F. Selective ultrafilters

This section is about the ultrafilters that are minimal, with respect to the RK-ordering, among non-principal ultrafilters on ω. ~~The construction of such ultrafilters;~~ ~~by~~ Such ultrafilters are constructed by a technique similar to that in B7, but an additional set-theoretic hypothesis is needed to make the construction work. This construction serves as a paradigm for many others in later chapters.

Proposition 1. Let $\mathcal{U}$ be a non-principal ultrafilter on I. The following are equivalent.

(a) Every function defined on I is either one-to-one on a set in $\mathcal{U}$ or constant on a set in $\mathcal{U}$.

(b) If $\mathcal{V} \leq \mathcal{U}$ then either $\mathcal{V} \cong \mathcal{U}$ or $\mathcal{V}$ is principal.

(c) The isomorphism class of $\mathcal{U}$ is RK-minimal among all isomorphism classes of non-principal ultrafilters.

(d) If I is partitioned into subsets, none of which are in $\mathcal{U}$, then $\mathcal{U}$ contains a set that has at most (or exactly) one member in common with each piece of the partition.

(e) For every function f defined on I, there exist a natural number n and a set $A \in \mathcal{U}$ such that f either takes at most n values on A or is at-most-n-to-one on A.

<u>Proof</u>. The "at most" and "exactly" versions of (d) are equivalent, since $\mathcal{U}$ is closed under supersets. ~~The equivalence between (a) and (d) arises from the observation~~ ~~For (a) $\Rightarrow$ (d), use the partition into~~ For (a) $\Rightarrow$ (d), given a partition of I, apply (a) to a function f that is constant on each piece of the partition and has different values on different pieces. As no piece is in $\mathcal{U}$, f is not constant on any set in $\mathcal{U}$, so by (a) it is ~~constant~~ one-to-one on some $A \in \mathcal{U}$. Then A contains at most one element from each piece. The converse, (d) $\Rightarrow$ (a), is equally easy, using the partition consisting of the fibers $f^{-1}\{x\}$ of the given function. Clearly (a) $\Rightarrow$ (e). For the converse, notice that if f takes at most n values (resp. is at-most-n-to-one) on $A \in \mathcal{U}$, then A can be partitioned into n sets on each of which f is constant (resp. one-to-one), and one of these n sets is in $\mathcal{U}$ by Corollary A5.

If f is one-to-one (resp. constant) on a set in $\mathcal{U}$, then $f(\mathcal{U})$ is isomorphic to $\mathcal{U}$ (resp. principal) by E10 (resp. E$1\frac{1}{2}$), so (a) $\Rightarrow$ (b). For (b) $\Rightarrow$ (a) notice first that if $f(\mathcal{U}) \cong \mathcal{U}$ then f is an isomorphism, by E13, and then apply the converse parts of E10 and E$1\frac{1}{2}$. Finally, (c) is just a translation of (b) into the language of the RK-ordering. $\square$

Definition 2. A non-principal ultrafilter on ω satisfying the equivalent conditions of Proposition 1 is called selective.

Remark 3. The word "selective" refers to Proposition 1(d), which says that $\mathcal{U}$ contains a selector for every partition of ω into small sets. Commonly used synonyms for "selective" are "RK-minimal" (or simply "minimal"), referring to Proposition 1(c), and "Ramsey", referring to below. Insert page $3\frac{1}{2}$ here (not a new paragraph).

Digression 4. The restriction, in Definition 2, to ultrafilters on ω raises the question of whether analogous sorts of ultrafilters exist on larger sets. Only uniform ultrafilters are of interest, since others are isomorphic to ultrafilters on smaller sets. It is not hard to check that any uniform ultrafilter on κ that satisfies the conditions in Proposition 1 must be $<\kappa$-complete; just apply 1(d) to a partition into fewer than κ pieces. Thus, if κ is uncountable, it must be a measurable cardinal.

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In French, some authors use "ultrafiltre absolu", but the English "absolute ultrafilter" has been used with a different meaning.

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It can be shown that, conversely, on every measurable cardinal there exist such ultrafilters (an[d] these may or may not all be isomorphic.)

