^* = i^*$ as $f(i) \sim i$.

Since $i \sim i^*$, let $m(i)$ be the smallest m such that, for some n , $f^m(i) = f^n(i^*)$, and let $n(i)$ be the smallest n such that $f^{m(i)}(i) = f^n(i^*)$.

Set $l(i) = m(i) + n(i)$.

Not a new paragraph

~~the~~ component of i in the

[View the graph as consisting of rooted trees ~~whose~~ whose roots lie on the cycle or the infinite path ~~starting~~ starting at i^* . Then $m(i)$ is the distance from i to the root of the tree in which it lies, and $n(i)$ is the distance backward ~~along~~ along the path or cycle from this root to i^* . ~~The rest of the proof is based on considering~~

Thus, $l(i)$ is the distance from i to i^* subject to the constraint that, after getting to the root of i 's tree, one must go backward ~~to~~ to i^* , even if i^* could be reached sooner by going forward, around a cycle.] ~~The rest of the proof is based on~~ a comparison between $l(i)$ and $l(f(i))$.

Case 1. $m(i) \geq 1$. [i is not on the ~~path or cycle~~ cycle or path starting from i^* . So $f(i)$ is one step closer to i^* .] ~~Then $m(f(i)) = m(i) - 1$~~ Then $m(f(i)) = m(i) - 1$ and $n(f(i)) = n(i)$, so $l(f(i)) = l(i) - 1$.

Case 2. $m(i) = 0$ and ~~the~~ $f(i) \neq i^*$. [i is on the cycle or path starting from i^* and is not the immediate predecessor of i^* or a cycle. ~~Then~~ So $f(i)$ is one step farther from i^* along this path or cycle.] Then $m(f(i)) = 0$.

also and $n(f(i)) = n(f(i)) + 1$, so $l(f(i)) = l(i) + 1$.

Case 3. $m(i) = 0$ and $f(i) = i^*$. ~~The~~ The former means that $i = f^{n(i)}(i^*)$, so the latter means that i^* is cyclic of order $n(i) + 1$ (not less, for then $n(i)$ wouldn't be minimal). [So i is the immediate predecessor of i^* on a cycle.] Then $m(f(i)) = 0$ also and $n(f(i)) = 0$, so $l(f(i)) = 0$.

The parity of $l(f(i))$ is different from that of $l(i)$ in all cases except Case 3 with the order of i^* odd. Set

$$A_0 = \{i \in I \mid f(i) = i\}$$

$$A_1 = \{i \in I \mid i \neq i^* \text{ and } l(i) \text{ is odd}\}$$

$$A_2 = \{i \in I \mid i \neq i^* \text{ and } l(i) \text{ is even}\}$$

$$A_3 = \{i^* \mid i \in I \text{ and } f(i^*) \neq i^*\}.$$

This is clearly a partition of I . Since ~~the~~ $l(f(i))$ and $l(i)$ have opposite parity except in some cases where $f(i) = i^*$, it follows that f maps ~~each of~~ each of A_1, A_2, A_3 into the union of the other two and A_0 . \square

Corollary 13. If $f(U) = g(U)$ and g is one-to-one on a set in U , then
 $(U_i) f(i) = g(i)$.

Proof. By Proposition 10, g is an isomorphism from U to $g(U)$. Let h be its inverse. Then $h(f(U)) = h(g(U)) = U$. Proposition 12, applied to the composite function hf , shows that $(U_i) h(f(i)) = i$. ^{so $(U_i) ghf(i) = g(i)$.} ~~But also $(g(U)_j) g(h(j)) = j$ as~~
 But also, as h is an inverse of g , $(g(U)_j) g(h(j)) = j$, i.e., $(f(U)_j) g(h(j)) = j$, i.e.,
 $(U_i) ghf(i) = f(i)$. Together with the preceding result, this means $(U_i) f(i) = g(i)$. \square

Example 14. The "one-to-one" hypothesis cannot be removed from Corollary 13.

Let \mathcal{V} be any non-principal ultrafilter on ω . ~~The sets $X \times Y \subseteq \omega^2$, with $X, Y \in \mathcal{V}$,~~

~~form a family with the~~ The family

$$\mathcal{S} = \{X \times Y \subseteq \omega \times \omega \mid X, Y \in \mathcal{V}\}$$

~~has the finite intersection~~

Let D be the diagonal $\{(x, x) \mid x \in \omega\}$ in ω^2 . The family

$$\mathcal{S} = \{(X \times Y) - D \subseteq \omega^2 \mid X, Y \in \mathcal{V}\}$$

has the finite intersection property; indeed, it is closed under finite intersection, and each

$(X+Y)-D$ in \mathcal{S} is infinite because X and Y are. Let \mathcal{U} be any ultrafilter that includes \mathcal{S} . Then the two projection maps $pr_1, pr_2: \omega^2 \rightarrow \omega$ both map \mathcal{U} to \mathcal{V} , for if $X \in \mathcal{V}$ then $pr_1^{-1}(X) = X \times \omega$ and $pr_2^{-1}(X) = \omega \times X$ are supersets of sets in \mathcal{S} . But (2i) $pr_1(i) \neq pr_2(i)$, since $\{i \mid pr_1(i) \neq pr_2(i)\} = \omega^2 - D \in \mathcal{S} \subseteq \mathcal{U}$.

Definition 15. For ~~any~~ ultrafilters \mathcal{U} and \mathcal{V} , let $\mathcal{U} \leq \mathcal{V}$ mean that $\mathcal{U} = f(\mathcal{V})$ for some function f . This binary relation on filters is called the Rudin-Kisler ordering or the RK-ordering.

Remark 16. (a) Proposition 5 implies that the RK-ordering is reflexive and transitive. It is not anti-symmetric, for isomorphic ultrafilters are \leq each other. So "pre-ordering" might be better terminology than "ordering". ~~The~~ Proposition 17 below shows that antisymmetry holds if isomorphic ultrafilters are identified.

(b) The ~~an~~ appropriate analog of the RK-ordering for ~~any~~ filters that are not necessarily ultrafilters is ~~given~~ given by: $\mathcal{F} \leq \mathcal{G}$ if $\mathcal{F} \subseteq f(\mathcal{G})$ for some f . This is sometimes called the Katětov ordering. Of course, if \mathcal{F} is an ultrafilter then $\mathcal{F} \subseteq f(\mathcal{G})$ implies $\mathcal{F} = f(\mathcal{G})$.

(c) Several other (pre-)orderings of ultrafilters have been introduced, and to distinguish the RK-ordering from these it is sometimes written \leq_{RK} .

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Proposition 17. ~~Let U and V be ultrafilters.~~ Let U and V be ultrafilters.

(a) If $f(U) = V$ and $g(V) = U$, then f and g are inverse isomorphisms.

(b) ~~$U \cong V$ if and only if $U \leq V \leq U$.~~ $U \cong V$ if and only if $U \leq V \leq U$.
(and Proposition 9(a))

Proof. (b) follows trivially from (a). So assume that U, V, f , and g are as in (a). Then, by Proposition 5, $U = gf(U)$. By Theorem 12, $(U_i) gf(i) = i$. Symmetrically, $(V_j) fg(j) = j$. This means that f and g are inverse isomorphisms. \square

Corollary 18. The RK-ordering induces a partial ordering of the isomorphism classes of ultrafilters.

This partial ordering is also called the RK-ordering and written \leq . All principal ultrafilters are isomorphic ~~and thus isomorphic~~ to each other and to no non-principal ultrafilters. The isomorphism class of principal ultrafilters ~~is at the bottom~~ is the smallest element of the RK-ordering, since constant functions map all ultrafilters to principal ultrafilters (Proposition 2(e)).