For non-measurable cardinals, one can define analogs of selective ultrafilters by weaken[ing] the requirements in Proposition 1, for example, by changing 1(a) to say that every function on I either is one-to-one on a set in $\mathcal{U}$ or takes fewer than κ values on a set in $\mathcal{U}$. (When $\kappa = \omega$ this change would make no difference, by 1(e).) Such ultrafilters are RK-minimal among the uniform ultrafilters on κ. An intermediate concept would have "countably many" in place of "fewer than κ"; such ultrafilters are called indecomposable.

~~To prove the existence of selective ultrafilters, it is necessary to make a set-theoretic assumption that goes beyond the usual axioms of set theory, the Zermelo F~~

The usual axioms of set-theory, the Zermelo-Fraenkel axioms including the axiom of choice (ZFC), do not imply the existence of selective ultrafilters, but such ultrafilters can be constructed if suitable (consistent) combinatorial assumptions are made. The next definition ~~gives two such hypotheses~~ gives two such hypotheses, CH and $P(\mathfrak{c})$.

Definition 5. (a) The continuum hypothesis, CH, is the assertion that the cardinality $\underset{\sim}{c} = 2^{\aleph_0}$ of the continuum equals $\aleph_1$.

(b) A is almost included in B, written A almost $\subseteq B$, if $A - B$ is finite.

(c) $P(\underset{\sim}{c})$ is the assertion that, if a family $\mathcal{S}$ of fewer than $\underset{\sim}{c}$ subsets of ω containing all the cofinite sets and has the finite intersection property, then there is an infinite $A \subseteq \omega$ almost included in every $S \in \mathcal{S}$.

~~Remark 6. In part (b), the requirement that $\mathcal{S}$ contains all the cofin~~

Remark 6 (a) In the definition of $P(\underset{\sim}{c})$, the hypotheses on $\mathcal{S}$ imply that the intersection of each finite subfamily of $\mathcal{S}$ is infinite. This property of $\mathcal{S}$ can be used in place of those hypotheses, for it implies that $\mathcal{S} \cup \{\text{cofinite sets}\}$ satisfies the stated hypotheses.

(b) For cardinals $\kappa < \underset{\sim}{c}$, $P(\kappa)$ can be defined analogously to $P(\underset{\sim}{c})$; just require $\mathcal{S}$ to have cardinality smaller than κ. In these terms, the proof of the next proposition shows that $P(\aleph_1)$ holds. The ~~smallest~~ largest κ such that $P(\kappa)$ holds is denoted by $\underset{\sim}{p}$; it is another of the cardinal invariants of the continuum, and it ~~Proposition. CH imp~~ satisfies $\aleph_1 \le \underset{\sim}{p} \le \underset{\sim}{c}$.

(c) CH is consistent relative to ZFC; that is, if ZFC is consistent then so is ZFC + CH. See . By the next proposition, it follows that $P(\underset{\sim}{c})$ is also consistent relative to ZFC.

Proposition 7. CH implies $P(\underset{\sim}{c})$.

Proof. Assume CH, and let $\mathcal{S}$ be as in the definition of $P(\underset{\sim}{c})$. As the cardinality of $\mathcal{S}$ is smaller than $\underset{\sim}{c}$, CH implies that it is countable. Let $\mathcal{S}$ be enumerated in an ω-sequence $S_0, S_1, \ldots$. Inductively choose a_n to be an element of $S_0 \cap \ldots \cap S_n - \{a_0, \ldots, a_{n-1}\}$; this can be done because $S_0 \cap \ldots \cap S_n$ is infinite. Let $A = \{a_n \mid n \in \omega\}$. If $x \in A - S_k$, then $x = a_n$ for some $n < k$, so $A - S_k$ is finite for each $S_k \in \mathcal{S}$. $\square$

Theorem 8. $P(\underset{\sim}{c})$ implies the existence of a selective ultrafilter.

Proof. Selectivity means that, for each $f: \omega \to \omega$, the ultrafilter $\mathcal{U}$ must satisfy

Requirement f: f is one-to-one or constant on some set in $\mathcal{U}$.

The construction of such an ultrafilter will be like Construction B7, with these new requirements added to those in B7, which had the form

Requirement X: $\mathcal{U}$ contains X or $\omega - X$.

for $X \subseteq \omega$. The number of f's and the number of X's are both $\mathfrak{c}$, so let the requirements be ~~well-ordered with~~ enumerated in order-type $\mathfrak{c}$. Thus, each requirement has fewer than $\mathfrak{c}$ predecessors. As in B7, the construction of the desired ultrafilter proceeds by inductively defining filters $\mathcal{F}_\alpha$ on ω for $\alpha \leq \mathfrak{c}$ such that:

(a) If $\alpha \leq \beta$ then $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$.

(b) $\mathcal{F}_0$ is the cofinite filter.

(c) If β is a limit ordinal, then $\mathcal{F}_\beta = \bigcup_{\alpha<\beta} \mathcal{F}_\alpha$.

(d) If $\alpha < \beta$ and the α^{th} requirement is $X \subseteq \omega$, then either $X \in \mathcal{F}_\beta$ or $\omega - X \in \mathcal{F}_\beta$.

In addition, the construction will handle the new requirements:

(e) If $\alpha < \beta$ and the α^{th} requirement is $f : \omega \to \omega$, then $\mathcal{F}_\beta$ contains a set on which f is one-to-one or constant.

Just as in B7, clauses (a), (d), and (e) follow from the special cases, referred to as (a′), (d′), and (e′), where $\beta = \alpha + 1$. The inductive definition of $\mathcal{F}_\beta$ is exactly as in B7, except for the case that $\beta = \alpha+1$ and the α^{th} requirement is a function $f : \omega \to \omega$. In this case, which had no analog in

B7, clause (e) demands that a set $A \subseteq \omega$ on which f is one-to-one or constant must be adjoined to $\mathcal{F}_\alpha$ in forming $\mathcal{F}_{\alpha+1}$; $\mathcal{F}_{\alpha+1}$ will be the filter generated by $\mathcal{F}_\alpha \cup \{A\}$. (For this definition of $\mathcal{F}_{\alpha+1}$ to make sense, $\mathcal{F}_\alpha \cup \{A\}$ must have the FIP.) The inductive construction of the $\mathcal{F}_\beta$'s will be complete once it is proved that such an A always exists. This is where the hypothesis $P(\mathfrak{c})$ is used

~~By Remark 8.8, $\mathcal{F}_\alpha$ is generated~~
~~Since at most~~
~~Each $\mathcal{F}_\alpha$ has a gener~~

Each $\mathcal{F}_\alpha$ is generated by a family $\mathcal{S}_\alpha$ consisting of the cofinite sets and at most $|\alpha|$ other sets. This is proved by induction on α, using the fact that at most one new generator is needed at any successor stage (and none at limit stages). In particular, $\mathcal{S}_\alpha$ has cardinality $< \mathfrak{c}$, so $P(\mathfrak{c})$ provides an infinite $B \subseteq \omega$ such that is almost included in every $S \in \mathcal{S}_\alpha$, hence also in every intersection of finitely many such S, hence in every $S \in \mathcal{F}_\alpha$. The same holds for any infinite subset A of B, and therefore $\mathcal{F}_\alpha \cup \{A\}$ has the FIP. Thus, to obtain an A to adjoin to $\mathcal{F}_\alpha$ in forming $\mathcal{F}_{\alpha+1}$, it suffices to find an infinite $A \subseteq B$ on which f, the α^{th} requirement, is one-to-one or constant. This is easy if f takes ~~a single~~ the same value at infinitely points in B, for then these points ~~and form A on which f is~~ form the desired A; f is constant on it. It is also easy if f takes infinitely many distinct values on B, for then A can consist of exactly one pre-image of each of those values; f is one-to-one on this A. As B is infinite, one of these two easy cases must occur. This completes the inductive construction of the filters $\mathcal{F}_\alpha$, $\alpha \le \underset{\sim}{c}$, satisfying (a) through (e).