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15 $\frac{1}{2}$

(d) There are several reasons for the convention, ~~implicit~~ in the definition of the RK-order, that $f(V)$ is considered smaller than V . One is that, in the category of filters ~~defined~~ in Digression 6 above, every morphism is an epimorphism; that is, if (V_i) $gf(i) = hf(i)$ then $(f(V)_j) g(j) = h(j)$ (by Proposition 3). Thus, $f(V)$ is a quotient of V and thus smaller than V . ~~Further~~ Other justifications for this convention are given by Proposition 19 and

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Proposition 19. Consider the two functions that assign to any ultrafilter \mathcal{U} (a) the smallest cardinality of any set in \mathcal{U} and (b) the smallest cardinality ~~of a \mathcal{U} -basis~~ of any generating set \mathcal{B} for \mathcal{U} . Both of these functions are non-decreasing with respect to the RK-ordering.

Proof. Suppose that $\mathcal{U} \leq \mathcal{V}$, and let f be such that $\mathcal{U} = f(\mathcal{V})$. Then, for any $A \in \mathcal{V}$, the set $f(A) \in \mathcal{U}$ has the same or smaller cardinality. ~~⊙~~ This proves monotonicity for (a). For (b), note first that "generating set" can be replaced by "basis", thanks to Proposition C 1 $\frac{1}{2}$. If \mathcal{B} is a basis for \mathcal{V} , then $\mathcal{B}' = \{f(B) \mid B \in \mathcal{B}\}$ is a basis for \mathcal{U} . Indeed, $\mathcal{B}' \subseteq \mathcal{U}$ by Proposition 2(c), and if $X \in \mathcal{U}$ then $f^{-1}(X) \in \mathcal{V}$, so, for some $B \in \mathcal{B}$, $f^{-1}(X) \supseteq B$ and therefore $X \supseteq f(B)$. ~~⊙~~ As \mathcal{B}' has the same cardinality as \mathcal{B} or smaller, this proves monotonicity for (b). \square

The following definition ~~⊙~~ will be used here to refine Theorem C8; it will have ~~more uses~~ more uses later.

Definition 20. A filter \mathcal{F} on ω is feeble if there is a finite-to-one function ~~such that~~ $f: \omega \rightarrow \omega$ such that $f(\mathcal{F})$ consists of cofinite sets.

Corollary 21. No ultrafilter is feeble.

Proof. If \mathcal{U} is an ultrafilter, then, by Proposition 2(b), so is $f(\mathcal{U})$ for any f , finite-to-one or not. ~~But then $f(\mathcal{U})$ cannot consist only of cofinite sets.~~ \square

But then, if we partition ω into two infinite pieces, one of the pieces is in $f(\mathcal{U})$ and is not cofinite. \square

Recall Definition C6 of the bounding number \mathfrak{b} , the smallest cardinality of any unbounded family of functions from ω to ω .

~~Theorem 22. Every filter generated by fewer than \mathfrak{b} sets is feeble~~

~~Proof. Suppose \mathcal{F} were a counterexample.~~

Theorem 22. Let \mathcal{F} be a filter on ω that contains no finite sets and is generated by fewer than \mathfrak{b} sets. Then \mathcal{F} is feeble.

Proof. Such a filter has, by Proposition C1 $\frac{1}{2}$, a base \mathcal{B} of ~~sets~~ ^{cardinality $< \mathfrak{b}$} , and ~~every~~ every set in \mathcal{B} is, by assumption, infinite. Proceed as in the proof of Theorem C8 to obtain a partition of ω into ~~blocks~~ blocks $[a_n, a_{n+1})$ such that, if $B \in \mathcal{B}$, then B intersects all but finitely many of these blocks. ~~Let g~~ Define $g: \omega \rightarrow \omega$

by $g(x) =$ the unique n such that $x \in [a_n, a_{n+1})$. Then g is finite-to-one and $g(B)$ is cofinite for every $B \in \mathcal{B}$. Every set in $g(\mathcal{F})$ has a subset $g(B)$ with $B \in \mathcal{B}$ and hence is cofinite. \square