As in B7, $\mathcal{U} = \mathcal{F}_{\underset{\sim}{c}}$ is an ultrafilter. The new clause (e) in the construction ensures that all the new requirements f are satisfied by $\mathcal{U}$. Thus, $\mathcal{U}$ is selective. $\square$

By combining the preceding construction with that of Proposition C4, one can obtain $2^{\aleph_1}$ selective ultrafilters. The following ~~theorem leads to~~ proposition is a slight improvement of this result.

~~Lemma 9. $P(\underset{\sim}{c})$ implies that no non-principal ultrafilter on ω can be generated by fewer than $\underset{\sim}{c}$ sets.~~

~~Proof. Suppose $\mathcal{U}$ were a non-principal counterexample, with generating set $\mathcal{S}$, which may be assumed to contain all cofinite sets ...~~

<u>Proposition 9.</u> $P(\underset{\sim}{c})$ implies the existence of $2^{\underset{\sim}{c}}$ pairwise non-isomorphic selective ultrafilters.

<u>Proof.</u> Let Q be the set of ordinals $< \underset{\sim}{c}$ such that the α^{th} requirement in the proof of Theorem 8 was one of the new requirements f. Since there are $\underset{\sim}{c}$ functions $f: \omega \to \omega$, Q has cardinality $\underset{\sim}{c}$, so there are $2^{\underset{\sim}{c}}$ functions $h: Q \to 2$. ~~Modify the construct~~ For each such h, let $\mathcal{F}_\alpha^h$ be the filters defined by the following modification of the proof of Theorem 8. The inductive steps defining $\mathcal{F}_\beta$ are unchanged except when $\beta = \alpha + 1$ for some $\alpha \in Q$, i.e., the steps where ~~functions are made or sets are added [illegible] on which functions~~ the new requirements are satisfied. At such a stage α, proceed as in Theorem 8 to obtain an infinite B almost included in each set in $\mathcal{F}_\alpha^h$. (It is assumed that a choice function, assigning such a set to each $< \underset{\sim}{c}$-generated filter, has been fixed in advance. Thus, B depends on h only via $\mathcal{F}_\alpha^h$.) Before continuing with the construction in Theorem 8, let B' consist of every second element of B, the ones in an even (resp. odd) position in B if $h(\alpha) = 0$ (resp. $h(\alpha) = 1$). Of course, B' is also almost included in every set in $\mathcal{F}_\alpha^h$, and an A, as in the proof of

Theorem 8, can be found with $A \subseteq B'$. (Again, A is to depend only on B' and the function f that is to be one-to-one or constant on A.)

This modified construction is clearly as good as the original, in that it produces a selective ultrafilter $\mathcal{U}^h = \mathcal{F}^h_{\underset{\sim}{c}}$. ~~The proof will be concluded by showing~~ As in Proposition C4, that $g \neq h$ implies $\mathcal{U}^g \neq \mathcal{U}^h$. Indeed, if α is the first element of Q such that $g(\alpha) \neq h(\alpha)$ then $\mathcal{F}^g_\alpha = \mathcal{F}^h_\alpha$ because only the earlier values of g and h have been used, and thus B is the same in stage α of the two constructions using g and h. (It is here that the parenthetical stipulations in the preceding paragraph are used.) But the two constructions have disjoint sets B' at stage α, for one B' consists of the elements in even positions in B, the other of those in odd positions, as $g(\alpha) \neq h(\alpha)$. And, of these disjoint B''s, one is in $\mathcal{U}^g$, the other in $\mathcal{U}^h$, so these ultrafilters are distinct.

Thus, there are $2^{\underset{\sim}{c}}$ distinct selective ultrafilters. Each one can be isomorphic to at most $\underset{\sim}{c}$ others, as there are only $\underset{\sim}{c}$ functions from ω to ω that can serve as isomorphisms. As $2^{\underset{\sim}{c}} > \underset{\sim}{c}$, there must be $2^{\underset{\sim}{c}}$ pairwise non-isomorphic selective ultrafilters. $\square$

<u>Corollary 10</u>. $P(\mathfrak{c})$ implies that the RK-ordering of isomorphism types of non-principal ultrafilters on ω has $2^{\mathfrak{c}}$ minimal elements. In particular, it is not a linear ordering.

<u>Proof</u>. Combine Propositions 1 and 9. $\square$